INTERVAL ARITHMETIC APPLIED TO STRUCTURAL DESIGN OF UNCERTAIN MECHANICAL SYSTEMS

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Uncertainties in mechanical systems

- Modeling: Finite Element Method

Data → Uncertain / Variable

Variations of the parameters, the geometry → Variations of the results
- constraints
- displacements
- eigenfrequencies

- Hypothesis = Deterministic Structure

Computations made for the mean value of the parameters

Different types of uncertainties

- stochastic parameters: Young's modulus, density, thickness...
- unknown parameters : boundary conditions, assembly...
- evolutive parameters : aging, variation in time
- model uncertainties: behavior law, choice of the mesh...

Modeling

- stochastic variables

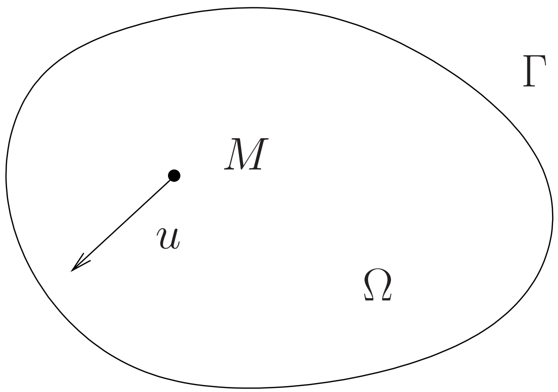
  generic laws (Gaussian, uniform)

  particular laws (bounded domain)

- Intervals

  uncertain and bounded variables

Finite Element Problems



Formulation : PDE

$$L(u) = 0 \qquad on \quad \Omega$$

$$C(u) = 0 \qquad on \quad \Gamma$$

Stationarity of an integral functional

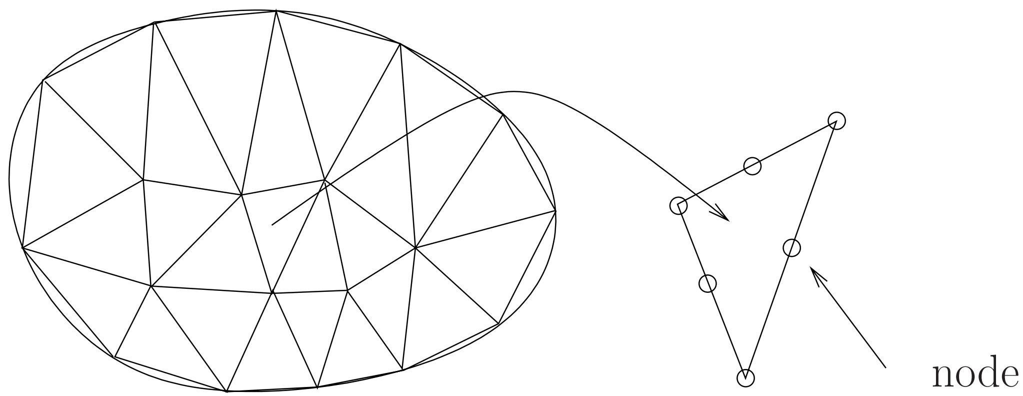
$$J(u) = \int R(u) d\Omega \qquad \delta J(u) = 0 \qquad u \in E_u$$

E_u space of the admissible fields

Discretization of E_u

sub domains (FE) and interpolation

$$\hat{u} = \sum_{nodes} U_i N_i(M)$$



Finite Element Problems

Elementary formulation $\longrightarrow$ Assembly $\longrightarrow$ global formulation

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

- Static problems

$$KU = F$$

- Dynamic problems

$$M\ddot{U} + R\dot{U} + KU = F$$

harmonic load $F = F_0 e^{\mathrm{i}\omega t}$

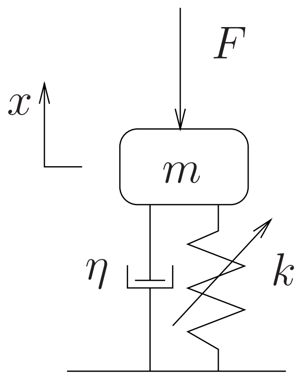
Frequency Response Function U_0

$$\left(K + \mathrm{i}\omega R - \omega^2 M\right) U_0 = F_0$$

- Properties of M and K

Symmetric, (semi) definite, positive

One DOF example



$F = 10$

$m = 1$

$k \in [99, 101]$

$\eta = 0.02$

Static problem

$$kx = F$$

$$x = \frac{F}{k} = \left[\frac{1}{11}, \frac{1}{9}\right]$$

Dynamic problem (Frequency response function)

$$\left(k(1 + \mathrm{i}\eta) - \omega^2 m\right) x = F_0$$

real problem: $\begin{bmatrix} k - \omega^2 m & -\eta k \\ \eta k & k - \omega^2 m \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$

k appears four times in the matrix!

Direct interval resolution (Intlab)

$$\begin{bmatrix} x_r \\ x_i \end{bmatrix} = \begin{bmatrix} [-0.3425, 0.3425] \\ [-0.6780, -0.3960] \end{bmatrix}$$

overestimation

Analytic solution: $\begin{bmatrix} x_r \\ x_i \end{bmatrix} = \begin{bmatrix} \frac{(k-\omega^2 m)F_0}{(\eta^2+1)k^2-2k\omega^2 m+\omega^4 m^2} \\ \frac{-\eta k F_0}{(\eta^2+1)k^2-2k\omega^2 m+\omega^4 m^2} \end{bmatrix}$

$$\begin{bmatrix} x_r \\ x_i \end{bmatrix} = \begin{bmatrix} [-0.204, 0.197] \\ [-0.501, -0.397] \end{bmatrix}$$

Formulation adapted to Finite Element Method

Young's modulus varying in $\boldsymbol{E}$

Stiffness matrix $\boldsymbol{E}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$

$$\begin{bmatrix} \boldsymbol{E}k_{11} & \boldsymbol{E}k_{12} \\ \boldsymbol{E}k_{21} & \boldsymbol{E}k_{22} \end{bmatrix} \curvearrowright \begin{bmatrix} E_1k_{11} & E_2k_{12} \\ E_3k_{21} & E_4k_{22} \end{bmatrix}$$

$E_i \in \boldsymbol{E}$ $\quad$ E_i independent

FEM
- Symmetric
- Definite (or semi-definite)
- Positive

General form:

$$[A] = [A_0] + \sum_{n=1}^{N} \boldsymbol{\epsilon_n}[A_n]$$

$$\{b\} = \{b_0\} + \sum_{p=1}^{P} \boldsymbol{\beta_p}\{b_p\}$$

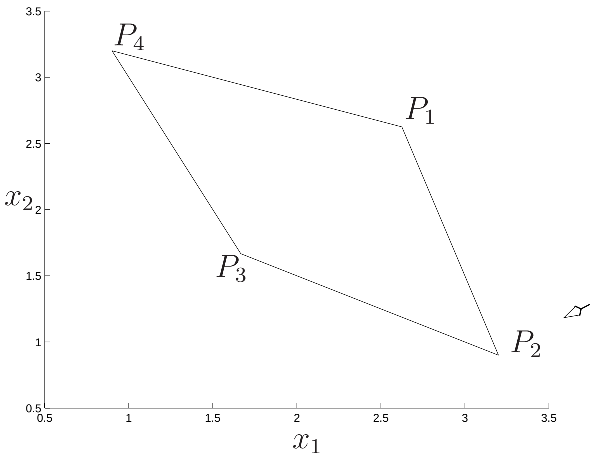
$$[K] = [K_0] + \sum_{n=1}^{N} \boldsymbol{\epsilon_n}[K_n]$$

for $\epsilon_n \in \boldsymbol{\epsilon_n}$

$[K]$ corresponds to a stiffness matrix

Symmetry, Vertex

$$[\boldsymbol{A}] = \begin{bmatrix} [3,4] & [1,2] \\ [1,2] & [3,4] \end{bmatrix}$$



$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \{x\} = \begin{Bmatrix} 10.5 \\ 10 \end{Bmatrix}$$

$$\left\{ \begin{bmatrix} 38 & -10 \\ -10 & 6 \end{bmatrix} + \boldsymbol{e_1} \begin{bmatrix} 10 & -2 \\ -2 & 2 \end{bmatrix} \right\} \boldsymbol{\{x\}} = \begin{Bmatrix} -44 \\ 20 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} [\frac{-1}{2}, \frac{-1}{3}] \\ [\frac{9}{5}, \frac{13}{3}] \end{Bmatrix}$$

$e_1 = -1 \longrightarrow$ $x_1 = \frac{-1}{3} \leftarrow$ sup bound, $x_2 = \frac{13}{3} \leftarrow$ sup bound

$e_1 = 0 \longrightarrow$ $x_1 = \frac{-1}{2} \leftarrow$ inf bound, $x_2 = \frac{5}{2}$

$e_1 = 1 \longrightarrow$ $x_1 = \frac{-7}{15}$, $x_2 = \frac{9}{5} \leftarrow$ inf bound

The algorithm of Rump

Rump's inclusion

Fix point theorem

$$[A]\{x\} = \{b\} \Leftrightarrow \{x\} = \{x_0\} + \{x^*\}$$

$$\{x^*\} = [G]\{x^*\} + \{g\}$$

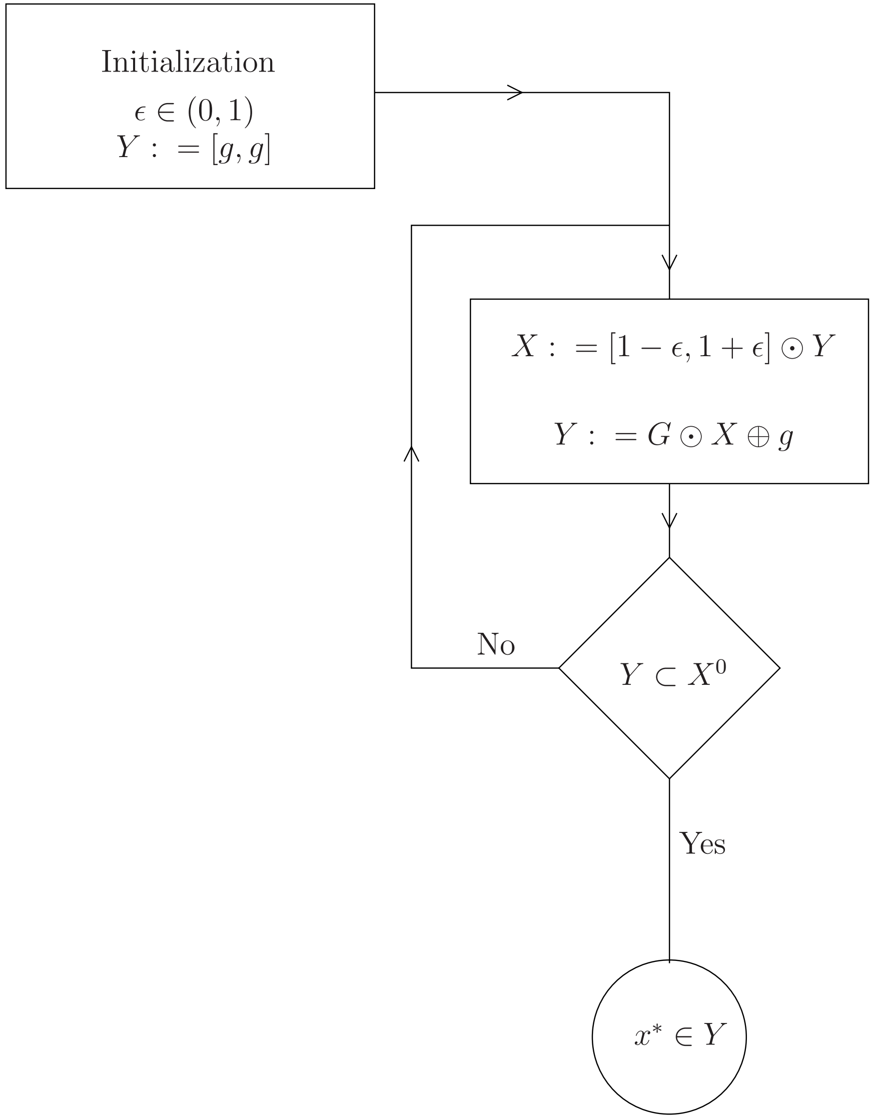
with $\quad [G] = [I] - [R][A]$

$$\{g\} = [R](\{b\} - [A]\{x_0\})$$

In practice

$$[R] \approx [A^{-1}]$$

$$\{x_0\} = [R]\{b\}$$



Algorithm adapted to the mechanical formulation

Taking into a account the factorization
example with one (centered) parameter $\boldsymbol{\alpha}$

$$[\boldsymbol{A}] = [A_0] + \boldsymbol{\alpha}[A_1]$$

$$[R] = mid([\boldsymbol{A}])^{-1} = [A_0]^{-1}$$

$$\{x_0\} = [R]\{b\} = [A_0]^{-1}\{b\}$$

$$\{\boldsymbol{x_1}\} = [\boldsymbol{G}]\{x_0\} + \{g\}$$

$$[\boldsymbol{G}] = [I] - [R][\boldsymbol{A}] = -\boldsymbol{\alpha}[A_0]^{-1}[A1]$$

$$\{g\} = [R](\{b\} - [\boldsymbol{A}]\{x_0\}) = -\boldsymbol{\alpha}[A_0]^{-1}[A_1]\{x_0\}$$

$$\{\boldsymbol{x_n}\} = \{\boldsymbol{x_{n-1}}\} + (-1)^{n+1}\boldsymbol{\alpha^{n+1}\epsilon^{n}}[B]^{n+1}\{x_0\}$$

$$[B] = [A_0]^{-1}[A_1]$$

Convergence

$$\rho(|G|) < 1$$

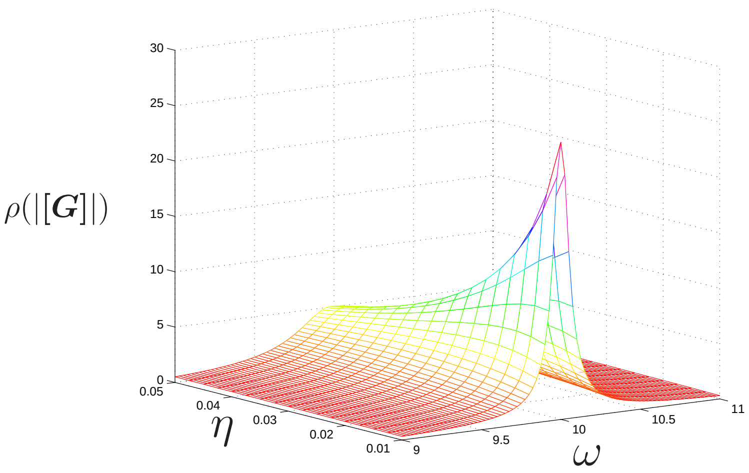
partition of $\boldsymbol{e}$:

$$\boldsymbol{e} = \cup_i \boldsymbol{e_i} \qquad \rho(|G(\boldsymbol{e_i})|) < 1 \qquad \forall \boldsymbol{e_i}$$

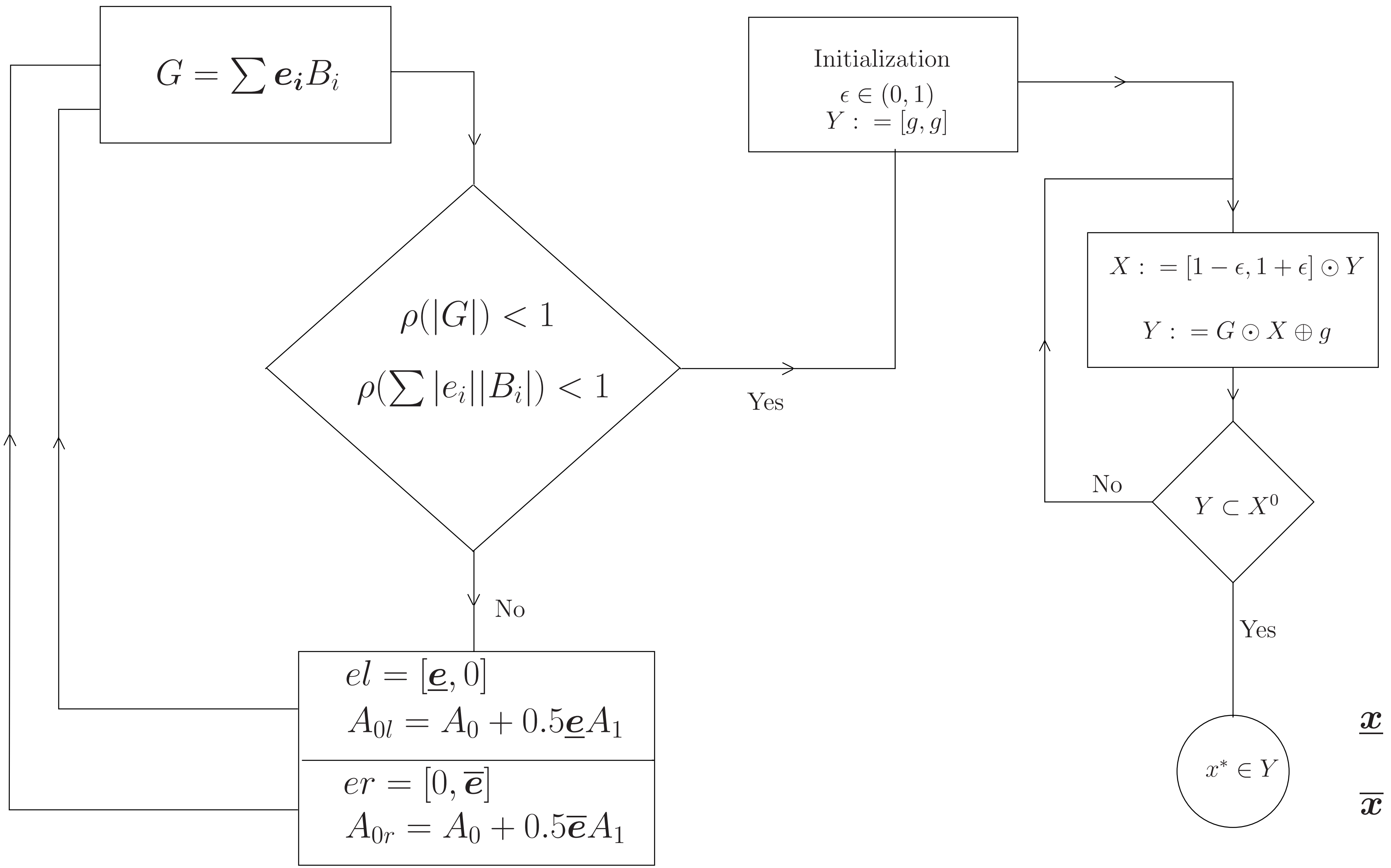
$$\boldsymbol{x} = \cup_i \boldsymbol{x_i}$$

$$[\boldsymbol{G}] = -\boldsymbol{\epsilon_1}[A_0]^{-1}[A_1]$$

$$\rho(|[\boldsymbol{G}]|) = \rho(\eta,\omega) = \frac{k_1|k_0(1+\eta^2)-\omega^2 m| + \eta k_1 \omega^2 m}{|(1+\eta^2)k_0^2 - 2k_0\omega^2 m + \omega^4 m^2|}$$



The proposed algorithm



$\underline{\boldsymbol{x}} = inf(\underline{\boldsymbol{x_i}})$

$\overline{\boldsymbol{x}} = sup(\overline{\boldsymbol{x_i}})$

Example

$$1e5\begin{bmatrix}[2.178, 2.360] & [-1.18, -1.08] \\ [-1.18, -1.08] & [0.72, 0.787]\end{bmatrix}\begin{Bmatrix}d\\ \theta\end{Bmatrix} = \begin{Bmatrix}[-102, -98]\\ [44.1, 45.9]\end{Bmatrix}$$

$$\boldsymbol{EI}\begin{bmatrix}12 & -6\\ -6 & 4\end{bmatrix}\begin{Bmatrix}d\\ \theta\end{Bmatrix} = \begin{Bmatrix}[-102, -98]\\ [44.1, 45.9]\end{Bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix}12 & -6L\\ -6L & 4L^2\end{bmatrix}\begin{Bmatrix}d\\ \theta\end{Bmatrix} = \begin{Bmatrix}F\\ M\end{Bmatrix}$$

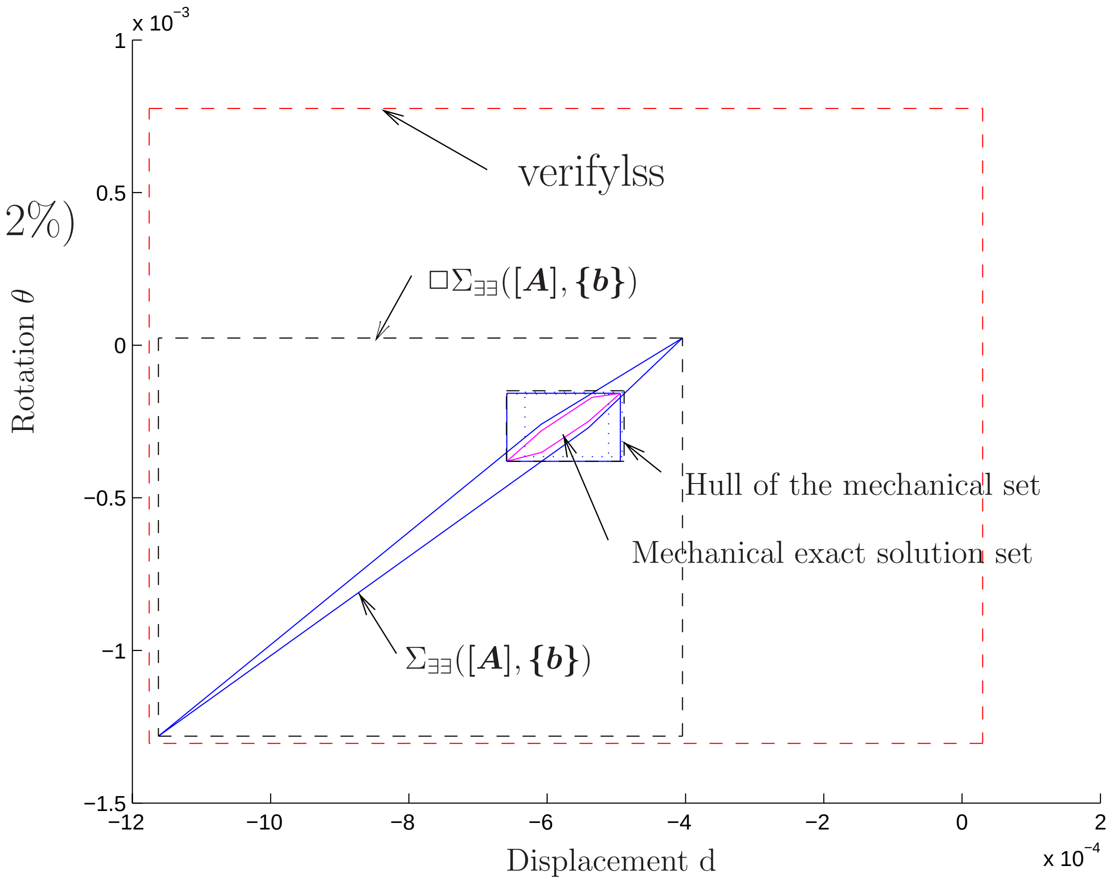
$EI \in [1.8151, 1.9664]e4 \quad (1.8908e4 \pm 2\%)$

$F \in [-102, -98] \qquad M \in [44.1, 45.9]$

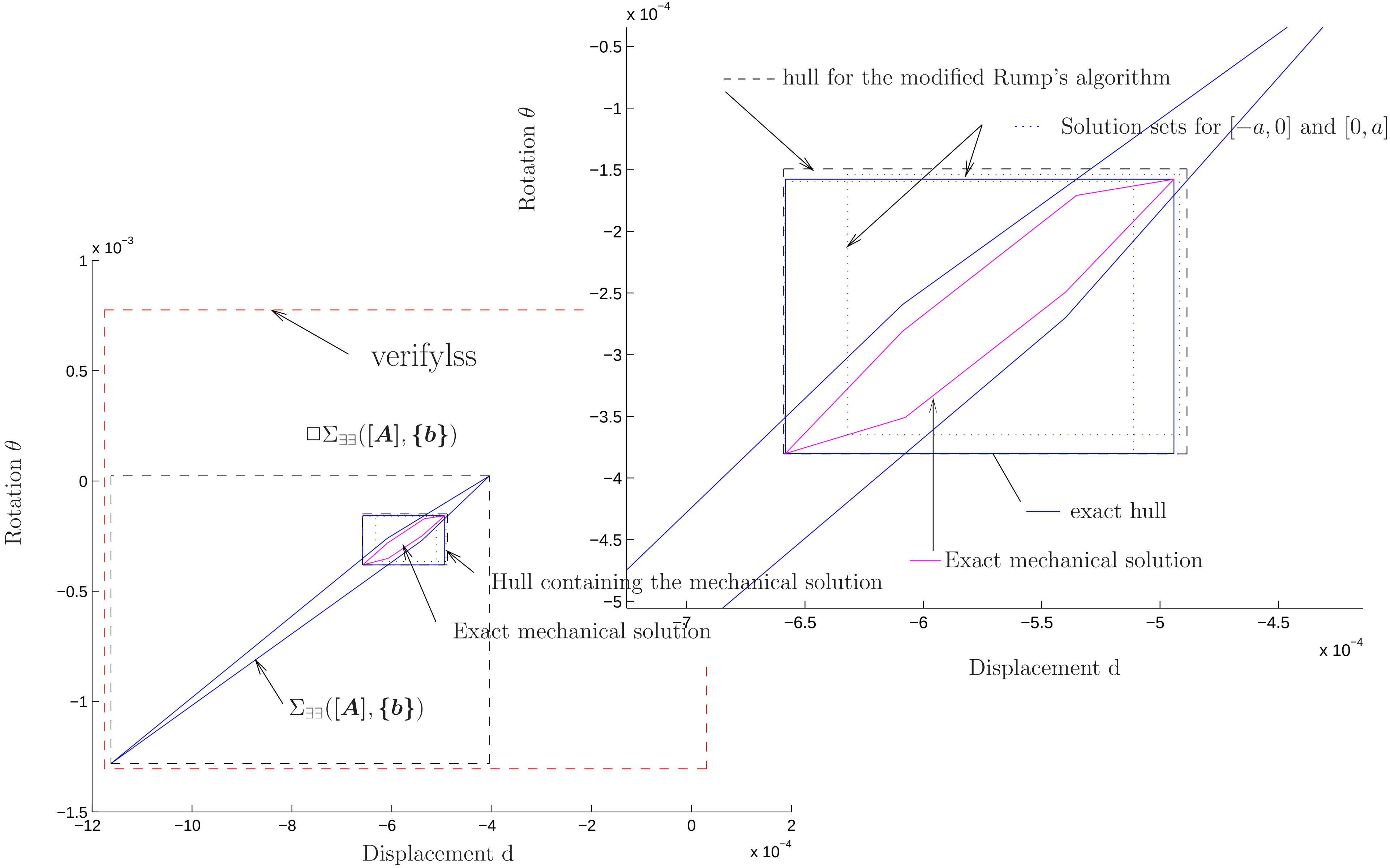
$L = 1$

$$\begin{Bmatrix}d\\ \theta\end{Bmatrix} = \begin{Bmatrix}[-1.163, -0.404]\\ [-1.282, 0.024]\end{Bmatrix}e-3$$

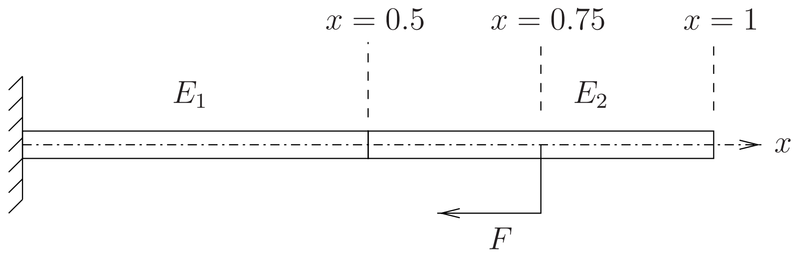
$$\begin{Bmatrix}d\\ \theta\end{Bmatrix} = \begin{Bmatrix}[-0.659, -0.494]\\ [-0.381, -0.157]\end{Bmatrix}e-3$$



Algorithm adapted to the mechanical formulation

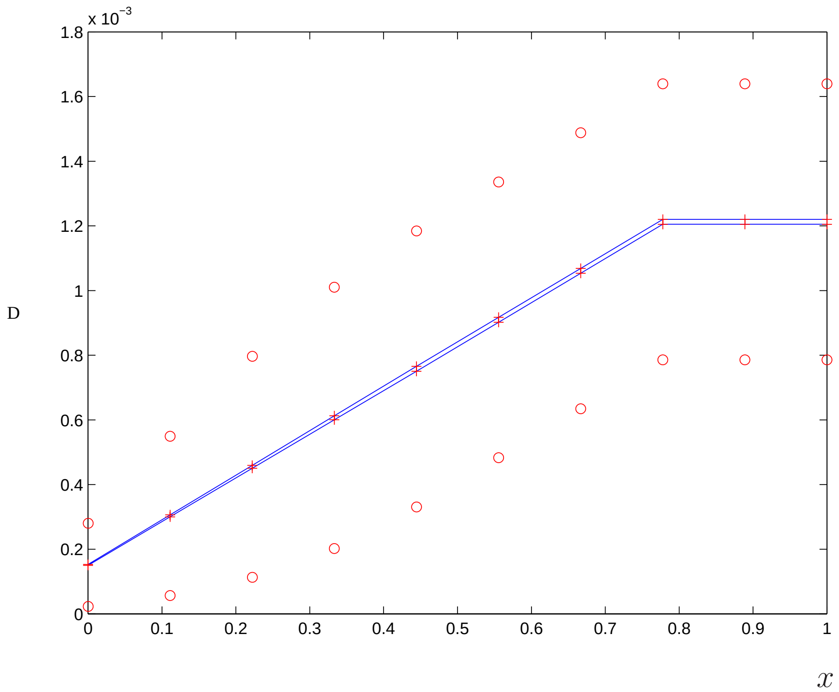


Clamped free Bar. Comparison

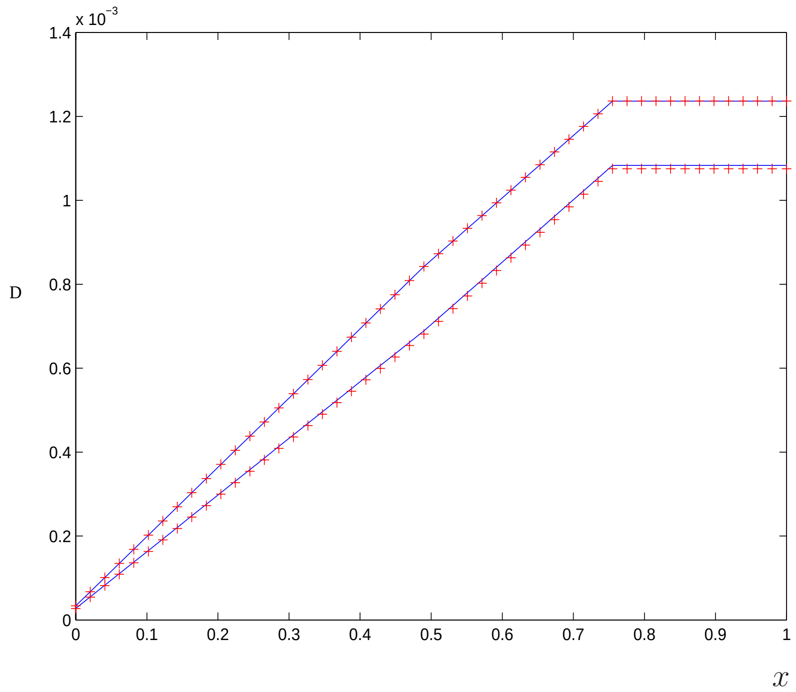


$$E_2 = 210 \pm 10\%\ GPa$$

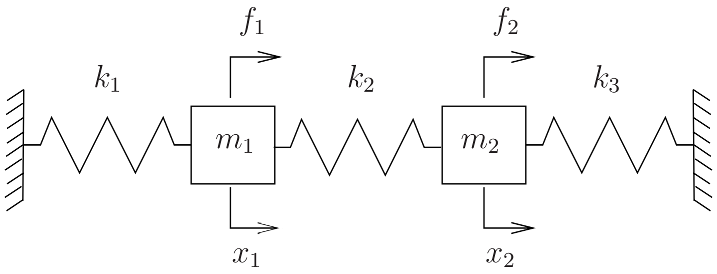
Original Rump's algorithm



Adapted algorithm



Example 1



Static Case

$$\left(\begin{bmatrix} k_1^0+k_2^0 & -k_2^0 \\ -k_2^0 & k_2^0+k_3^0 \end{bmatrix} + e_1\begin{bmatrix} k_1^1 & 0 \\ 0 & 0 \end{bmatrix} + e_2\begin{bmatrix} k_2^1 & -k_2^1 \\ -k_2^1 & k_2^1 \end{bmatrix} + e_3\begin{bmatrix} 0 & 0 \\ 0 & k_3^1 \end{bmatrix}\right)\begin{Bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Monte Carlo 100000

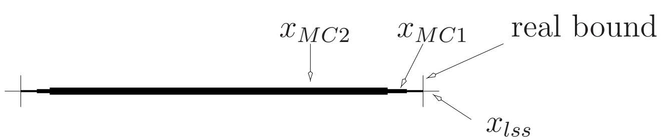
$$x_{MC1} = \begin{Bmatrix} [0.010374, 0.011333] \\ [0.018436, 0.019962] \end{Bmatrix}$$

Monte Carlo 1000

$$x_{MC2} = \begin{Bmatrix} [0.010399, 0.011306] \\ [0.018454, 0.019925] \end{Bmatrix}$$

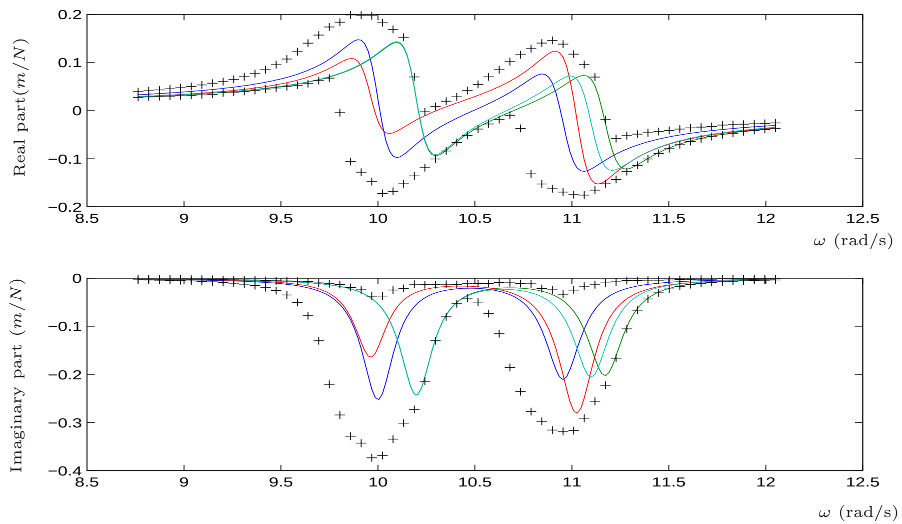
Proposed algorithm

$$x_{lss} = \begin{Bmatrix} [0.010331, 0.011353] \\ [0.018394, 0.019968] \end{Bmatrix}$$

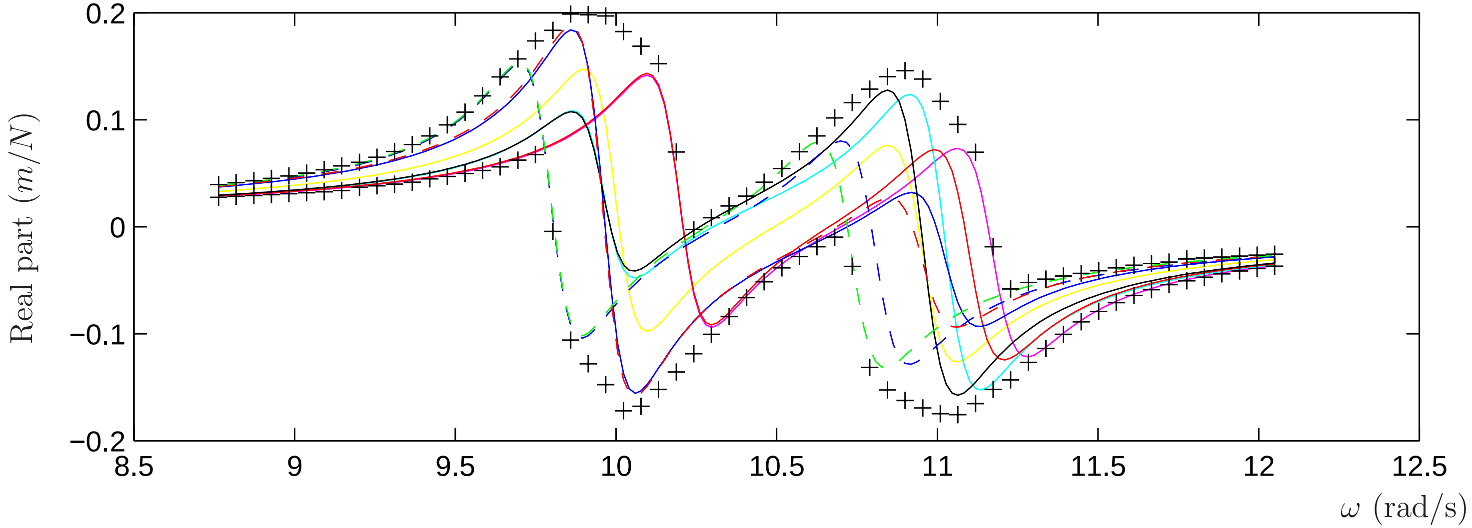


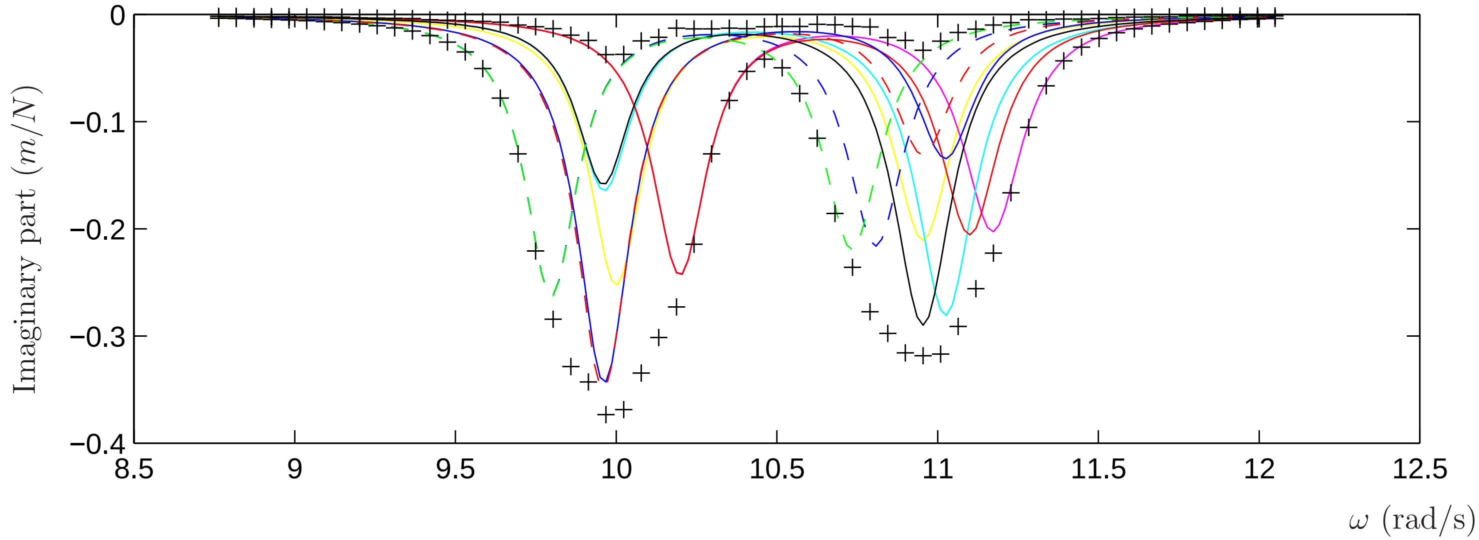
Dynamic Case

$$\left(\begin{bmatrix} (1+i\eta_1)k_1^0+(1+i\eta_2)k_2^0 & -(1+i\eta_2)k_2^0 \\ -(1+i\eta_2)k_2^0 & (1+i\eta_2)k_2^0+(1+i\eta_3)k_3^0 \end{bmatrix} + \boldsymbol{e_1}\begin{bmatrix} (1+i\eta_1)k_1^1 & 0 \\ 0 & 0 \end{bmatrix} + \boldsymbol{e_2}\begin{bmatrix} (1+i\eta_2)k_2^1 & -(1+i\eta_2)k_2^1 \\ -(1+i\eta_2)k_2^1 & (1+i\eta_2)k_2^1 \end{bmatrix} + \boldsymbol{e_3}\begin{bmatrix} 0 & 0 \\ 0 & (1+i\eta_3)k_3^1 \end{bmatrix} - \omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right)\begin{Bmatrix} \boldsymbol{H_1} \\ \boldsymbol{H_2} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

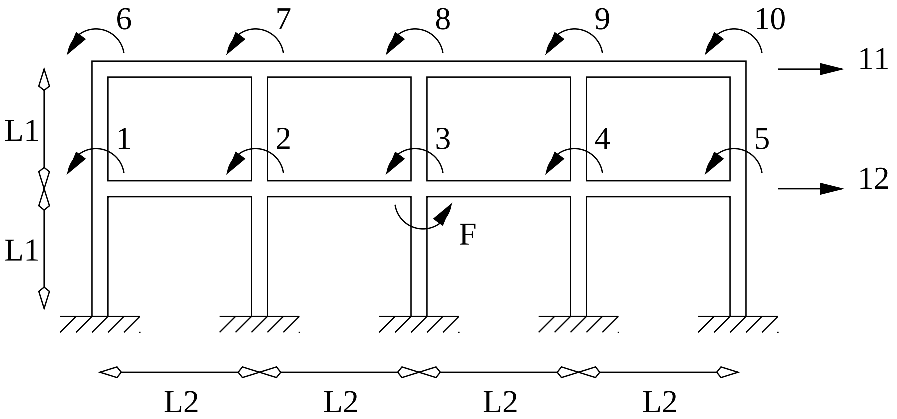


Example 1


0.2
0.1
0
−0.1
−0.2
Real part (m/N)
8.5
9
9.5
10
10.5
11
11.5
12
12.5
ω (rad/s)


0
−0.1
−0.2
−0.3
−0.4
Imaginary part (m/N)
8.5
9
9.5
10
10.5
11
11.5
12
12.5
ω (rad/s)

Example 2 : Truss structure



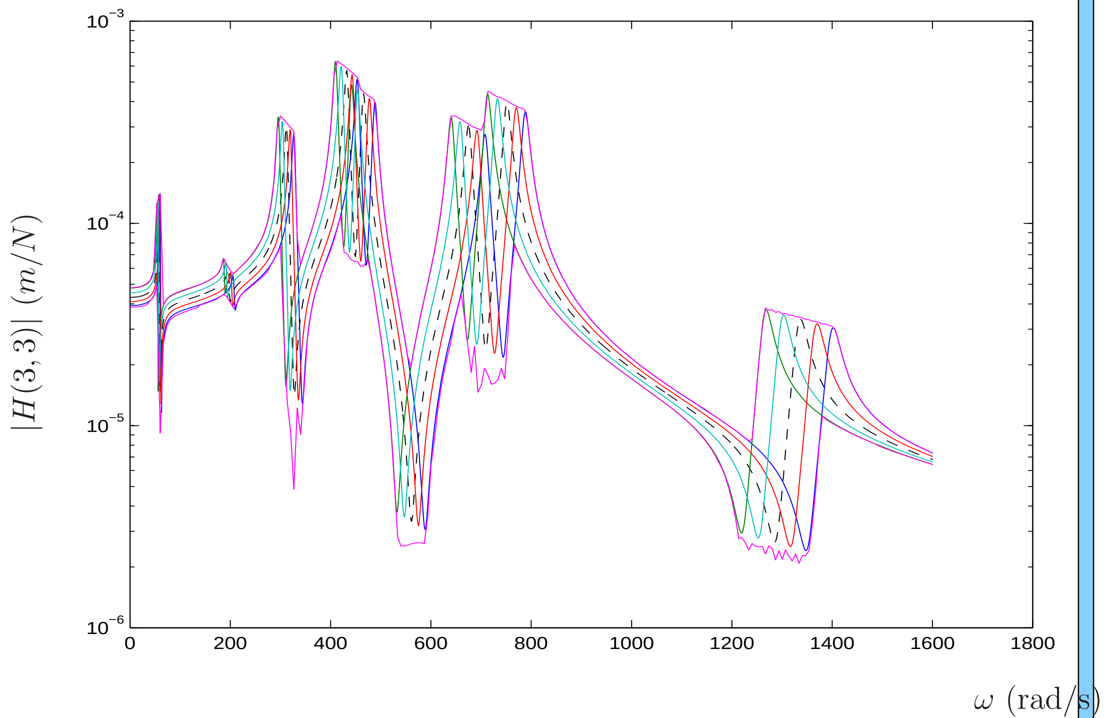
$L1 = L2 = 1m$

$F = 10^3\ Nm$

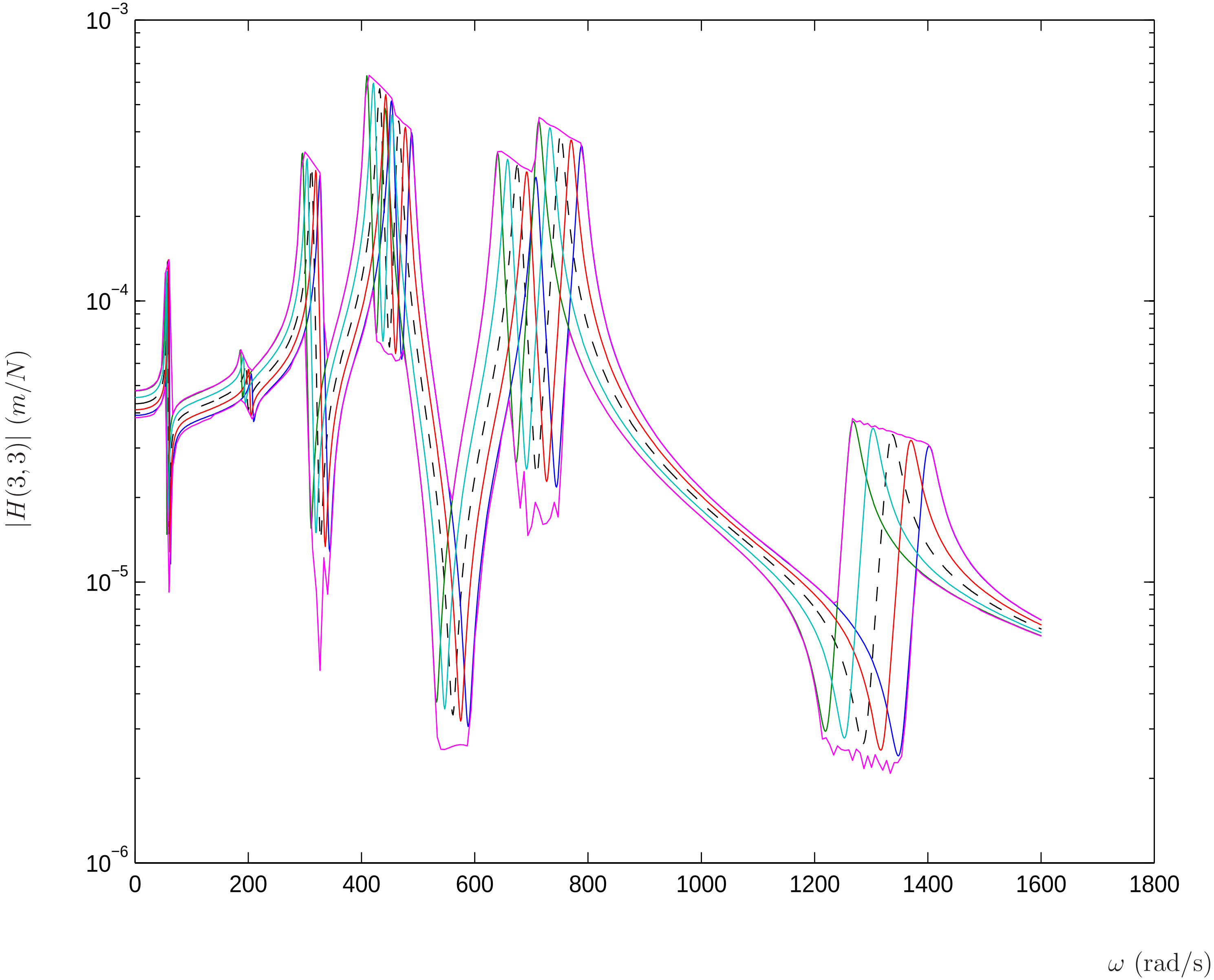
$I = \pi\ 10^{-8}/4\ m^4$

$S = \pi\ 10^{-4}\ m^2$

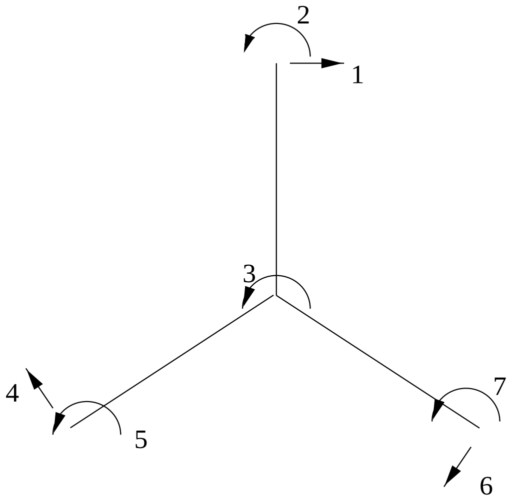
$E = 210 \pm 10\%\ GPa$



Modulus of the FRF $H(3,3)$ of the truss.
The stiffness is uncertain ($E = 210 \pm 10\%$ GPa).
Min and max values, FRF for several values of E



Example 3: Multiple eigenvalues system

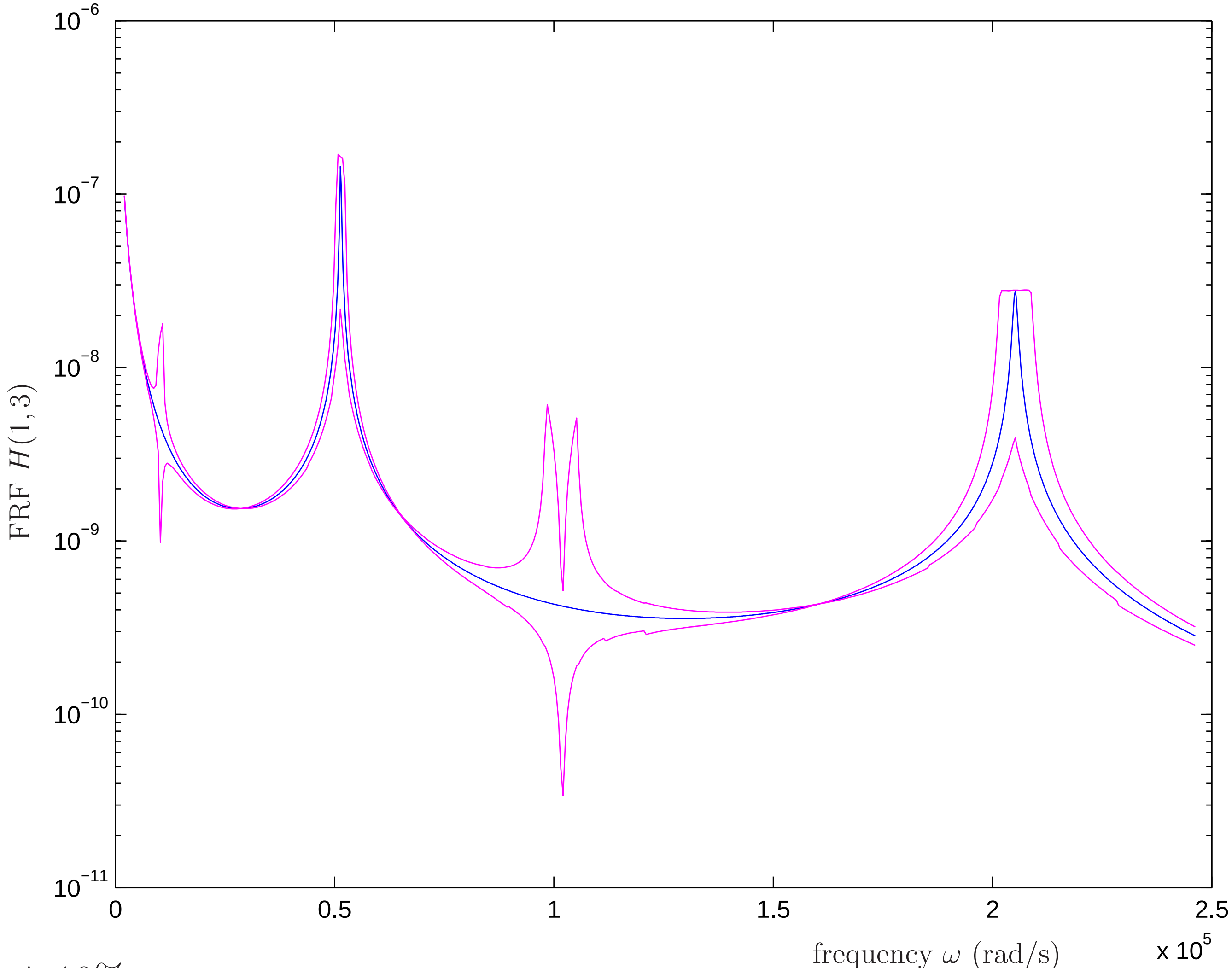


$L = 1\ m \quad S = \pi 10^{-4}\ m^2$

$\rho = 7800\ kg/m^3$

$E_0 = 210\ GPa$

$\eta = 2\%$

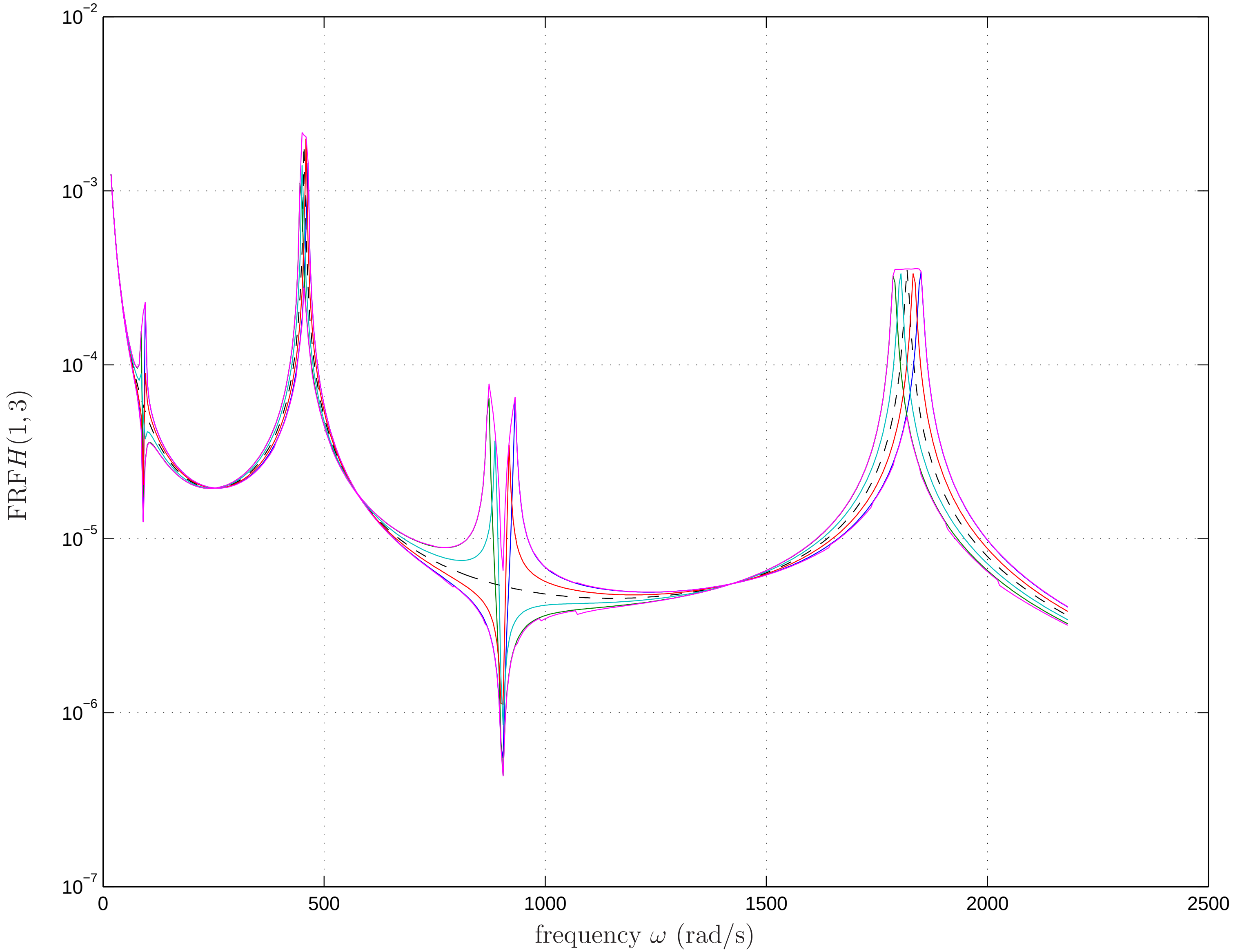


For the first blade $E = E_0 \pm 10\%$

Modulus of the transfer function $H(1,3)$

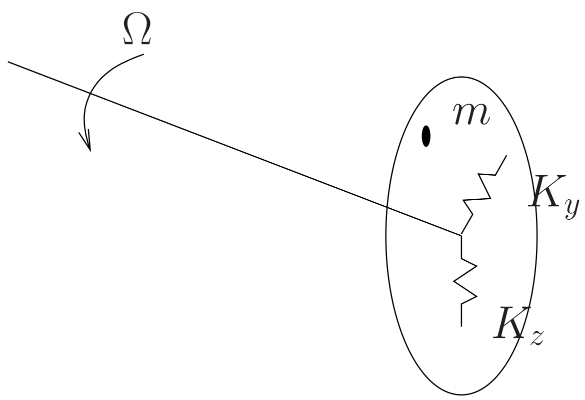
$E = E_0 \pm 10\%$ for the first blade

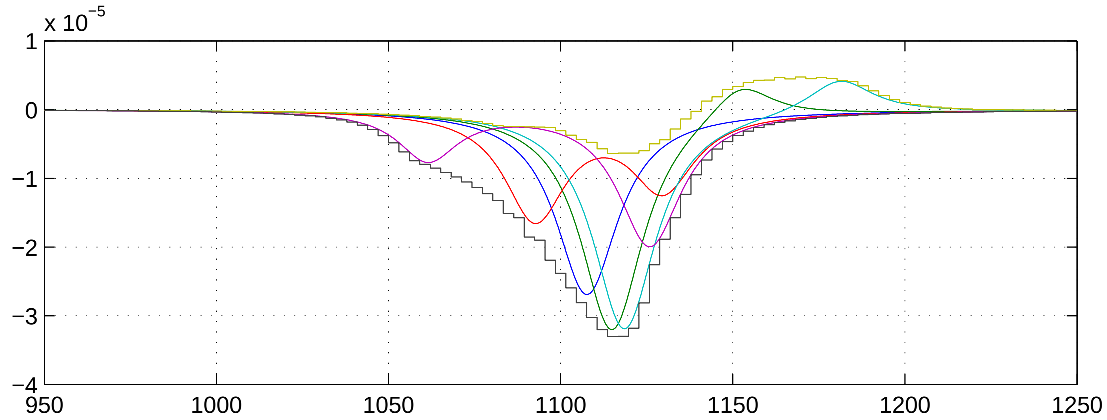
Crisp cases, envelope calculated with the modified algorithm



Gyroscopic System

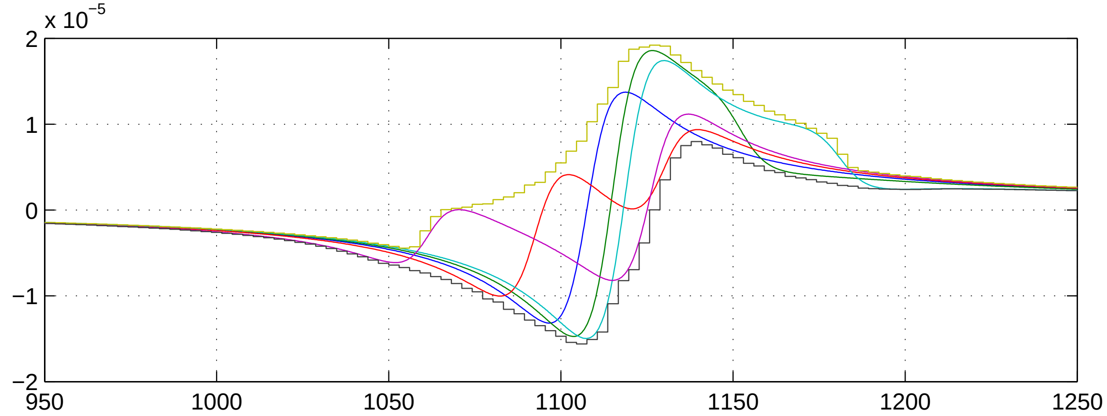
$$(J' + J_B)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\Psi}\\ \ddot{\Theta}\end{Bmatrix} + J'a\Omega\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{Bmatrix}\dot{\Psi}\\ \dot{\Theta}\end{Bmatrix} + K_z L^2\begin{bmatrix}\mu & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\Psi\\ \Theta\end{Bmatrix} = J_B\frac{r}{L}\Omega^2\begin{Bmatrix}\cos\Omega t\\ \sin\Omega t\end{Bmatrix}$$





unbalanced rotating disk

Ω and $\mu = \frac{K_y}{K_z}$ intervals



$$\left(\begin{bmatrix}-\Omega_0 * M + K_0 & -\Omega_0 * G - \eta K_0\\ \Omega_0 * G + \eta K_0 & -\Omega_0 * M + K_0\end{bmatrix} + \boldsymbol{e_1}\begin{bmatrix}-\Omega_r M & -\Omega_r G\\ \Omega_r G & -\Omega_r M\end{bmatrix} + \boldsymbol{e_2}\begin{bmatrix}K_1 & -\eta K_1\\ \eta K_1 & K_1\end{bmatrix}\right)\begin{Bmatrix}\Psi\\ \Theta\end{Bmatrix} = \Omega_0\begin{Bmatrix}F_r\\ F_i\end{Bmatrix} + \boldsymbol{e_1}\Omega_r\begin{Bmatrix}F_r\\ F_i\end{Bmatrix}$$

Conclusion

- Use of Interval Arithmetic
  - ☐ overestimations
  - ☐ resolution of linear systems
- Mechanical Formulation
  - ☐ $[A] = [A_0] + \boldsymbol{\alpha}[A_1]$
- Modification of the algorithm of Rump
  - ☐ $\longrightarrow$ "smallest" Hull
  - ☐ convergence
  - ☐ efficiency
- "robust" envelopes
  - ☐ Static cases
  - ☐ Dynamic cases (FRF)