# Tolerable Solution Sets to Interval Linear Systems with Dependent (Tied) Coefficients

**Irene A. Sharaya**

**Sergey P. Shary**

*Institute of Computational Technologies*

*Novosibirsk, Russia*

There are quite a lot of papers devoted to interval linear systems with dependent (tied) data:

- Jiří Rohn — *Linear Algebral and its Appls.*, 1981
- Ch. Jansson — *Computing*, 1991
- Siegfried M. Rump — *Linear Algebra and its Appls.*, 1998
- Lubomir V. Kolev — *Reliable Computing*, 2004
- Evgenija D. Popova — 2001, 2005
- Sergey P. Shary — *Siberian J. Numer. Math.*, 2004
- Ramil R. Akhmerov — *Reliable Computing*, 2005
- G. Alefeld, V. Kreinovich, G. Mayer — a series of papers of 1993, 1996, 1997, 1998, 2001, 2003

All of them deal with **united** solution set.

We shall consider **tolerable** solution set.

# Notation

Boldface letters — for intervals.

Calligraphic letters — for sets.

Lower indices — for projections of a set onto coordinate subspaces.

$\odot$ — symbol of memberwise multiplication (for sets).

**Examples.**

If $\mathcal{A} \subset \mathbb{R}^{m \times n}$, then

$$\mathcal{A}_{i:} = \{(A_{i1}, \ldots, A_{in}) \mid A \in \mathcal{A}\}.$$

For $x \in \mathbb{R}^n$,

$$\mathcal{A} \odot x = \{Ax \mid A \in \mathcal{A}\}.$$

# Tolerable solution set of ILAS

$$\boldsymbol{A} \in \mathbb{IR}^{m\times n}, \quad \boldsymbol{b} \in \mathbb{IR}^{m}$$

| ILAS | | family of LASes |
|---|---|---|
| $\boldsymbol{A}x = \boldsymbol{b}$ | is | $Ax = b,\ A \in \boldsymbol{A},\ b \in \boldsymbol{b}$ |

Tolerable solution set (TSS) of ILAS is

$$\Xi_{tol}(\boldsymbol{A}, \boldsymbol{b}) \ := \ \left\{ x \in \mathbb{R}^n \,\middle|\, (\forall A \in \boldsymbol{A})\,(\exists b \in \boldsymbol{b})\,(Ax = b) \right\}$$

$$= \ \left\{ x \in \mathbb{R}^n \,\middle|\, \boldsymbol{A} \odot x \ \subseteq \ \boldsymbol{b} \right\}.$$

# Tolerable solution set of ILAS with dependent (tied) coefficients

A dependence (tie) on coefficients is a set $\mathcal{G} \subset \mathbb{R}^{m \times n}$.

A system $\boldsymbol{A}x = \boldsymbol{b}$ with tie $\mathcal{G}$ on coefficients is the family of

$$Ax = b, \qquad A \in \boldsymbol{A} \cap \mathcal{G}, \;\; b \in \boldsymbol{b}.$$

Tolerable solution set of the system $\boldsymbol{A}x = \boldsymbol{b}$ with a tie $\mathcal{G}$ on coefficients is defined as

$$\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) := \left\{ x \in \mathbb{R}^n \;\middle|\; \big(\forall A \in (\boldsymbol{A} \cap \mathcal{G})\big)\,(\exists b \in \boldsymbol{b})\,(Ax = b) \right\}$$

$$= \left\{ x \in \mathbb{R}^n \;\middle|\; (\boldsymbol{A} \cap \mathcal{G}) \odot x \subseteq \boldsymbol{b} \right\}.$$

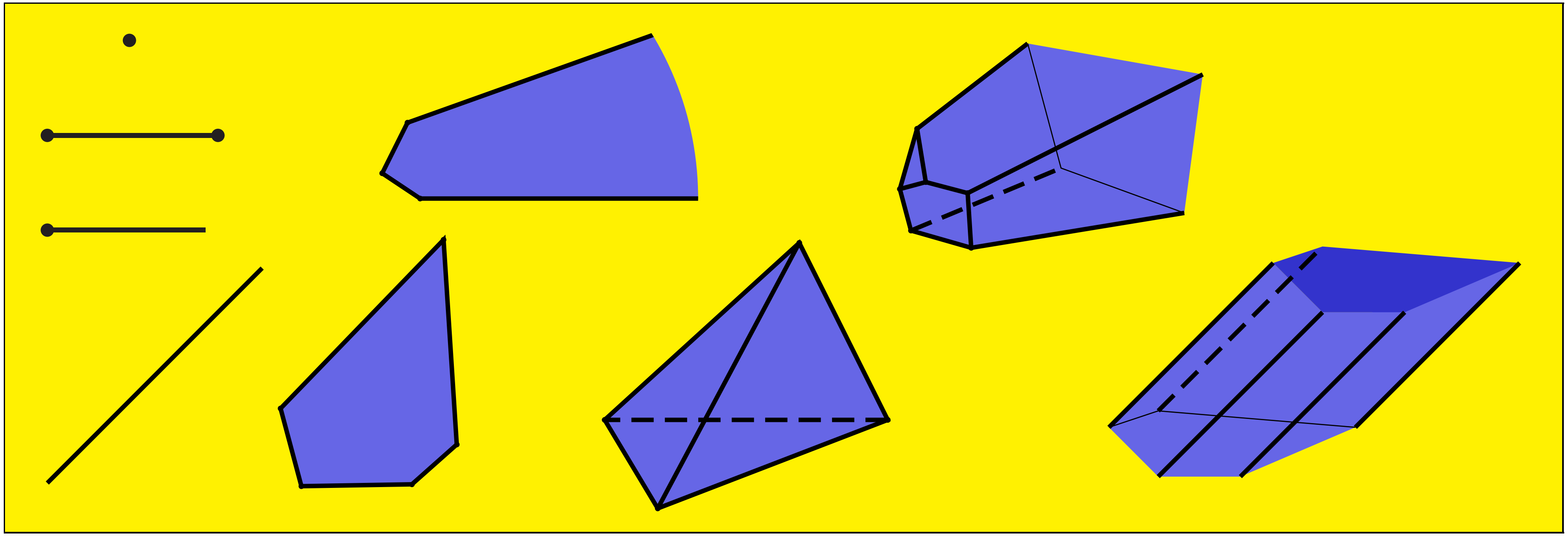
# Convex polyhedral ties on coefficients

Let $\mathcal{G}$ be a convex polyhedron

= intersection of finite number of half-spaces

= solution set to a system of linear inequalities

**Examples of convex polyhedrons in $\mathbb{R}^3$:**

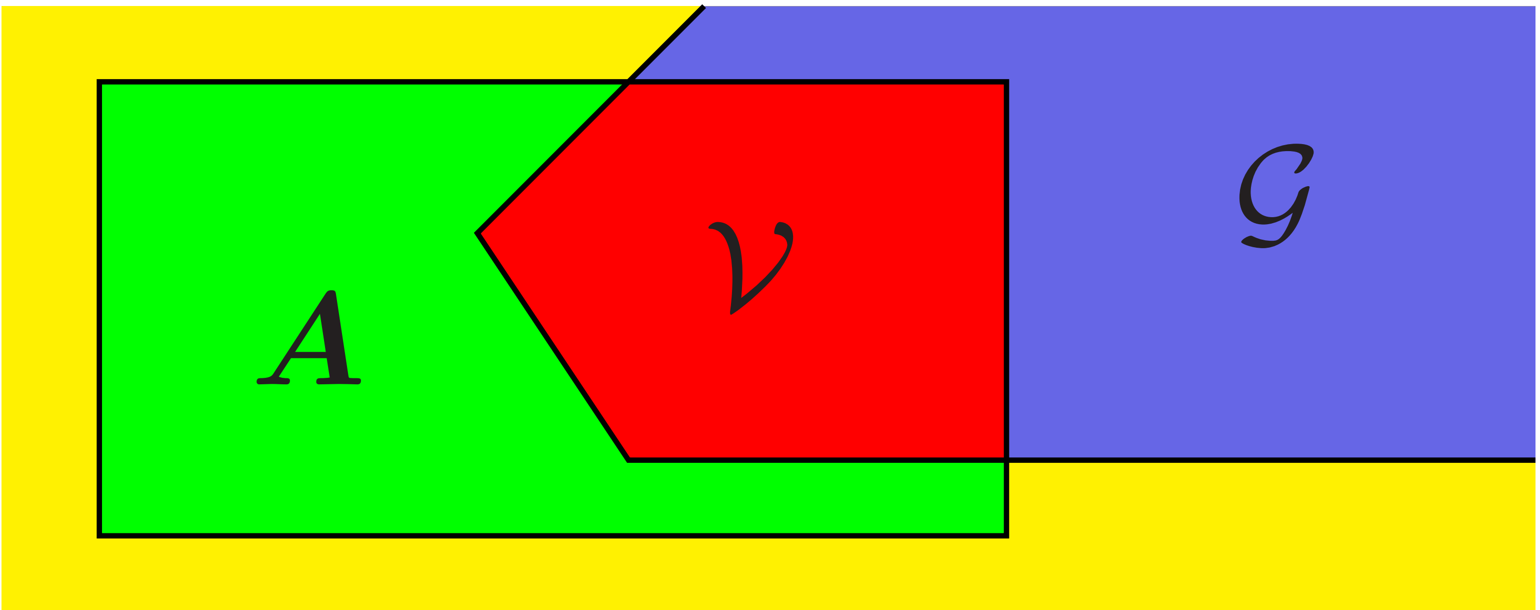


## $\boldsymbol{A} \cap \mathcal{G} = \textbf{?}$

$\mathcal{G}$ and $\boldsymbol{A}$ are convex polyhedrons, and $\boldsymbol{A}$ is bounded.

We denote $\mathcal{V} := \boldsymbol{A} \cap \mathcal{G}$.

Clearly, $\mathcal{V}$ is a bounded convex polyhedron.



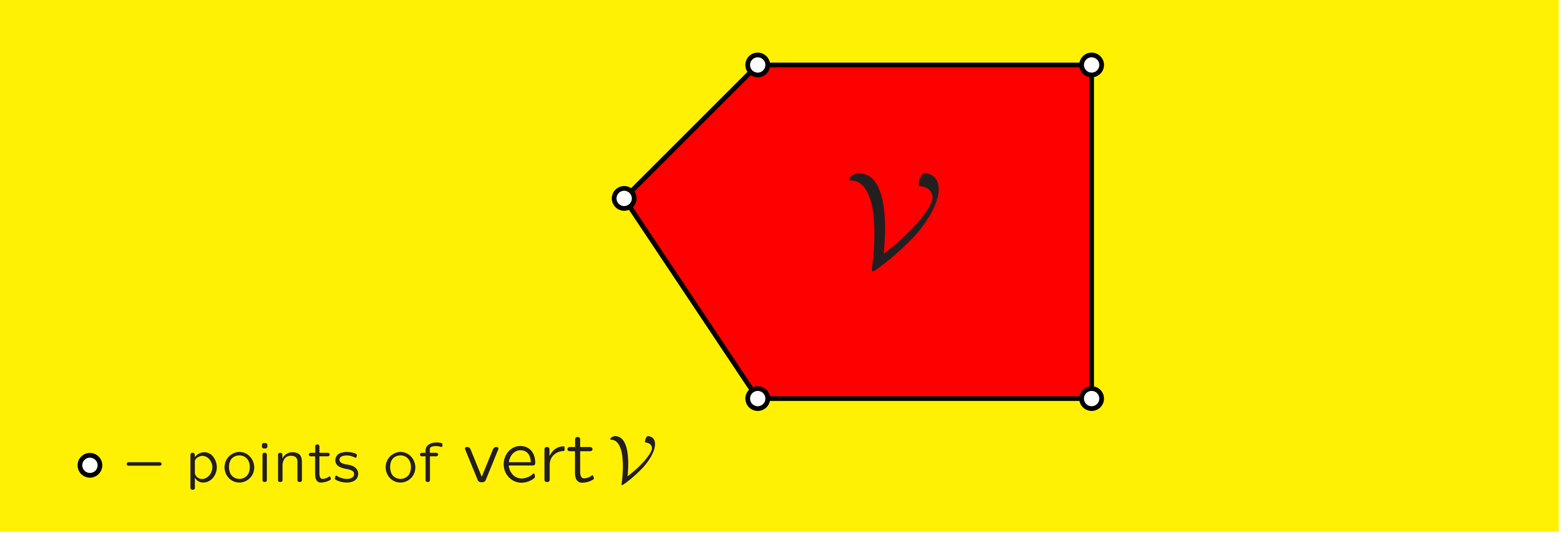
$\mathcal{V}$ is a bounded convex polyhedron

$=$ convex polytope

$=$ convex hull of its vertex set.

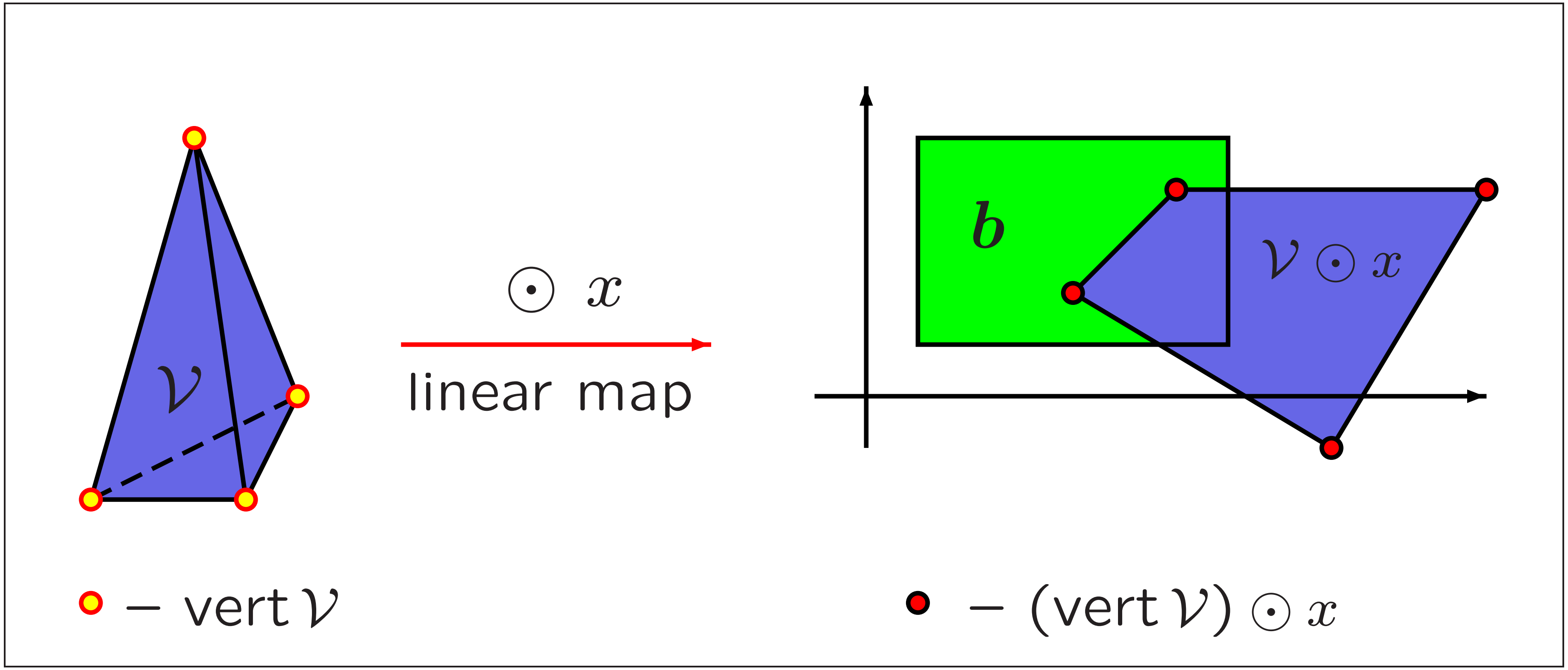
The set of vertices of $\mathcal{V}$ is denoted as $\operatorname{vert}\mathcal{V}$.

$\operatorname{vert}\mathcal{V}$ has finitely many members.



## Lemma 1

$$\mathcal{V} \odot x \subseteq \boldsymbol{b} \iff (\operatorname{vert} \mathcal{V}) \odot x \subseteq \boldsymbol{b}.$$



## Lemma 2

$$\mathcal{V} \odot x \subseteq \boldsymbol{b} \iff \mathop{\&}_{i} \Big( \big(\mathrm{vert}(\mathcal{V}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i \Big).$$

Proof:

1) $\boldsymbol{b} \in \mathbb{IR}^m$ implies

$$\mathcal{V} \odot x \subseteq \boldsymbol{b} \iff \mathop{\&}_{i} \big( (\mathcal{V} \odot x)_i \subseteq \boldsymbol{b}_i \big).$$

2) According to definitions,

$$(\mathcal{V} \odot x)_i = \mathcal{V}_{i:} \odot x.$$

3) Applying Lemma 1 to convex polytope $\mathcal{V}_{i:}$ results in

$$\mathcal{V}_{i:} \odot x \subseteq \boldsymbol{b}_i \iff \big(\mathrm{vert}(\mathcal{V}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i.$$

## Similarity of Lemmata 1 and 2

Both lemmata reduce the inclusion $\mathcal{V} \odot x \subseteq \boldsymbol{b}$ to finite systems of two-sided linear inequalities:

$$\mathcal{V} \odot x \subseteq \boldsymbol{b}$$

Lemma 1

Lemma 2

**System 1**

$$\mathop{\&}_{V \in \mathrm{vert}\,\mathcal{V}} \left(\underline{b} \le Vx \le \overline{b}\right)$$

**System 2**

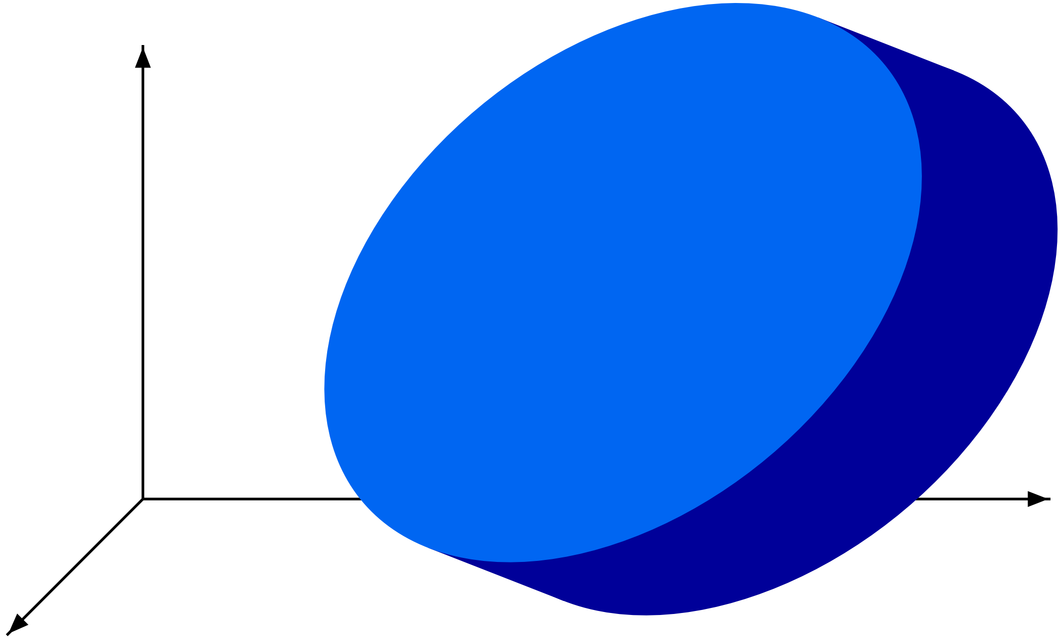
$$\mathop{\&}_{i} \mathop{\&}_{v \in \mathrm{vert}(\mathcal{V}_{i:})} \left(\underline{b}_i \le vx \le \overline{b}_i\right)$$

Solution set of a two-sided linear inequality

is a set, bounded by two

parallel hyperplanes

= **hyperstripe**.

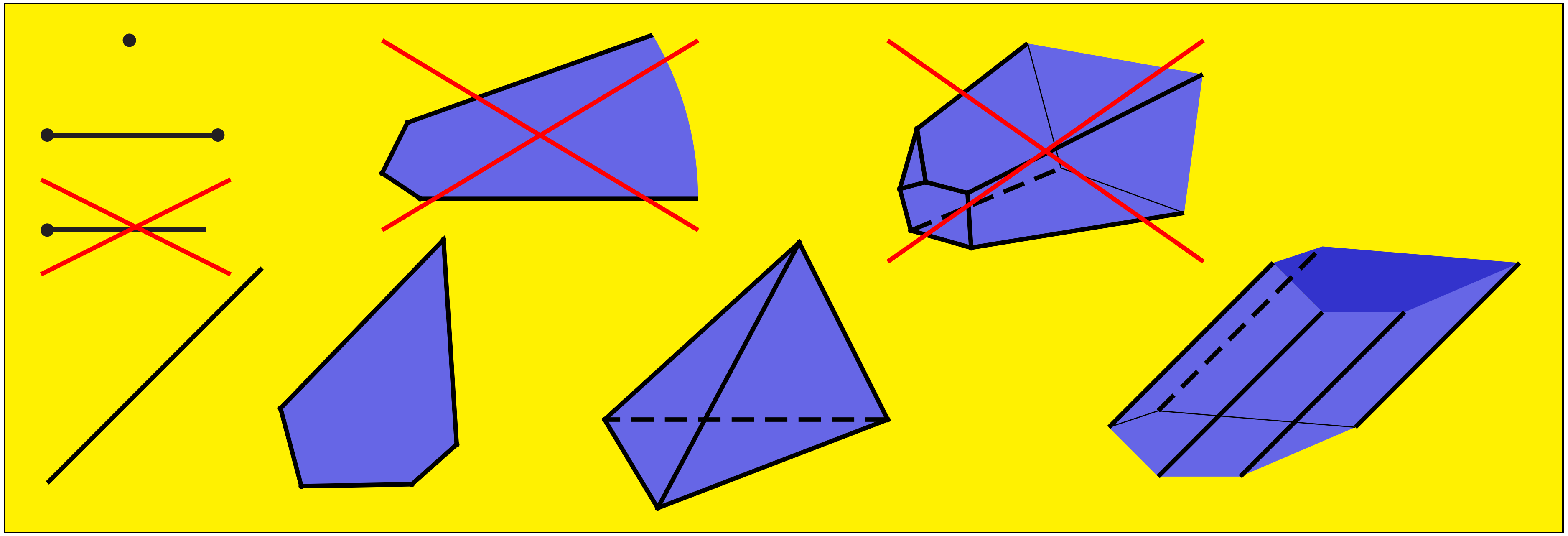


## Shape of $\varXi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$

For convex polyhedron $\mathcal{G}$, $\varXi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$ is solution set of a two-sided linear inequalities system

= intersection of a finite number of hyperstripes
(particular case of convex polyhedron)

**Examples in $\mathbb{R}^3$:**



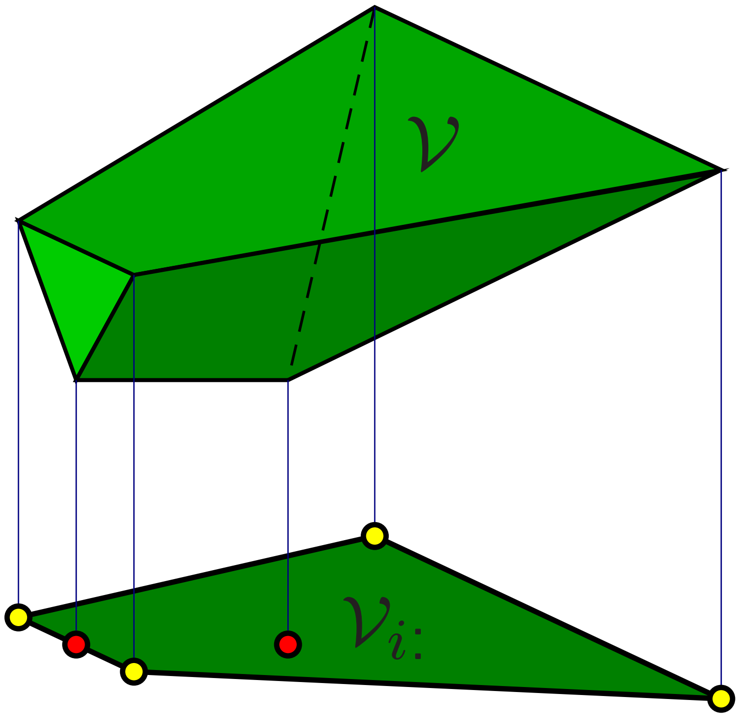
# Distinction of Lemmata 1 and 2

**System 1** $\supseteq$ **System 2**

$$\underset{i}{\&} \, (\mathrm{vert}\,\mathcal{V})_{i:} \odot x \subseteq \boldsymbol{b}_i \qquad \underset{i}{\&} \, \big(\mathrm{vert}(\mathcal{V}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i$$

due to

$$(\mathrm{vert}\,\mathcal{V})_{i:} \supseteq \mathrm{vert}(\mathcal{V}_{i:}).$$



## Computation of tolerable solution set for ILAS with a convex polyhedron tie on coefficients

$$\Xi_{tol}(\mathbf{A} \cap \mathcal{G}, \mathbf{b}) = ?$$

### Algorithm

1) Find the sets $\mathrm{vert}((\mathbf{A} \cap \mathcal{G})_{i:}),\ i = 1, \ldots, m.$

2) Find or estimate the solution set of the system

$$\mathop{\&}_{i} \mathop{\&}_{v \in \mathrm{vert}((\mathbf{A} \cap \mathcal{G})_{i:})} vx \subseteq \mathbf{b}_i \, .$$

## Examples

We assume that efficient methods for the solution of two-sided linear inequalities system are available.

In examples, we shall fulfil only the first step of the method:

$$\text{Find } \operatorname{vert}((\boldsymbol{A} \cap \mathcal{G})_{i:})$$

Compexity of the first step of our algorithm depends on

- shape of the set $\mathcal{G}$ (linear subspace, interval,. . . ),
- way of description of $\mathcal{G}$ (inequalities system, etc.,. . . ),
- location of $\mathcal{G}$ with respect to $\boldsymbol{A}$ ($\boldsymbol{A} \subseteq \mathcal{G}$, $\mathcal{G} \subseteq \boldsymbol{A}$,. . . )
- . . .

# Example 1

**Problem.** $\Xi_{tol}(\boldsymbol{A}, \boldsymbol{b}) = \,?$

We may think that $\boldsymbol{A} \subseteq \mathcal{G}$, and then

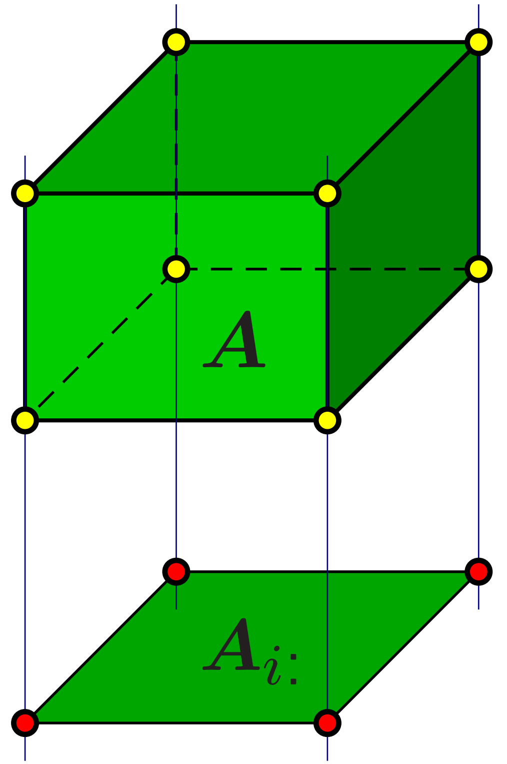
$$(\boldsymbol{A} \cap \mathcal{G})_{i:} = \boldsymbol{A}_{i:}$$

$$\mathrm{vert}\big((\boldsymbol{A} \cap \mathcal{G})_{i:}\big) = \mathrm{vert}(\boldsymbol{A}_{i:})$$

**System 2:** $\underset{i}{\&} \; (\mathrm{vert}\,\boldsymbol{A}_{i:}) \odot x \subseteq \boldsymbol{b}_i$

$$\Xi_{tol}(\boldsymbol{A}, \boldsymbol{b}) = ?$$

| System | Form | Number of rows |
|---|---|---|
| 1 | $\mathop{\&}\limits_{i} \; (\text{vert}\, \boldsymbol{A})_{i:} \odot x \subseteq \boldsymbol{b}_i$ | $m \cdot 2^{mn}$ |
| 2 | $\mathop{\&}\limits_{i} \; \big(\text{vert}(\boldsymbol{A}_{i:})\big) \odot x \subseteq \boldsymbol{b}_i$ | $m \cdot 2^{n}$ |




Due to replications, the number of rows in system 1 is $2^{(m-1)n}$ times larger than that in system 2.

## Example 2

**Problem.**

Matrices from the set $\mathcal{G}$:

- symmetric
- skew-symmetric
- circulant
- Toeplitz
- Hankel
- ...

A more general requirement on coefficients of the system:

- all the coefficients are divided into groups,
- coefficients from a group are proportional,
- there are no coefficients from one row in a group.

$$\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) = ?$$

$$(\boldsymbol{A} \cap \mathcal{G})_{i:} = \widetilde{\boldsymbol{A}}_{i:}\,,$$

where $\widetilde{\boldsymbol{A}}$ is an inclusion maximal interval matrix contained in $\boldsymbol{A}$ that meets the elementwise proportionality requirement on $\mathcal{G}$.

$\widetilde{\boldsymbol{A}}$ is easy to find.

If, for instance, $\mathcal{G}$ represents symmetric matrices, then $\widetilde{\boldsymbol{A}}_{ij} = \boldsymbol{A}_{ij} \cap \boldsymbol{A}_{ji}$.

Finally, $\text{vert}\big((\boldsymbol{A} \cap \mathcal{G})_{i:}\big) = \text{vert}\big(\widetilde{\boldsymbol{A}}_{i:}\big)$ — vertices of the box.

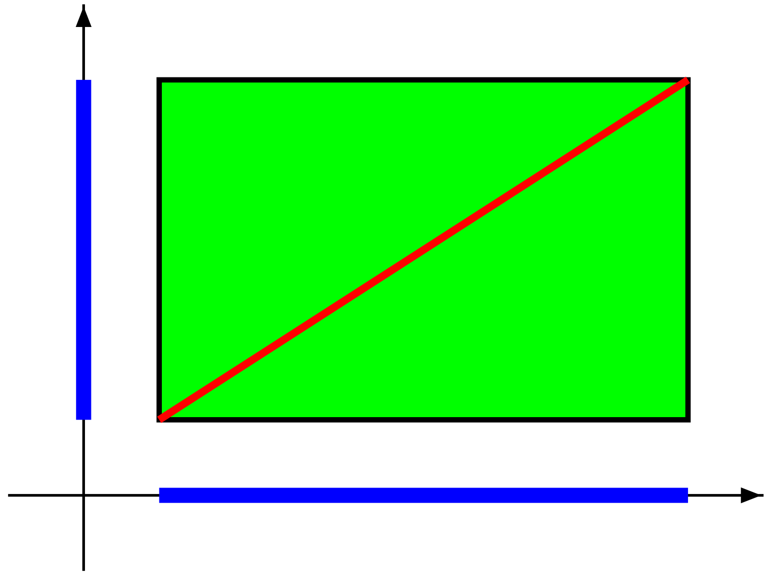
**System 2:** $$\mathop{\&}_{i} \Big(\text{vert}\big(\widetilde{\boldsymbol{A}}_{i:}\big)\Big) \odot x \subseteq \boldsymbol{b}_i.$$

**Remark.** In Example 2,

$$\boldsymbol{A} \cap \mathcal{G} \neq \widetilde{\boldsymbol{A}},$$

**but**

$$(\boldsymbol{A} \cap \mathcal{G})_{i:} = \widetilde{\boldsymbol{A}}_{i:}.$$



Anyway, this is sufficient for the equality

$$\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) \; = \; \Xi_{tol}(\widetilde{\boldsymbol{A}}, \boldsymbol{b}).$$

## Numerical problem for Examples 1 and 2

**Given:** $\boldsymbol{A}x = \boldsymbol{b}$,

$$\boldsymbol{A} = \begin{pmatrix} 1 & [0,1] \\ [0,1] & [-4,-1] \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix},$$

$\mathcal{G}$ is the set of symmetric matricies.

**Required:** $\Xi_{tol}(\boldsymbol{A}, \boldsymbol{b})$, $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$.

United solution set for this system with symmetric tie

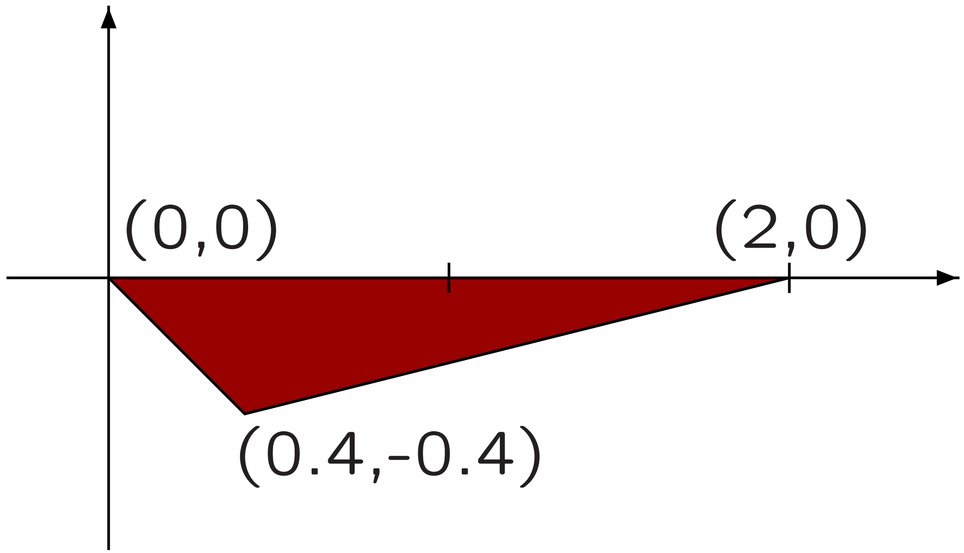
was considered in

*Alefeld G., Kreinovich V., Mayer G.* On symmetric solution sets // *Inclusion methods for nonlinear problems with applications in engineering, economics, and physics* / Herzberger J., ed. – Wien, New York: Springer, 2003. – P. 1–23. – (Computing Supplement; 16)

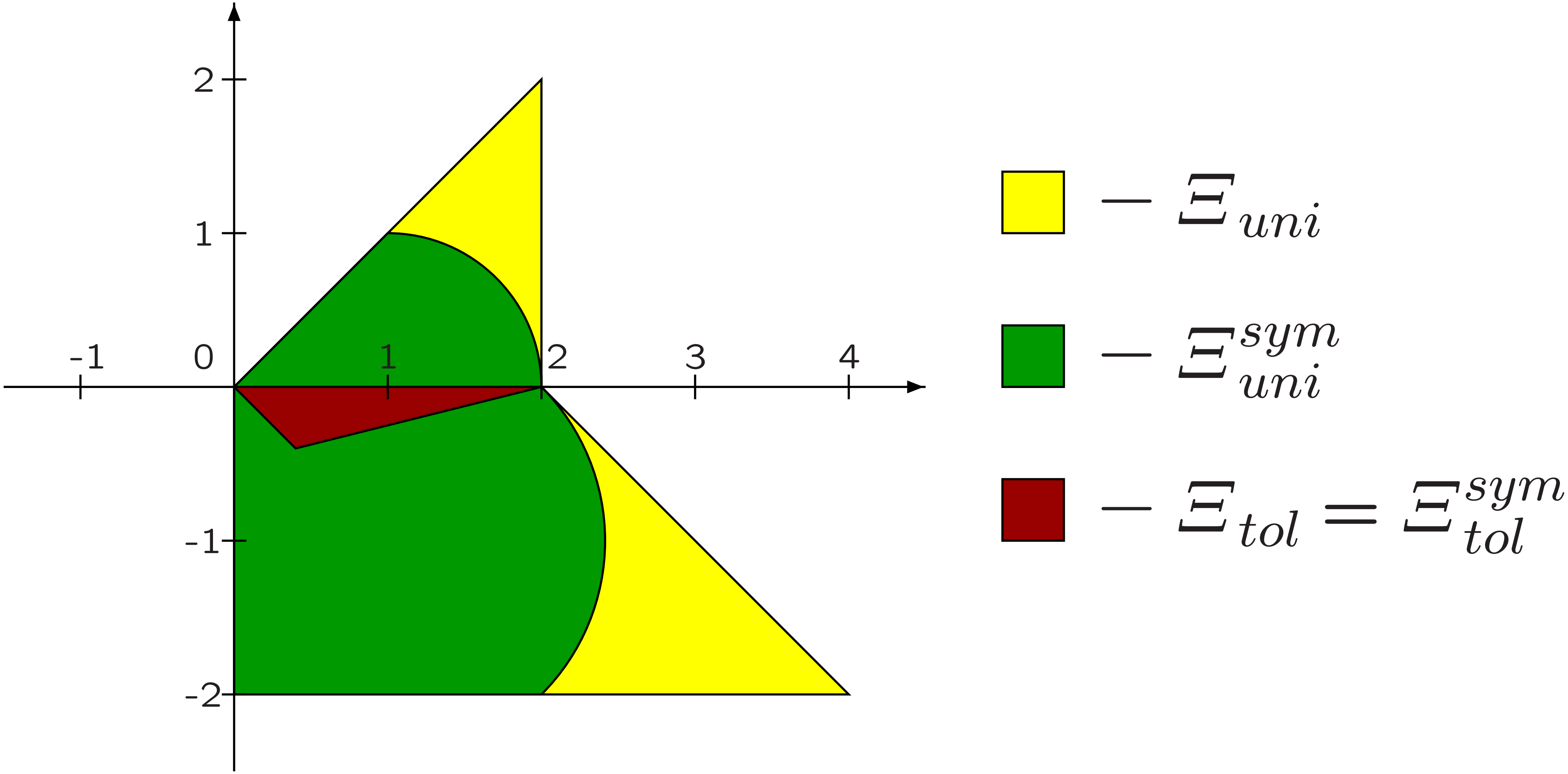
1. $\varXi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b}) = \varXi_{tol}(\boldsymbol{A}, \boldsymbol{b})$, since $\boldsymbol{A}$ is symmetric.

2. **System 2:**

$$\left\{\begin{array}{rrcl} x_1 & + & & \subseteq [0,2] \\ x_1 & + & x_2 & \subseteq [0,2] \\ & - & 4x_2 & \subseteq [0,2] \\ & - & x_2 & \subseteq [0,2] \\ x_1 & - & 4x_2 & \subseteq [0,2] \\ x_1 & - & x_2 & \subseteq [0,2] \end{array}\right.$$



# Comparison of results for united and tolerable solution sets



## Conclusions

1. $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$, for a convex polyhedral set $\mathcal{G}$, is intersection of finite number of hyperstripes.

2. $\Xi_{tol}(\boldsymbol{A} \cap \mathcal{G}, \boldsymbol{b})$ can be found from the system

$$\mathop{\&}_{i} \mathop{\&}_{v \in \mathrm{vert}((\boldsymbol{A} \cap \mathcal{G})_{i:})} vx \subseteq \boldsymbol{b}_i \,.$$

**Thank you for your attention**

## Example 3

**Problem.** $\boldsymbol{B} \in \mathbb{IR}^{m\times m}$, $\boldsymbol{C} \in \mathbb{IR}^{n\times n}$, $\boldsymbol{D} \in \mathbb{IR}^{m\times n}$, $X \in \mathbb{R}^{m\times n}$

$$\{X \mid (\forall B \in \boldsymbol{B})(\forall C \in \boldsymbol{C})\ (BX + XC \in \boldsymbol{D})\} - ?$$

$$\sum_{\substack{k=1\\ k\neq i}} B_{ik}X_{kj} + \sum_{\substack{l=1\\ l\neq j}} C_{lj}X_{il} + (B_{ii} + C_{jj})X_{ij} \in \boldsymbol{D}_{ij}$$

**System 2:** $$\mathop{\&}_{i,j} \begin{pmatrix} \Big(\text{vert}(\boldsymbol{B}_{i,\neq i})\Big) \odot X_{\neq i,j} + & \\ X_{i,\neq j} \odot \Big(\text{vert}(\boldsymbol{C}_{\neq j,j})\Big) + & \subseteq \boldsymbol{D}_{ij} \\ \Big(\text{vert}(\boldsymbol{B}_{ii} + \boldsymbol{C}_{jj})\Big) \odot X_{ij} & \end{pmatrix}.$$