Stress analysis of a doubly reinforced concrete beam with uncertain structural parameters

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   - Non-deterministic modelling in engineering
   - Fuzzy uncertainty analysis in engineering
   - Implementation strategies

2. Response surface approaches
   - Objective
   - Taylor expansion response approximation
   - Kriging surrogate modelling approach

3. Application: doubly reinforced concrete beam
   - Description of the deterministic model
   - Introduction of uncertainty in the model
   - Results of the response surface surface approaches

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Introduction
Non-deterministic modelling in engineering

Non-determinism in modelling
Every aspect in the numerical model or solution procedure that introduces doubt or scatter on the outcome of the analysis

- in design procedures
  - present in all phases of design process
  - appearing in different forms
- has to be taken into account
  → represented using non-deterministic mathematical concepts
- parametric approach: uncertainty on parameters is processed to uncertainty on output quantities
Introduction

Non-deterministic modelling in engineering

- scatter: caused by variation in design (variability, aleatory)
Introduction

Non-deterministic modelling in engineering

- scatter: caused by variation in design (*variability, aleatory*)
- doubt: caused by uncertainty of designer (*uncertainty, epistemic*)
Introduction
Non-deterministic modelling in engineering

different tools for dealing with parametric non-determinism:
- variability
  - sampling methods
  - random processes
  - stochastic methods
  - ...
- uncertainty
  - interval methods
  - fuzzy approach
  - interval probabilities
  - info-gap
  - ...

ND modelling
Fuzzy analysis
Implementation
RSA
Application
Conclusions
Introduction
Non-deterministic modelling in engineering

different tools for dealing with parametric non-determinism:

- variability
  - sampling methods
  - random processes
  - stochastic methods
  - ...

- uncertainty
  - interval methods
  - fuzzy approach
  - interval probabilities
  - info-gap
  - ...

Fuzzy uncertainty analysis

- input: membership functions of fuzzy model properties
- output: membership function of output quantity

implementation: using $\alpha$-sublevel technique

added value in design procedures
  - definition of tolerances
  - design for robustness
**Introduction**

Fuzzy uncertainty analysis in engineering

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### $\alpha$-level strategy

- Interval analysis is core of fuzzy analysis.
- Selection of number of $\alpha$-levels is a trade-off between accuracy and numerical cost.

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![Diagram illustrating $\alpha$-level strategy with intervals and function $f(\tilde{a}, \tilde{b})$.](image-url)
Introduction

Fuzzy uncertainty analysis in engineering

**Definition of tolerances**

- Interval concept very appropriate representation of tolerances in early design stage
- Derive allowable tolerance intervals:

  \[ \text{fuzzy input} \rightarrow \text{critical value} \rightarrow \text{safety margin} \rightarrow \text{fuzzy output} \]
Introduction

Fuzzy uncertainty analysis in engineering

Definition of tolerances

- interval concept very appropriate representation of tolerances in early design stage
- derive allowable tolerance intervals:

fuzzy input

fuzzy output

critical value

safety margin

m

a
Introduction

Fuzzy uncertainty analysis in engineering

Definition of tolerances

- interval concept very appropriate representation of tolerances in early design stage
- derive allowable tolerance intervals:
Implementation strategies

Basic interval problem

If deterministic FE analysis is represented by \( f(x) \), uncertain parameters bounded by \( x^I \), find the set of possible outputs \( y \):

\[
y^S = \{ y \mid (x \in x^I)(y = f(x)) \}
\]

output set \( y^S \) can adopt any form in output space

\[ \rightarrow \text{hycprcubic approximation} \]
Implementation strategies

**interval arithmetic**

- approaching the exact interval result from outside
- based on interval arithmetics
- can be subject to conservatism due to dependency problem

**optimization**

- approaching the exact interval result from inside

\[
\begin{align*}
\underbar{y}_i &= \min_{x \in \mathbf{x}} f_i(x), \quad i = 1 \ldots n \\
\overbar{y}_i &= \max_{x \in \mathbf{x}} f_i(x), \quad i = 1 \ldots n
\end{align*}
\]

- global optimization $\rightarrow$ computationally expensive
- approximate optimization schemes
  - DOE: full factorial (vertex method), 3-level FF, ...
  - local series expansion about interval mid-value
  - using global surrogate models
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Response surface approaches

Objective

principal idea RSA for interval analysis

1. to build a surrogate model based on limited information of the goal function \( f(x) \)
2. to use knowledge from this surrogate model to speed up optimization

Two methods are used in this work:

1. **Taylor expansion** in interval mid-point to perform monotonicity check (sensitivity analysis)

   *see also A. Pownuk, “General Interval FEM Program Based on Sensitivity Analysis”, NSF workshop on Reliable Engineering Computing, February 20-22, 2008, Savannah, Georgia, USA, pp.397-428*

2. **Kriging approach** for global response surface modelling

   *see also M. De Munck, D. Moens, W. Desmet, and D. Vandepitte, “A kriging based optimization algorithm for interval and fuzzy frf analysis,” in 8th. World Congress on Computational Mechanics (WCCM8), (Venice), 2008*
Response surface approaches

Taylor expansion: procedure

1. build first order approximation of partial derivative of output function in mid-point of the uncertainty space

\[
\frac{\partial f(x)}{\partial x_i} \approx \frac{\partial f(x_0)}{\partial x_i} + \sum_j \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} (x_j - x_{j0})
\]

2. evaluate partial derivatives in vertex points

\[
\left( \frac{\partial f}{\partial x_i} \right)^l \approx \frac{\partial f(x_0)}{\partial x_i} - \sum_j \left| \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right| \Delta x_j
\]

\[
\left( \frac{\partial f}{\partial x_i} \right)^u \approx \frac{\partial f(x_0)}{\partial x_i} + \sum_j \left| \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right| \Delta x_j
\]

where \( \Delta x_j = \frac{x_{j}^u - x_{j}}{2} \)

3. condition for monotonicity

\[
0 \notin \left[ \left( \frac{\partial f}{\partial x_i} \right)^l , \left( \frac{\partial f}{\partial x_i} \right)^u \right]
\]

4. if monotonicity is detected, function is evaluated at optimal vertex combinations
Response surface approaches

Taylor expansion
Response surface approaches

Taylor expansion
Response surface approaches

Taylor expansion

vertex locations represent exact interval result

\( f(x) \)

\( x^l \)  \( x^u \)

\( x_0 \)
Response surface approaches
Taylor expansion

$f(x)$

vertex locations
not correct!
Response surface approaches

Taylor expansion

properties

+ very efficient calculation
+ improved confidence in vertex results
+ reliable for low order behavior
- does not capture higher order behavior
- no solution when non-monotonicity is detected
Response surface approaches

Kriging approach: principle

- Kriging (DACE) approximation
- Objective: build a reliable response surface with a limited number of samples
- Principle idea: errors are no longer considered independent
  - Model:
    \[ \tilde{f}(x_i) = \sum_{k=1}^{n} a_k \tilde{f}_k(x_i) + \epsilon(x_i) \]
  - Correlation:
    \[ \text{Corr}(\epsilon(x_i), \epsilon(x_j)) = \exp\left(-d(x_i, x_j)\right) \]
    \[ d(x_i, x_j) = \sum_{h=1}^{k} \theta_h |x_{ih} - x_{jh}|^{\rho_h} \]
- DACE stochastic process model
  \[ \tilde{f}(x_i) = \mu + \epsilon(x_i) \]
Riging model: example
Response surface approaches

Kriging approach

Kriging approach: procedure

1. build initial Kriging RS $\tilde{f}(x)$ based on space filling design
2. repeat until convergence:
   - generate set of candidate points $x_i$ for improved RS, and calculate $\tilde{f}(x_i)$ and maximum error $\Delta \tilde{f}(x_i)$
   - search for point(s) with the maximum expected improvement

$$\begin{align*}
\min_i & \left[ \tilde{f}(x_i) - \Delta \tilde{f}(x_i) \right] \\
\max_i & \left[ \tilde{f}(x_i) + \Delta \tilde{f}(x_i) \right]
\end{align*}$$

- rebuild $\tilde{f}(x)$ by adding selected point(s) to Kriging RS
3. perform optimization on $\tilde{f}(x)$
Response surface approaches

Kriging approach

\[ f(x) \]

step1: initial Kriging RS

- Requires limited number of sampling points
- Finds local extrema
- Easily extendable towards multi-objective optimization
- Surface can be re-used at different \( \alpha \)-levels

- Extrema evaluated on surrogate model
Response surface approaches

Kriging approach

step 2.1: candidate response points
Response surface approaches

Kriging approach

step 2.2: maximum expected improvement

\( f(x) \)

\[ x' \]

\[ x^u \]
Response surface approaches

Kriging approach

- properties
  + requires limited number of sampling points
  + finds local extrema
  + easily extendable towards multi-objective optimization
  + surface can be re-used at different \( \alpha \)-levels
- extrema evaluated on surrogate model

step 2.3: add selected points to Kriging RS
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Doubly reinforced beam

Doubly reinforced beam: deterministic case

- objective: given geometry and material properties, determine the allowable external bending moment

\[ N_s = A_s f_s \]
Doubly reinforced beam: deterministic case

→ implicit non-linear problem solved using deterministic iterative solution scheme on $\epsilon_{cc}$

- for increasing $\epsilon_{cc}$
  1. express all forces $N_{sc}$, $N_s$ and $N_c$ as a function of $x$
  2. horizontal equilibrium yields quadratic equation in $x$:

$$Ax^2 + Bx + C = 0 \quad (1)$$

with $A(\epsilon_{cc})$, $B(A_S, A_{SC}, E_S)$ and $C(A_S, A_{SC}, E_S)$

- from $x$, the location of the resultant compression force in concrete is determined
- finally, the internal moment of resistance $M_R$ is determined

- stop when either $\epsilon_{cc}$, $f_s$ or $f_{sc}$ reaches physical limit
Doubly reinforced beam: problem with uncertainty

- fuzzy uncertainty has been defined on geometric and material properties of steel: $A_s, A_{sc}, E_s$
- triangular fuzzy membership functions, with a base interval $[-5\%, +5\%]$
- objective: given an external bending moment, determine the tolerances on geometry and the allowable interval on the Young’s modulus
- solved using interval arithmetic approach, Taylor and Kriging approach
Doubly reinforced beam

Results

Taylor expansion: $1 + 2 \times 10 = 21$ function evaluations

Kriging approach: 20 function evaluations
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Conclusions

- 2 response surface methodologies were presented to approximate interval solutions in numerical analysis.
- The Taylor approach proved to be a very efficient calculation scheme, resulting in improved confidence in vertex results.
- The Kriging approach is applicable for non-monotonic functions, requiring a minimal amount of sampling points.
- For the doubly reinforced concrete beam problem, both methods yield very accurate interval results.