# Verification Methods for the Linear Complementarity Problem with Interval Data

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Let $M \in \mathbf{R}^{n\times n}$ and $q \in \mathbf{R}^n$. Then the linear complementarity problem is defined as follows: Determine (or conclude that there is no) $z \in \mathbf{R}^n$ with

$$q + Mz \geq 0, \quad z \geq 0, \quad (q + Mz)^T z = 0. \tag{1}$$

(Here, matrix inequalities are understood componentwise, [2].) Linear complementarity problems model many important mathematical problems. The article [3] gave an extensive documentation of complementarity problems in engineering and equilibrium modeling.

Meanwhile, verification methods have been found to give guaranteed bounds on the distance between the numerical solution and the exact solution of the linear complementarity problem (see e.g. [1]).

In this talk we extend this idea to the case where the input data itself are not known exactly but can only be enclosed in intervals. This situation arises for example from the following application.

Let $f : [0, \infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function and $y_0 > 0$. Then the free boundary problem is defined as follows:

$$\left.\begin{array}{rcl} \text{Find } c \in \mathbf{R} & \text{and} & y(\cdot) : [0,\infty) \to \mathbf{R} \text{ with} \\ y''(x) = f(x, y(x), y'(x)) & & \text{if } x \in [0, c], \\ y(x) > 0 & & \text{if } x \in [0, c), \\ y(x) = 0 & & \text{if } x \in [c, \infty), \\ y'(c) = 0, & & y(0) = y_0. \end{array}\right\} \tag{2}$$

**Theorem 1.** ([4]) *The free boundary problem (2) is considered. It is assumed that (2) has a unique solution* $\left(\tilde{c}, \tilde{y}(\cdot)\right)$ *and it is assumed that an* $a \in \mathbf{R}$ *with* $\tilde{c} \leq a$ *is known.* $n + 2$ *points are determined by* $x_0 := 0$, $h := a/(n + 1)$, $x_{i+1} := x_i + h$, $i = 0, ..., n$. *Let* $f$ *fulfill the following conditions:*

- $f(x, s, t) : [0, \infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *is continuously differentiable.*

- *There exists an interval* $[F] = [\underline{F}, \overline{F}]$ *with* $\{\mu \in \mathbf{R} : \mu = f(x, \tilde{y}(x), \tilde{y}'(x)),\ x \in [0,a]\} \subseteq [F]$ *and* $\underline{F} \geq 0$.
- *There exists* $D \in \mathbf{R}$ *with* $|f_x(x, \tilde{y}(x), \tilde{y}'(x)) + f_s(x, \tilde{y}(x), \tilde{y}'(x))\tilde{y}'(x) + f_t(x, \tilde{y}(x), \tilde{y}'(x))\tilde{y}''(x)| \leq D$, $x \in [0, \tilde{c}]$.

*Then there exists a vector* $q \in \mathbf{R}^n$ *contained in the interval vector*

$$[q] = \frac{1}{2}\begin{pmatrix} \left[\frac{1}{2}, 1\right] h^2[F] + \frac{1}{2}h^3[-D, D] - y_0 \\ \left[\frac{1}{2}, 1\right] h^2[F] + \frac{1}{2}h^3[-D, D] \\ \vdots \\ \left[\frac{1}{2}, 1\right] h^2[F] + \frac{1}{2}h^3[-D, D] \end{pmatrix}$$

*and the vector*

$$\tilde{y} := \begin{pmatrix} \tilde{y}(x_1) \\ \vdots \\ \tilde{y}(x_n) \end{pmatrix}$$

*is the unique solution of the linear complementarity problem defined by that* $q$ *and*

$$M = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{pmatrix} \in \mathbf{R}^{n \times n}.$$

The proof uses Taylor's formula with remainder where one has to take into account that it is not necessary that $\tilde{y}(x) : [0, \infty) \to \mathbf{R}$ is twice differentiable at $x = \tilde{c}$. In addition, we want to emphasize that it is not possible to prove Theorem 1 with $M \in \mathbf{R}^{n \times n}$ and $[q] = q \in \mathbf{R}^n$ (even if $f \equiv 1$).

In order to verify that a given interval vector $[z]$ includes $\tilde{y}$ of Theorem 1 we have to consider a family of linear complementarity problems. The presented verification methods are based on the following equivalence

$$(1) \Leftrightarrow H(z) := \min(z, q + Mz) = 0$$

and the knowledge of a slope matrix $G(x, y)$ satisfying

$$H(x) - H(y) = G(x, y)(x - y) \quad \text{for all } x, y \in [z].$$

For the matrix $M$ and the interval vector $[q]$ of Theorem 1 we define $H(z; [q])$ componentwise by

$$(H(z; [q]))_i = \left\{ \begin{array}{ccc} z_i & \text{if} & z_i < \underline{q_i} + (Mz)_i, \\ [q_i] + (Mz)_i & \text{if} & z_i > \overline{q_i} + (Mz)_i, \\ [\underline{q}_i + (Mz)_i, z_i] & \text{if} & z_i \in [q_i] + (Mz)_i, \end{array} \right\} i = 1, .., n,$$

and we present an algorithm that calculates an interval matrix $G(x,[z],[q])$ with arbitrary $x \in [z]$ satisfying

$$G(x,y) \in G(x,[z],[q]) \text{ for all } y \in [z] \text{ and for all } q \in [q].$$

Then we define the operator

$$N(x,[z],[q]) := x - IGA(G(x,[z],[q]), H(x;[q]))$$

($IGA$ means interval Gaussian algorithm) and the operator

$$L(x,A,[z],[q]) := x - A \cdot H(x;[q]) + \Big(I - A \cdot G(x,[z],[q])\Big)([z] - x)$$

with arbitrary $x \in [z]$ and an arbitrary nonsingular matrix $A$. Using the Brouwer fixed point theorem we include $\tilde{y}$ of Theorem 1. We present some examples.

## References


[1] G. Alefeld, X. Chen, and F. Potra, "Numerical validation of solutions of linear complementarity problems", *Numer. Math.*, 1999, Vol. 83, pp. 1–23.

[2] R. W. Cottle, J. S. Pang, and R. E. Stone, *The linear complementarity problem*, Academic Press, 1992.

[3] M. C. Ferris and J. S. Pang, "Engineering and economic applications of complementarity problems", *SIAM Rev.*, 1997, Vol. 39, No. 4, pp. 669–713.

[4] U. Schäfer, "An enclosure method for free boundary problems based on a linear complementarity problem with interval data", *Numerical Functional Analysis and Optimization*, 2001, Vol. 22, No. 7-8, pp. 991–1011.