Validated computations for elliptic systems of FitzHugh-Nagumo type

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Reaction-Diffusion system

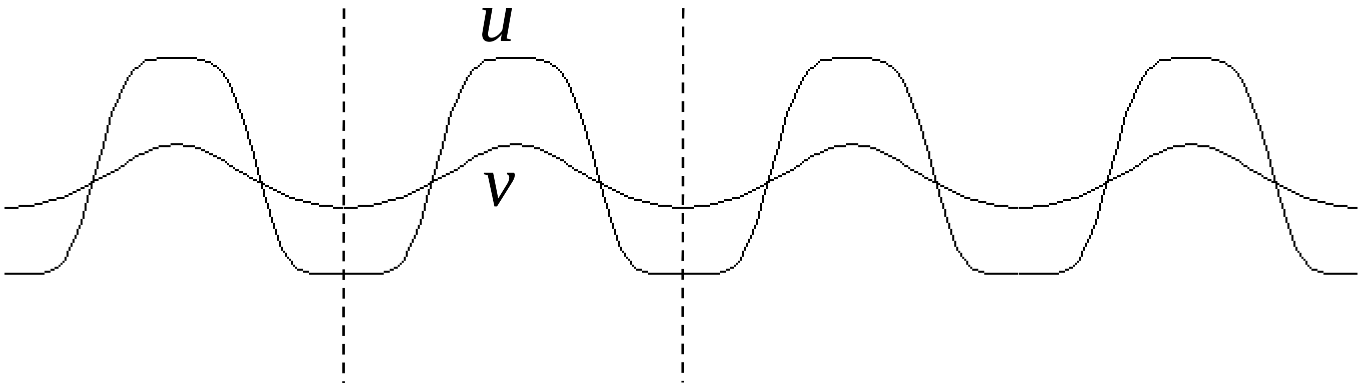
$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \dfrac{\partial^2 u}{\partial x^2} + f(u,v), \\[2ex] \dfrac{\partial v}{\partial t} = D_v \dfrac{\partial^2 v}{\partial x^2} + g(u,v) \end{cases} \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+$$

u, v : density or population

D_u, D_v : diffusion coefficients

f, g : nonlinear terms

Enclosing of (steady-state) solution, linearized eigenvalue problem
(Infinite dimensional problem, coupling type)



e.g. Stability of the periodic solution ･･･ Difference between bounded domain and unbounded domain ?

FitzHugh-Nagumo equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_u \dfrac{\partial^2 u}{\partial x^2} + f(u,v), \\ \dfrac{\partial v}{\partial t} = D_v \dfrac{\partial^2 v}{\partial x^2} + g(u,v) \end{cases}$$

with $f(u,v) = u(1-u)(1+u) - v,$

$g(u,v) = \varepsilon(u - \gamma v)$

$(\varepsilon, \gamma :$ constants $)$

Steady-state solution in 1D bounded domain, Neumann B.C.

$$\Omega = (0,b) \subset \mathbb{R} \qquad \varepsilon, \gamma > 0$$

$$\begin{cases} D_u u'' + u(1-u)(1+u) - v = 0 \quad \text{in } \Omega \\ D_v v'' + \varepsilon(u - \gamma v) = 0 \quad \text{in } \Omega \\ u' = v' = 0 \quad \text{on } \partial\Omega \end{cases}$$

Cf. Dirichlet type：
Watanabe, Y., A numerical verification method of solutions for an elliptic system of reaction-diffusion equations, submitted.

$$\begin{cases} -\varepsilon^2 u'' = u(1-u)(u-a) - \delta v \quad \text{for } -1 \le x \le 1 \\ -v'' = u - \gamma v \quad \text{for } -1 \le x \le 1 \\ u = v = 0 \quad \text{for } x \in \{-1, 1\} \end{cases}$$

$$\begin{cases} -\varepsilon^2 \Delta u = u - u^3 - \delta v \quad \text{in } \Omega \qquad \Omega = (0,1) \times (0,1) \\ -\Delta v = u - \gamma v \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega \end{cases}$$

Function spaces

$$X^k = \left\{ v = \sum_{n=0}^{\infty} c_n \varphi_n \;\middle|\; c_n \in \mathbb{R}, \sum_{n=0}^{\infty} n^{2k} c_n^2 < \infty \right\} \subset H^k(\Omega)$$

$$X_N = \left\{ v = \sum_{n=0}^{N} c_n \varphi_n \;\middle|\; c_n \in \mathbb{R} \right\} \subset X^1$$

where $\varphi_n = \cos \dfrac{n\pi x}{b}$

$$\left\langle z_1, z_2 \right\rangle_{X^1} = \left(z_1', z_2' \right)_{L^2} + \left(z_1, z_2 \right)_{L^2} \qquad z_1, z_2 \in X_1$$

$$\left\| z \right\|_{X^1} = \sqrt{\left\| z' \right\|_{L^2}^2 + \left\| z \right\|_{L^2}^2} \qquad z \in X_1$$

$$L\psi = -\psi'' + \psi$$

Lemma 1.

For all $\phi \in X^0$ there exists a unique solution $\psi \in X^2$ of $L\psi = \phi$.

(Proof)

For $\phi = \sum_{n=0}^{\infty} \phi_n \varphi_n \quad (\phi_n \in \mathbb{R}, n = 0, 1, 2, \cdots)$

let $\psi = \sum_{n=0}^{\infty} \psi_n \varphi_n, \quad \psi_n = \dfrac{\phi_n}{(n\pi / b)^2 + 1}$

$\Rightarrow$ $L\psi = \phi$

$\psi \in X^2 \quad \because \sum_{n=0}^{\infty} n^4 \psi_n^2 = \dfrac{b^4}{\pi^4} \sum_{n=0}^{\infty} \phi_n^2 < \infty$ ■

Projections

Define a H^1- projection $\tilde{P}_N : X^1 \to X_N$ by

$$\left\langle z - \tilde{P}_N z, z_N \right\rangle_{X^1} = 0 \quad z \in X^1, \forall z_N \in X_N$$

$P_N : X \to X_N$ s.t. $P_N \left(\sum_{n=0}^{\infty} c_n \varphi_n \right) = \sum_{n=0}^{N} c_n \varphi_n$

Note that $\tilde{P}_N = P_N$

Define $P : X \times X \to X_N \times X_N$ by

$$P(z_1, z_2) = (P_N z_1, P_N z_2), \; z_1, z_2 \in X^1$$

Error estimates

Lemma 2.

For $z \in X^2$

$$\left\| z - P_N z \right\|_{X^1} \le C(N) \left\| Lz \right\|_{L^2}$$

where

$$C(N) = \frac{b}{\sqrt{b^2 + (N+1)^2 \pi^2}}$$

Moreover

$$\left\| z - P_N z \right\|_{L^2} \le C(N) \left\| z - P_N z \right\|_{X^1}$$

(Proof of Lemma 2)

For $z = \sum_{n=0}^{\infty} c_n \varphi_n$

$$
\begin{aligned}
\| z - P_N z \|_{X^1}^2 &= \| z' - (P_N z)' \|_{L^2}^2 + \| z - P_N z \|_{L^2}^2 \\
&= \sum_{n=N+1}^{\infty} c_n^2 \| \varphi_n' \|_{L^2}^2 + \sum_{n=N+1}^{\infty} c_n^2 \| \varphi_n \|_{L^2}^2 \\
&= \sum_{n=N+1}^{\infty} \left\{ 1 + \frac{n^2 \pi^2}{b^2} \right\} c_n^2 \| \varphi_n \|_{L^2}^2 \\
&\le \frac{1}{1 + \frac{(N+1)^2 \pi^2}{b^2}} \sum_{n=N+1}^{\infty} \left\{ 1 + \frac{n^2 \pi^2}{b^2} \right\}^2 c_n^2 \| \varphi_n \|_{L^2}^2 \\
&\le \frac{b^2}{b^2 + (N+1)^2 \pi^2} \| L z \|_{L^2}^2
\end{aligned}
$$

Consider $\Phi \in X^2$ s.t. $\begin{cases} L\Phi = z - P_N z & \text{in } \Omega \\ \Phi' = 0 & \text{on } \partial\Omega \end{cases}$

$$\begin{aligned} \|z - P_N z\|_{L^2}^2 &= \left(z - P_N z, z - P_N z\right)_{L^2} \\ &= \left(z - P_N z, L\Phi\right)_{L^2} \\ &= \left\langle z - P_N z, \Phi\right\rangle_{X^1} \\ &= \left\langle z - P_N z, \Phi - P_N\Phi\right\rangle_{X^1} \\ &\le \|z - P_N z\|_{X^1} \|\Phi - P_N\Phi\|_{X^1} \\ &\le \|z - P_N z\|_{X^1} C(N)\|L\Phi\|_{L^2} \\ &\le \|z - P_N z\|_{X^1} C(N)\|z - P_N z\|_{L^2} \end{aligned}$$

$$\therefore \ \|z - P_N z\|_{L^2} \le C(N)\|z - P_N z\|_{X^1} \quad \blacksquare$$

Embedding constant

Lemma 3.

Let $I = (a,b),\ a < b.$

For $u \in H^1(I)$

$$\|u\|_{L^p} \le K_p \|u\|_{H^1} \quad (2 \le p \le \infty)$$

where

$$K_p = \begin{cases} C_{a,b} & (p = \infty) \\ (b-a)^{1/p} C_{a,b} & (2 \le p < \infty) \end{cases}$$

$$C_{a,b} = \sqrt{\max\left\{\frac{2}{b-a}, 2(b-a)\right\}}$$

(proved by a partial differentiation and Schwarz's inequality)

Fixed point formulation

$$(1)\begin{cases} D_u u'' + u(1-u)(1+u) - v = 0 \\ D_v v'' + \varepsilon(u - \gamma v) = 0 \end{cases} \Leftrightarrow \begin{cases} D_u L u = u(1-u)(1+u) - v + D_u u \\ D_v L v = \varepsilon(u - \gamma v) + D_v v \end{cases}$$

Define $F : X^1 \times X^1 \to X^1 \times X^1$ by

$$F(u,v) = \begin{pmatrix} \dfrac{1}{D_u} L^{-1}\left\{u(1-u)(1+u) - v + D_u u\right\} \\ \dfrac{1}{D_v} L^{-1}\left\{\varepsilon(u - \gamma v) - v + D_v v\right\} \end{pmatrix}$$

compact operator

$(1) \Leftrightarrow (u,v) = F(u,v)$

Verification conditions

$$w = (u, v)$$

$$w = F(w) \rightarrow \begin{cases} Pw = PF(w) & \text{Newton's method} \\ (I - P)w = (I - P)F(w) & \text{Error estimate} \end{cases}$$

$\hat{w}_N = (u_N, v_N) \in X_N \times X_N$ approximate solution

$$N(w) = Pw - \left[I - F'(\hat{w}_N)\right]_N^{-1} \left(Pw - PF(w)\right)$$

Here we assumed that the restriction to $X_N \times X_N$ of the operator

$$P\left[I - F'(\hat{w}_N)\right] : X^1 \times X^1 \rightarrow X_N \times X_N$$

has an inverse

$$\left[I - F'(\hat{w}_N)\right]_N^{-1} : X_N \times X_N \rightarrow X_N \times X_N .$$

This assumption can be checked in the actual computation.

Define $T : X^1 \times X^1 \to X^1 \times X^1$ by

$$T(w) = N(w) + (I - P)F(w)$$

Then T is a compact operator and

$$w = F(w) \quad \Leftrightarrow \quad w = T(w)$$

holds.

By Schauder's fixed point theorem, if there exists a non-empty, convex, bounded and closed set $W \subset X^1 \times X^1$ s.t.

$$T(W) \subset W,$$

then there exists a fixed point of T in W .

Candidate set

$$W = U \times V \subset X^1 \times X^1$$

$$U = u_N + U_N + U_\perp,$$

$$V = v_N + V_N + V_\perp$$

$$U_N = \left\{ \phi_N \in X_N \;\middle|\; \|\phi_N\|_{X^1} \le \alpha_1 \right\},$$

$$U_\perp = \left\{ \phi_\perp \in X_N^\perp \;\middle|\; \|\phi_\perp\|_{X^1} \le \alpha_2 \right\}$$

$$V_N = \left\{ \phi_N \in X_N \;\middle|\; \|\phi_N\|_{X^1} \le \beta_1 \right\},$$

$$V_\perp = \left\{ \phi_\perp \in X_N^\perp \;\middle|\; \|\phi_\perp\|_{X^1} \le \beta_2 \right\}$$

⇒ Find $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$

Sufficient conditions for $T(W) \subset W$

$$T(W) \subset W$$

$$\Uparrow$$

$$\begin{cases} N(W) \subset P(W) \\ (I-P)F(W) \subset (I-P)W \end{cases}$$

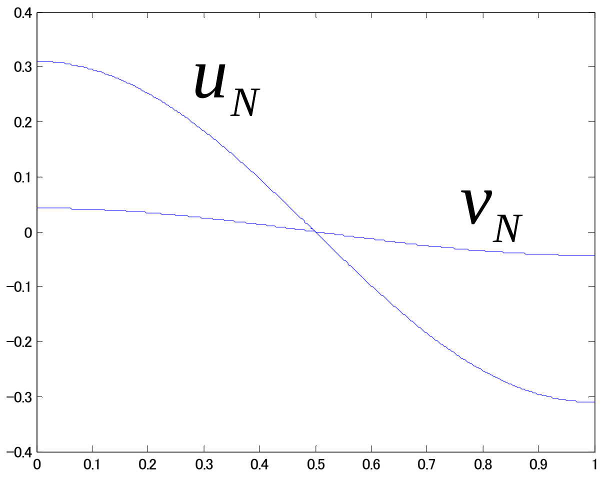
$$\Uparrow$$

$$\forall \left(\phi_1, \phi_2\right) \in N(W)$$

$$\begin{cases} \left\| \phi_1 - u_N \right\|_{X^1} \le \alpha_1 \\ \left\| \phi_2 - v_N \right\|_{X^1} \le \beta_1 \\ \dfrac{1}{D_u} C(N) \sup\limits_{(u,v)\in W} \left\| u(1-u)(1+u) - v + D_u u \right\|_{L^2} \le \alpha_2 \\ \dfrac{1}{D_v} C(N) \sup\limits_{(u,v)\in W} \left\| \varepsilon(u - \gamma v) + D_v v \right\|_{L^2} \le \beta_2 \end{cases}$$

Numerical results

$$D_u = 0.08,\ D_v = 2.0,\ \varepsilon = 3.0,\ \gamma = 2/3$$



$$N = 400$$

$$\alpha_1 = 3.716165 \times 10^{-3},\ \alpha_2 = 3.036165 \times 10^{-3}$$

$$\beta_1 = 1.190685 \times 10^{-3},\ \beta_2 = 1.190685 \times 10^{-3}$$

Future works

- 2D or 3D problems
- Periodic boundary condition
- Extension to an unbounded domain
  (stability analysis)
- Non-steady-type solution