On the interval Cholesky method

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1. Introduction

Given: \( [A] = [A]^T = ([a]_{i,j}) \in \mathbb{R}^{n \times n} \) regular, \( [b] = ([b]_i) \in \mathbb{R}^n \)

Enclose the symmetric solution set

\[
\Sigma_{\text{sym}} := \{ x \in \mathbb{R}^n \mid Ax = b, \ A = A^T \in [A] = [A]^T, \ b \in [b] \}
\]

More frequently: Enclose the solution set

\[
\Sigma := \{ x \in \mathbb{R}^n \mid Ax = b, \ A \in [A], \ b \in [b] \}
\]
Example 1

\[ [A] = \begin{pmatrix} 1 & [0, 1] \\ [0, 1] & [-4, -1] \end{pmatrix}, \quad [b] = \begin{pmatrix} [0, 2] \\ [0, 2] \end{pmatrix}. \]
Example 2

\[
[A] = \begin{pmatrix}
1 & [-1, 1] \\
[-1, 1] & -1
\end{pmatrix}, \quad [b] = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.
\]

\([A]\) contains two sing. matrices but no sing. symmetric matrix.

\([-1, 1] \mapsto [-1 + \varepsilon, 1 - \varepsilon] : \text{ Arbitrarily large overestimation!} \]
Thus the hull inverse is not adapted to the optimal treatment of symmetric matrices.

However, at present, no special methods have been devised for this case, and we shall content ourselves with the unsymmetric treatment of symmetric matrices.


Unless you are able to handle dependent data, you will never get interest of the engineers.

Publications on $\Sigma_{\text{sym}}$ (Cholesky method highlighted)

Neumaier 1985, Dec. 23, letter to Rohn
1990

Rohn 1986 (talk 1990; published 2004)

Jansson 1990 (talk; published 1991)


Rump 1994


Schäfer 2001

Hladík 2007, 2008


Popova / Krämer 2007, 2008

Sharaya / Shary 2008

1. $LL^T$ decomposition

\[
\text{for } j := 1 \text{ to } n \text{ do } \\
[l]_{jj} := (a)_{jj} - \sum_{k=1}^{j-1} [l]_{jk}^2 \right)^{1/2} \\
\text{for } i := j + 1 \text{ to } n \text{ do } \\
[l]_{ij} := ([a]_{ij} - \sum_{k=1}^{j-1} [l]_{ik}[l]_{jk}) / [l]_{jj}
\]

2. Forward substitution

\[
\text{for } i := 1 \text{ to } n \text{ do } \\
[y]_i := ([b]_i - \sum_{j=1}^{i-1} [l]_{ij}[y]_j) / [l]_{ii}
\]

3. Backward substitution

\[
\text{for } i := n \text{ downto } 1 \text{ do } \\
[x]_i^C := ([y]_i - \sum_{j=i+1}^{n} [l]_{ji}[x]_j^C) / [l]_{ii}
\]
**Representation of** \( [x]^C = \text{ICh}([A],[b]) \supseteq \Sigma_{\text{sym}} \)

a) **Standard formulae**

b) **Multiple product**

\[
[x]^C = [D]^{(1)}([L]^{(1)}T([D]^{(2)}([L]^{(2)}T(\ldots([L]^{(n-1)}T([D]^{(n)})

\[ [D]^{(n)}([L]^{(n-1)}([D]^{(n-1)}(\ldots([L]^{(1)}([[D]^{(1)}[b]]))\ldots)\]

c) **Recurrence**

\[
[A] = \begin{pmatrix}
[a]_{11} & [c]^T \\
[c] & [A]' \\
\end{pmatrix}
\text{ with } [c] \in \mathbb{IR}^{n-1}.
\]

The Cholesky decomposition \(([L],[L]^T)\) of \([A]\) exists, if \(0 < a_{11}\) and if either \(n = 1\), \([L] := (\sqrt{[a]_{11}}),\) or \(n > 1\) and the Cholesky decomposition \(([L]',[L]'^T)\) of \([A]' - \frac{1}{[a]_{11}}[c][c]^T\) exists. Then

\[
[L] := \begin{pmatrix}
\sqrt{[a]_{11}} & 0 \\
[c]/\sqrt{[a]_{11}} & [L]' \\
\end{pmatrix}; \quad [x]^C \text{ as in a).}
Example 3 \hspace{2pt} ([A] \hspace{2pt} H\text{-}matrix)

\[
[A] := \begin{pmatrix} 4 & [-1,1] \\ [-1,1] & 4 \end{pmatrix}, \quad [b] := \begin{pmatrix} 6 \\ 6 \end{pmatrix}
\]

\[\Sigma_{\text{sym}} \neq \Sigma \]
\[\Sigma_{\text{sym}} \neq [x]^C \]
\[\Sigma \not\subseteq [x]^C \]
\[[x]^C \not\subseteq [x]^G \]
3. Known Results (Alefeld/M. 1993)

Theorem 1

If $[A] \in \mathbb{IR}^{n \times n}$ is an $H$–matrix with $[A] = [A]^T$ and $0 < a_{ii}$ for $i = 1, \ldots, n$, then $[x]^C$ exists.

Theorem 2

Let $[A] = [A]^T \in \mathbb{IR}^{n \times n}$ be tridiagonal and let there exist a symmetric positive definite matrix $\tilde{A} \in [A]$ with $\langle \tilde{A} \rangle = \langle [A] \rangle$. Then $[x]^C$ exists.

Theorem 3 (Quality of enclosure)

Let $[A] = [A]^T \in \mathbb{IR}^{n \times n}$ be an $M$–matrix and let $[b] \in \mathbb{IR}^n$ satisfy $b \geq 0$ or $0 \in [b]$ or $\bar{b} \leq 0$. Then $[x]^C = \Sigma_{\text{sym}} = \Sigma = [x]^G$. 
4. New Results (Alefeld/M. 2008)

Theorem 4
Let \([A] = [A]^T \in \mathbb{IR}^{n \times n}\) contain a symmetric positive definite matrix. If \([x]^G\) exists then \([x]^C\) exists.

Theorem 5
Let \([A] = [A]^T \in \mathbb{IR}^{n \times n}\). If all symmetric matrices in \([A]\) are positive definite – in particular, if \([x]^C\) exists – then \(x^G\) exists for each matrix \(\tilde{A} \in [A]\).

Theorem 6
Let \([A] = [A]^T \in \mathbb{IR}^{n \times n}\) contain a symmetric and positive definite matrix \(\tilde{A}\) and let \(n \leq 3\). Then \([x]^C\) exists if and only if \([x]^G\) exists.
The converse of Theorem 4 does not hold!

**Example 4**  (Counterexample)

Let \( [A] = \begin{pmatrix} 1 & [-1, 1] & 0 & 0 \\ [-1, 1] & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 2 & 16/3 \end{pmatrix} \). Then \( [x]^C \) exists since

\[
[L] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ [-1, 1] & [1, \sqrt{2}] & 0 & 0 \\ 0 & [1/\sqrt{2}, 1] & [1, \sqrt{3}/2] & 0 \\ 0 & [2/\sqrt{2}, 2] & [0, 1] & [\sqrt{1/3}, \sqrt{10/3}] \end{pmatrix},
\]

while \( [x]^G \) does not exist because of

\[
[\hat{U}] = \begin{pmatrix} 1 & [-1, 1] & 0 & 0 \\ 0 & [1, 3] & 1 & 2 \\ 0 & 0 & [1, 5/3] & [0, 4/3] \\ 0 & 0 & 0 & [-4/9, 4] \end{pmatrix}.
\]
Theorem 7  (For a variant cf. Frommer 2001)

Let \( [A] = [A]^T \in \mathbb{R}^{n \times n} \) contain a symmetric positive definite matrix. If the (undirected) graph of \( \langle [A] \rangle \) is a tree and if it is ordered according to the minimum-degree algorithm, then the following properties are equivalent.

(i) \( [x]^G \) exists.

(ii) \( [x]^C \) exists.

(iii) Each symmetric matrix in \([A]\) is positive definite.

Example 5

Let

\[
[A] = \begin{pmatrix}
2 & 0 & [-1, 1] \\
0 & 2 & [-1, 1] \\
[-1, 1] & [-1, 1] & 2
\end{pmatrix}.
\]

Each symmetric matrix \( \tilde{A} \in [A] \) is positive definite, hence \([x]^C\) exists.
Theorem 8

If $[A] = I + [-R, R]$, $O \leq R = R^T$, $0 < a_{ii}$, $i = 1, \ldots, n$, then $[x]^C$ exists if and only if $\rho(R) < 1$, hence if and only if $[A]$ is an $H$–matrix. Thus $[x]^C$ exists if and only if $[x]^G$ exists.

Definition 1

Let $[A] \in \mathbb{R}^{n \times n}$.

a) Sign matrix $S = (\text{sign} \ (\text{mid}([a]_{ij})))$.

b) Extended sign matrix $S'$:

$$S' = S$$

for $k = 1 : (n - 1)$

for $i = (k + 1) : n$

for $j = (k + 1) : n$

if $s'_{ij} == 0$ then $s'_{ij} = -s'_{ik}s'_{kj}s'_{kj}$.
Definition 2

a) \([A] \in \mathbb{IR}^{n \times n}\) is irreducible if \(|[A]|\) is irreducible.

b) \([A] \in \mathbb{IR}^{n \times n}\) is generalized diagonally dominant if there is a vector \(x > 0\) such that \(\langle [A] \rangle x \geq 0\).

Theorem 9

Let \([A] = [A]^T \in \mathbb{IR}^{n \times n}\) be irreducible and generalized diagonally dominant with \(0 < a_{ii}, i = 1, \ldots, n\). Define \(S'\) as above. Then \([x]^C\) exists if and only if \([x]^G\) exists if and only if \([A]\) is generalized irreducibly diagonally dominant or the sign condition

\[ s'_{ij} s'_{ik} s'_{kj} = 1 \]

holds for some triple \((i, j, k)\) with \(k < j < i\).
Example 6

\[
[A] = \begin{pmatrix}
4 & 0 & [0, 2] & [-2, 0] \\
0 & 4 & [0, 2] & [0, 2] \\
[0, 2] & [0, 2] & [6, 9] & [-2, 2] \\
[-2, 0] & [0, 2] & [-2, 2] & [6, 9]
\end{pmatrix}
\]

is irreducible and diagonally dominant with

\[
S = \begin{pmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
-1 & 1 & 0 & 1
\end{pmatrix} \neq S' = \begin{pmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{pmatrix}
\]

\([x]^C\) exists by Theorem 9 with \((i, j, k) = (4, 3, 2)\).