# On the interval Cholesky method

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## Outline

1. Introduction
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## 1. Introduction

Given: $[A] = [A]^T = ([a]_{ij}) \in \mathbb{IR}^{n\times n}$ regular, $[b] = ([b]_i) \in \mathbb{IR}^n$

Enclose the symmetric solution set

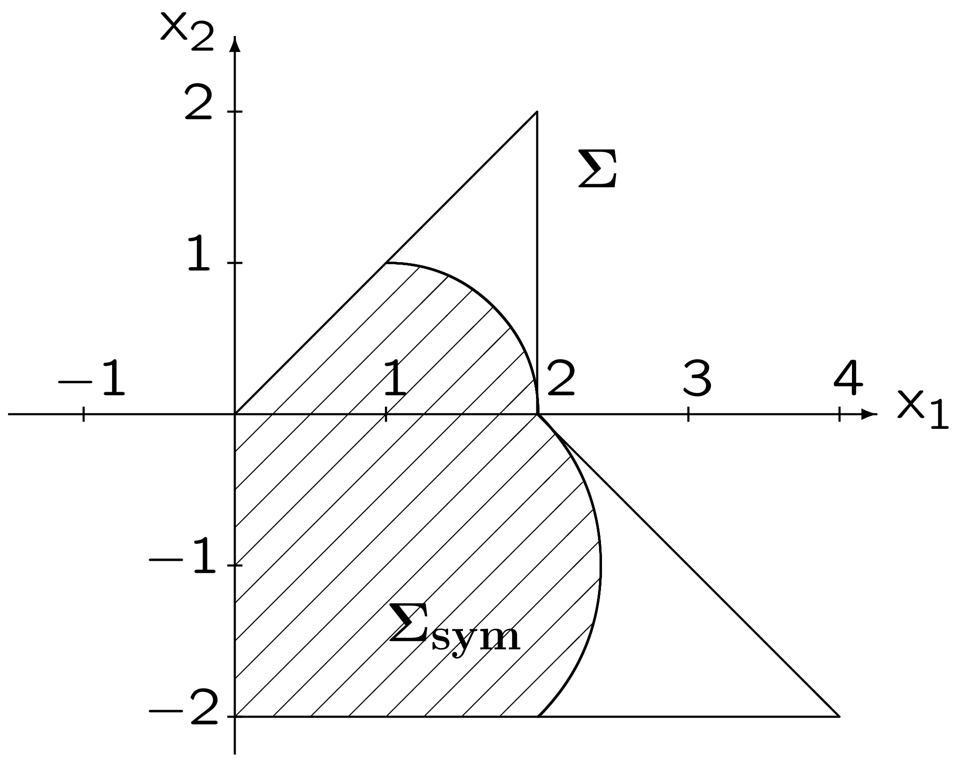
$$\Sigma_{\mathrm{sym}} \;:=\; \{\, x \in \mathbb{R}^n \mid Ax = b,\ A = A^T \in [A] = [A]^T,\ b \in [b] \,\}$$

More frequently: Enclose the solution set

$$\Sigma \;:=\; \{\, x \in \mathbb{R}^n \mid Ax = b,\ A \in [A],\ b \in [b] \,\}$$

## Example 1

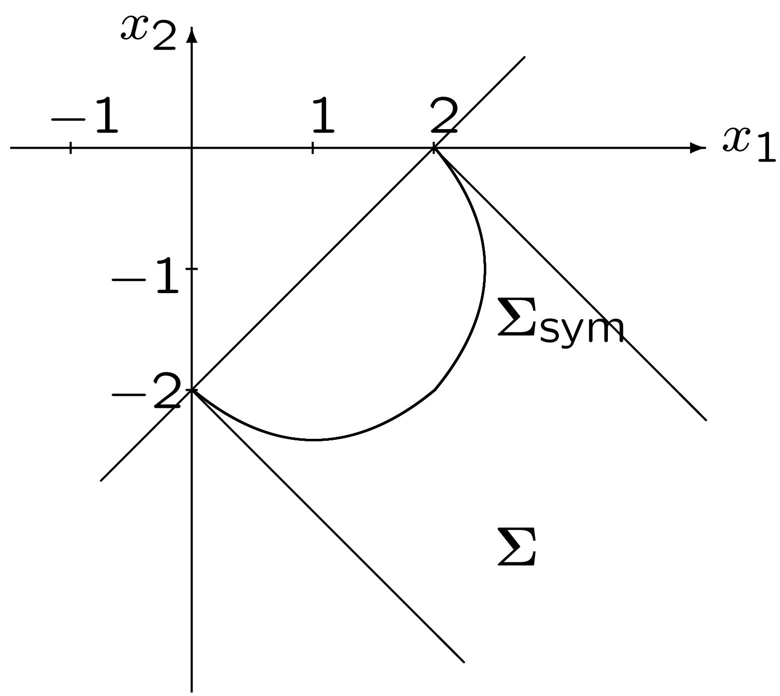
$$[A] = \begin{pmatrix} 1 & [0,1] \\ [0,1] & [-4,-1] \end{pmatrix}, \quad [b] = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix}.$$



**Example 2**

$$[A] = \begin{pmatrix} 1 & [-1,1] \\ [-1,1] & -1 \end{pmatrix}, \qquad [b] = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

$[A]$ contains two sing. matrices but no sing. symmetric matrix.




$[-1,1] \rightsquigarrow [-1+\varepsilon, 1-\varepsilon]$ : Arbitrarily large overestimation!

*Thus the hull inverse is not adapted to the optimal treatment of symmetric matrices.*

*However, at present, no special methods have been devised for this case, and we shall content ourselves with the unsymmetric treatment of symmetric matrices.*

Neumaier, Interval Methods for Systems of Equations, 1990, p. 95.

*Unless you are able to handle dependent data, you will never get interest of the engineers.*

Babuška, conversation with J. Rohn, 1992.

**Publications on $\Sigma_{\text{sym}}$** (Cholesky method highlighted)

| | |
|---|---|
| Neumaier | 1985, Dec. 23, letter to Rohn |
| | 1990 |
| Rohn | 1986 (talk 1990; published 2004) |
| Jansson | 1990 (talk; published 1991) |
| Alefeld / M. | 1993, 1995, 2008 |
| Rump | 1994 |
| Alefeld / Kreinovich / M. | 1995, 1996, 1997, 1998, 2001, 2003 |
| M. | 2001, 2007 |
| Schäfer | 2001 |
| Hladík | 2007, 2008 |
| | |
| Popova | 2002, 2004, 2007 |
| Popova / Krämer | 2007, 2008 |
| | |
| Sharaya / Shary | 2008 |

## **2. Interval Cholesky method** (Alefeld/M. 1993, 1995, 2008)

1. $LL^T$ decomposition

**for** $j := 1$ **to** $n$ **do**

$$[l]_{jj} := ([a]_{jj} - \sum_{k=1}^{j-1} [l]_{jk}^2)^{1/2}$$

**for** $i := j + 1$ **to** $n$ **do**

$$[l]_{ij} := ([a]_{ij} - \sum_{k=1}^{j-1} [l]_{ik}[l]_{jk}) \;/\; [l]_{jj}$$

2. Forward substitution

**for** $i := 1$ **to** $n$ **do**

$$[y]_i := ([b]_i - \sum_{j=1}^{i-1} [l]_{ij}[y]_j)/[l]_{ii}$$

3. Backward substitution

**for** $i := n$ **downto** 1 **do**

$$[x]_i^C := ([y]_i - \sum_{j=i+1}^{n} [l]_{ji}[x]_j^C)/[l]_{ii}$$

**Representation of $[x]^C = \mathrm{ICh}([A],[b]) \supseteq \Sigma_{\mathrm{sym}}$**

a) Standard formulae

b) Multiple product

$$\begin{aligned}[x]^C \;=\; & [D]^{(1)}([L]^{(1)T}([D]^{(2)}([L]^{(2)T}(\ldots([L]^{(n-1)T}([D]^{(n)}(\\ & [D]^{(n)}([L]^{(n-1)}([D]^{(n-1)}(\ldots([L]^{(1)}([D]^{(1)}[b]))\ldots)\end{aligned}$$

c) Recurrence

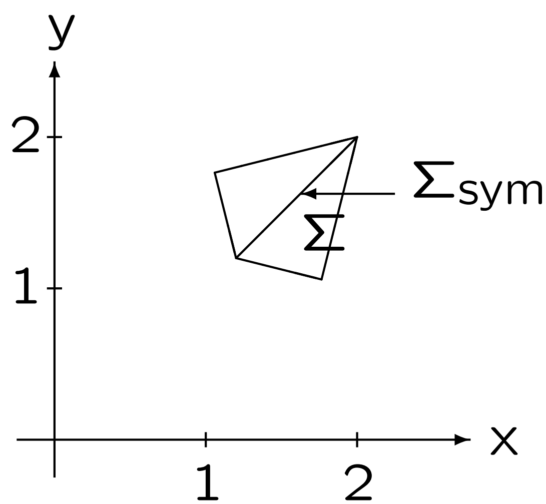
$$[A] = \begin{pmatrix} [a]_{11} & [c]^T \\ [c] & [A]' \end{pmatrix} \quad \text{with } [c] \in \mathbb{IR}^{n-1}.$$

The Cholesky decomposition $([L],[L]^T)$ of $[A]$ exists, if $0 < \underline{a}_{11}$ and if either $n = 1$, $[L] := (\sqrt{[a]_{11}})$, or $n > 1$ and the Cholesky decomposition $([L]',[L]'^T)$ of $[A]' - \frac{1}{[a]_{11}}[c][c]^T$ exists. Then

$$[L] := \begin{pmatrix} \sqrt{[a]_{11}} & 0 \\ [c]/\sqrt{[a]_{11}} & [L]' \end{pmatrix}; \quad [x]^C \text{ as in a)}.$$

**Example 3** $([A]\ H\text{–matrix})$

$$[A] := \begin{pmatrix} 4 & [-1,1] \\ [-1,1] & 4 \end{pmatrix}, \quad [b] := \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$




$$\begin{aligned}
\square\Sigma_{\text{sym}} &\neq \square\Sigma \\
\square\Sigma_{\text{sym}} &\neq [x]^C \\
\square\Sigma &\neq [x]^G \\
\Sigma &\not\subseteq [x]^C \\
[x]^C &\subsetneq [x]^G
\end{aligned}$$

## 3. Known Results (Alefeld/M. 1993)

**Theorem 1**

If $[A] \in \mathbb{IR}^{n\times n}$ is an $H$–matrix with $[A] = [A]^T$ and $0 < \underline{a}_{ii}$ for $i = 1, \ldots, n$, then $[x]^C$ exists.

**Theorem 2**

Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ be tridiagonal and let there exist a symmetric positive definite matrix $\tilde{A} \in [A]$ with $\langle \tilde{A} \rangle = \langle [A] \rangle$. Then $[x]^C$ exists.

**Theorem 3** (Quality of enclosure)

Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ be an $M$–matrix and let $[b] \in \mathbb{IR}^n$ satisfy $\underline{b} \geq 0$ or $0 \in [b]$ or $\overline{b} \leq 0$.
Then $[x]^C = \square\Sigma_{\text{sym}} = \square\Sigma = [x]^G$.

## 4. New Results (Alefeld/M. 2008)

**Theorem 4**

Let $[A] = [A]^T \in \mathbb{IR}^{n \times n}$ contain a symmetric positive definite matrix. If $[x]^G$ exists then $[x]^C$ exists.

**Theorem 5**

Let $[A] = [A]^T \in \mathbb{IR}^{n \times n}$. If all symmetric matrices in $[A]$ are positive definite – in particular, if $[x]^C$ exists – then $x^G$ exists for each matrix $\tilde{A} \in [A]$.

**Theorem 6**

Let $[A] = [A]^T \in \mathbb{IR}^{n \times n}$ contain a symmetric and positive definite matrix $\tilde{A}$ and let $n \leq 3$. Then $[x]^C$ exists if and only if $[x]^G$ exists.

The converse of Theorem 4 does not hold!

**Example 4** (Counterexample)

Let $[A] = \begin{pmatrix} 1 & [-1,1] & 0 & 0 \\ [-1,1] & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 16/3 \end{pmatrix}$. Then $[x]^C$ exists since

$$[L] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ [-1,1] & [1,\sqrt{2}] & 0 & 0 \\ 0 & [1/\sqrt{2},1] & [1,\sqrt{3/2}] & 0 \\ 0 & [2/\sqrt{2},2] & [0,1] & [\sqrt{1/3},\sqrt{10/3}] \end{pmatrix},$$

while $[x]^G$ does not exist because of

$$[\hat{U}] = \begin{pmatrix} 1 & [-1,1] & 0 & 0 \\ 0 & [1,3] & 1 & 2 \\ 0 & 0 & [1,5/3] & [0,4/3] \\ 0 & 0 & 0 & [-4/9,4] \end{pmatrix}.$$

**Theorem 7** (For a variant cf. Frommer 2001)

Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ contain a symmetric positive definite matrix. If the (undirected) graph of $\langle [A] \rangle$ is a tree and if it is ordered according to the minimum–degree algorithm, then the following properties are equivalent.

(i) $[x]^G$ exists.

(ii) $[x]^C$ exists.

(iii) Each symmetric matrix in $[A]$ is positive definite.

**Example 5**

Let

$$[A] = \begin{pmatrix} 2 & 0 & [-1,1] \\ 0 & 2 & [-1,1] \\ [-1,1] & [-1,1] & 2 \end{pmatrix} .$$

Each symmetric matrix $\tilde{A} \in [A]$ is positive definite, hence $[x]^C$ exists.

**Theorem 8**

If $[A] = I + [-R, R]$, $O \leq R = R^T$, $0 < \underline{a}_{ii}$, $i = 1, \ldots, n$, then $[x]^C$ exists if and only if $\rho(R) < 1$, hence if and only if $[A]$ is an $H$−matrix. Thus $[x]^C$ exists if and only if $[x]^G$ exists.

**Definition 1**

Let $[A] \in \mathbb{IR}^{n \times n}$.

a) Sign matrix $S = \Big(\text{sign}\,(\text{mid}([a]_{ij}))\Big)$.

b) Extended sign matrix $S'$:

$S' = S$
**for** $k = 1 : (n-1)$
  **for** $i = (k+1) : n$
    **for** $j = (k+1) : n$
      **if** $s'_{ij} == 0$ **then** $s'_{ij} = -s'_{ik} s'_{kk} s'_{kj}$ .

**Definition 2**

a) $[A] \in \mathbb{IR}^{n\times n}$ is irreducible if $|[A]|$ is irreducible.

b) $[A] \in \mathbb{IR}^{n\times n}$ is generalized diagonally dominant if there is a vector $x > 0$ such that $\langle [A] \rangle x \geq 0$.

**Theorem 9**

Let $[A] = [A]^T \in \mathbb{IR}^{n\times n}$ be irreducible and generalized diagonally dominant with $0 < \underline{a}_{ii},\ i = 1, \ldots, n$. Define $S'$ as above. Then $[x]^C$ exists if and only if $[x]^G$ exists if and only if $[A]$ is generalized irreducibly diagonally dominant or the sign condition

$$s'_{ij}\, s'_{ik}\, s'_{kj} = 1$$

holds for some triple $(i, j, k)$ with $k < j < i$.

## Example 6

$$[A] = \begin{pmatrix} 4 & 0 & [0,2] & [-2,0] \\ 0 & 4 & [0,2] & [0,2] \\ [0,2] & [0,2] & [6,9] & [-2,2] \\ [-2,0] & [0,2] & [-2,2] & [6,9] \end{pmatrix}$$

is irreducible and diagonally dominant with

$$S = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \neq S' = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

$[x]^C$ exists by Theorem 9 with $(i, j, k) = (4, 3, 2)$.