NONNEGATIVE INTERVAL LINEAR SYSTEMS AND THEIR SOLUTION

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Interval linear systems of equations

$$\left\{\begin{array}{ccccccccc} \boldsymbol{a}_{11}x_1 & + & \boldsymbol{a}_{12}x_2 & + & \ldots & + & \boldsymbol{a}_{1n}x_n & = & \boldsymbol{b}_1, \\ \boldsymbol{a}_{21}x_1 & + & \boldsymbol{a}_{22}x_2 & + & \ldots & + & \boldsymbol{a}_{2n}x_n & = & \boldsymbol{b}_2, \\ & \vdots & & & \ddots & & & \vdots & \\ \boldsymbol{a}_{m1}x_1 & + & \boldsymbol{a}_{n2}x_2 & + & \ldots & + & \boldsymbol{a}_{mn}x_n & = & \boldsymbol{b}_m, \end{array}\right.$$

or, briefly,

$$\boldsymbol{A}x = \boldsymbol{b}$$

with interval matrix $\boldsymbol{A} = (\,\boldsymbol{a}_{ij})$ and vector $\boldsymbol{b} = (\,\boldsymbol{b}_i)$.

Interval linear systems of equations

$$\boldsymbol{A}x = \boldsymbol{b}$$

— a family of point linear systems $Ax = b$ with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

Solution set

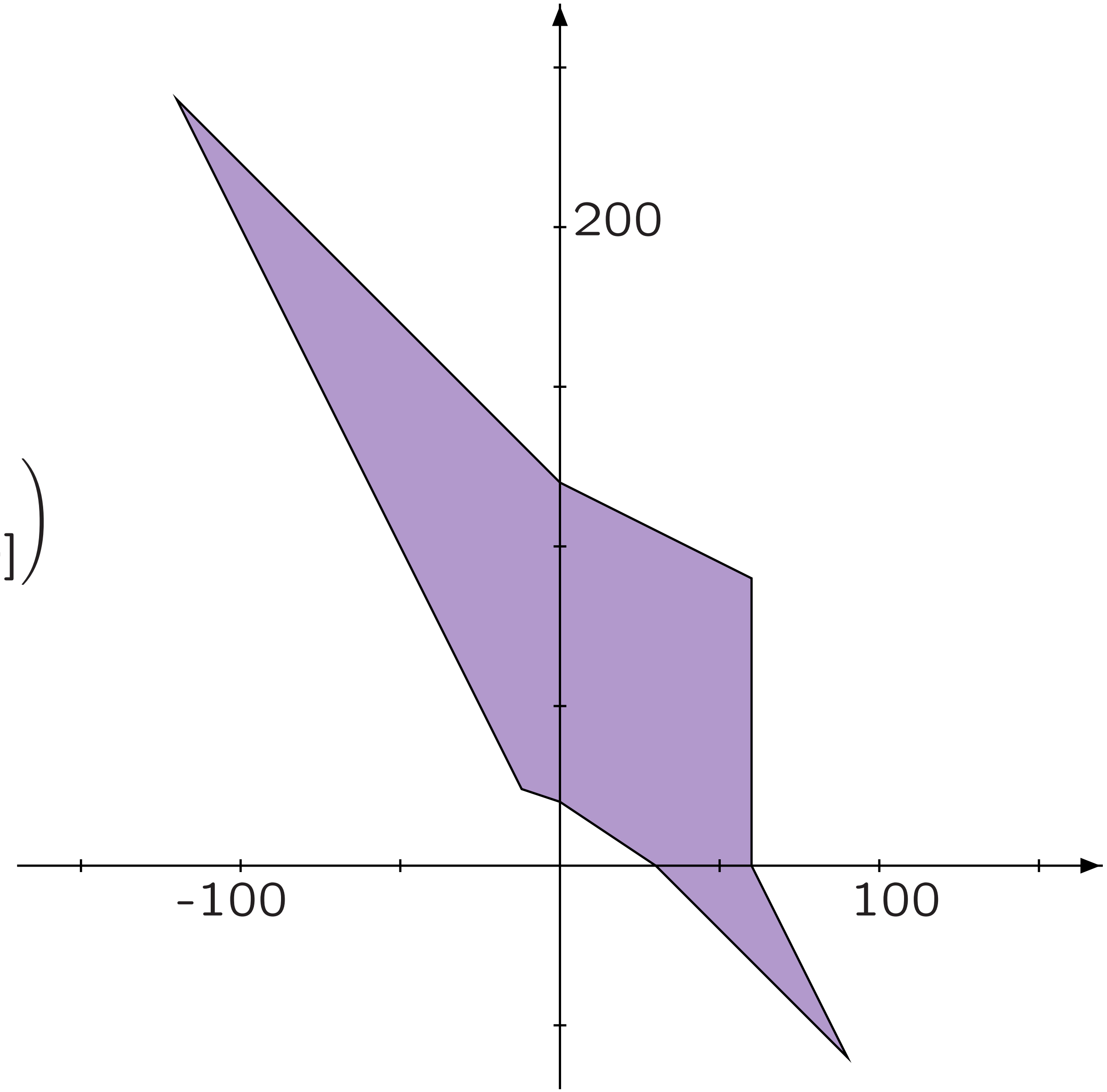
of interval linear system of equations —

$$\Xi(\boldsymbol{A}, \boldsymbol{b}) \;=\; \left\{ x \in \mathbb{R}^n \mid (\exists\, A \in \boldsymbol{A})(\exists\, b \in \boldsymbol{b})(\, Ax = b \,) \right\}$$

Also *united solution set* . . .

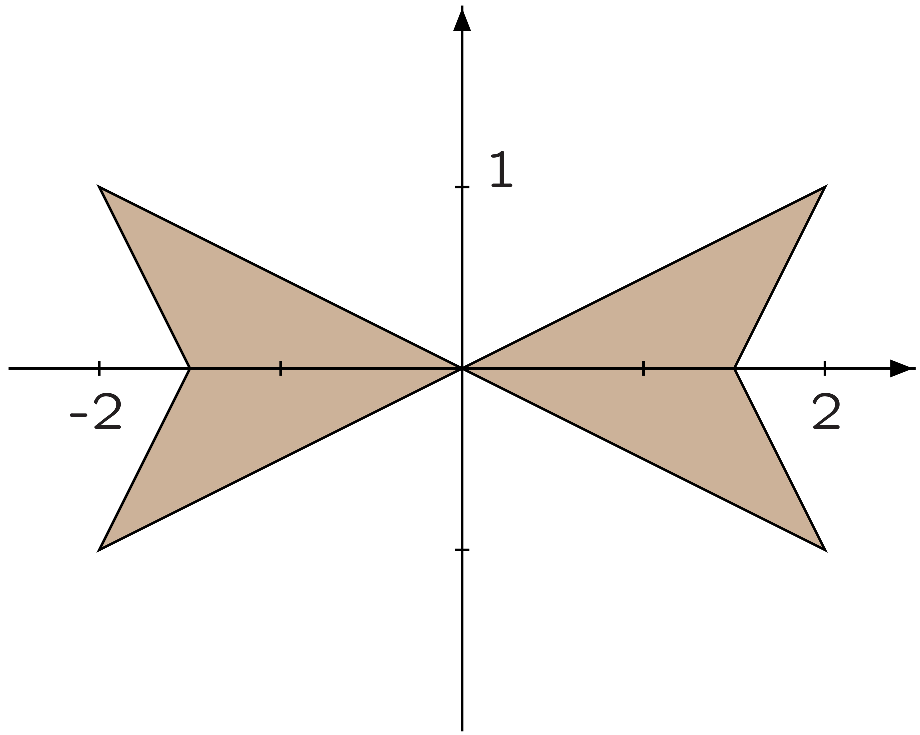
Example — Hansen system

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$

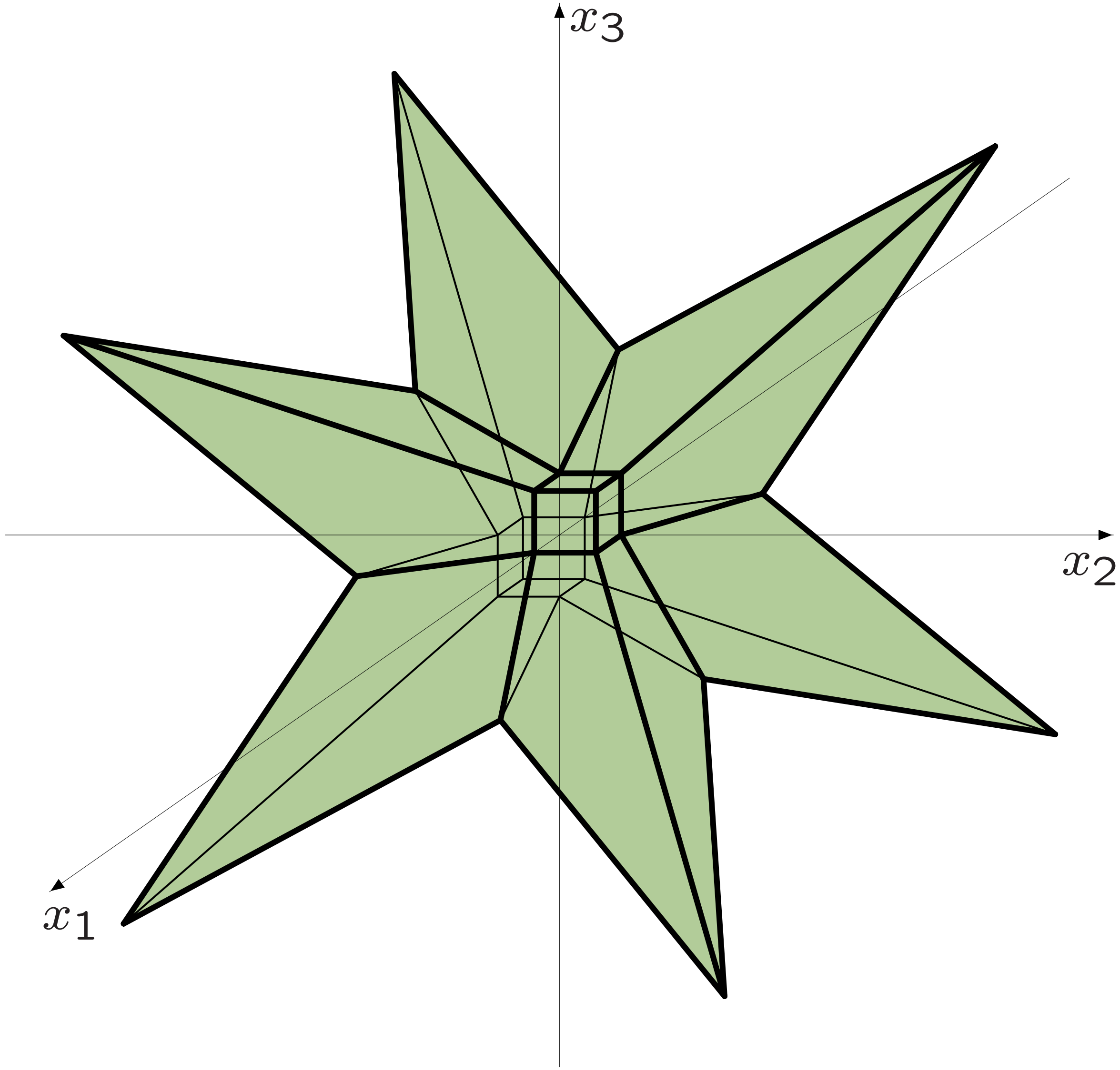


Example — almost disconnected solution set

$$\begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-3,3] \\ 0 \end{pmatrix}$$



Example — Neumaier system



$$\begin{pmatrix} 3.5 & [0,2] & [0,2] \\ [0,2] & 3.5 & [0,2] \\ [0,2] & [0,2] & 3.5 \end{pmatrix} x = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}$$

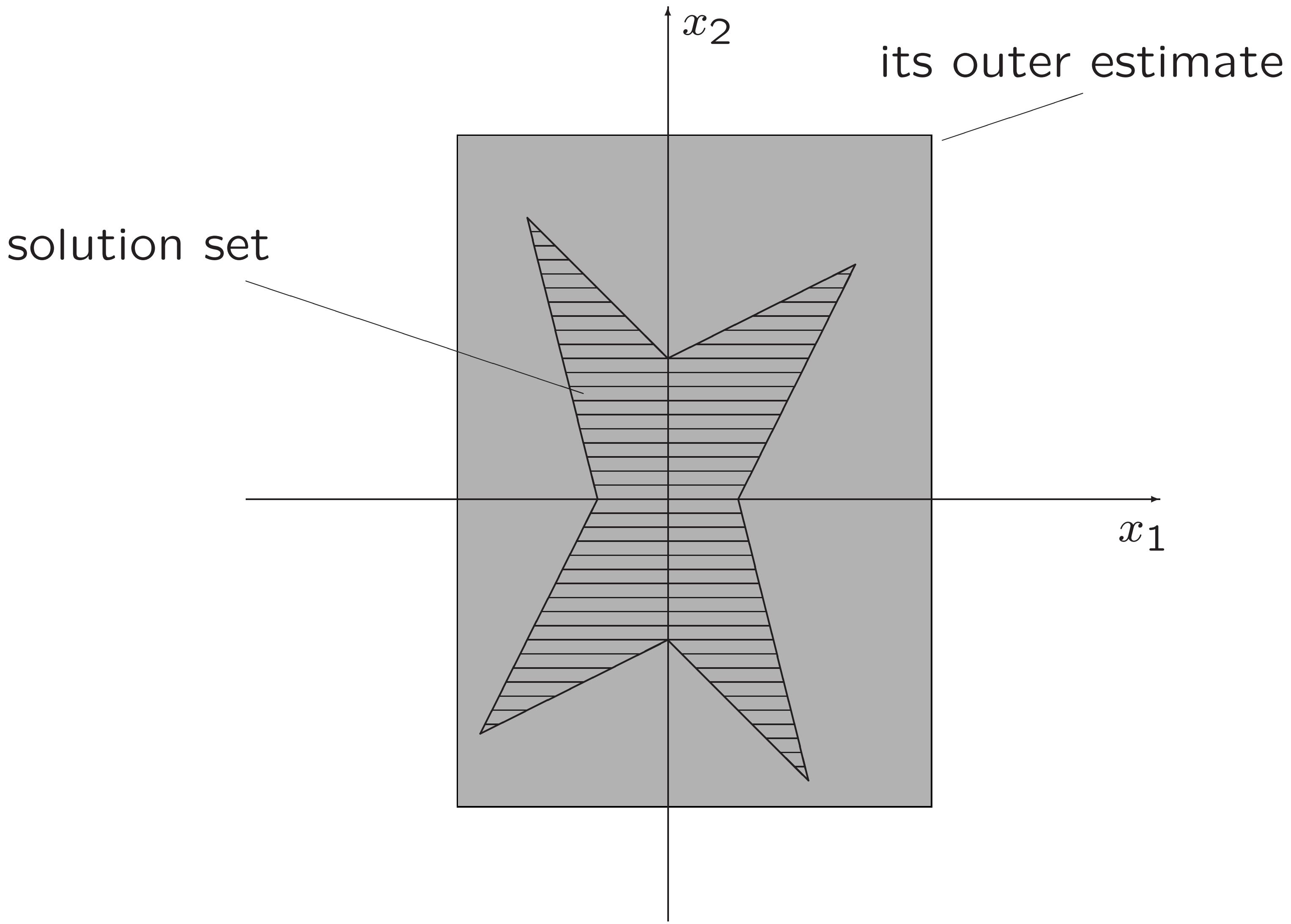
Interval linear systems of equations

Exact and complete description of the solution set $\Xi(\mathbf{A}, \mathbf{b})$ is

- practically impossible due to its enormous complexity,
- not necessary in reality.

In most cases, it suffices to know an *approximate description*,
or *estimate* of the solution set by simpler sets
i.e. having less constructive complexity.

“Outer problem”



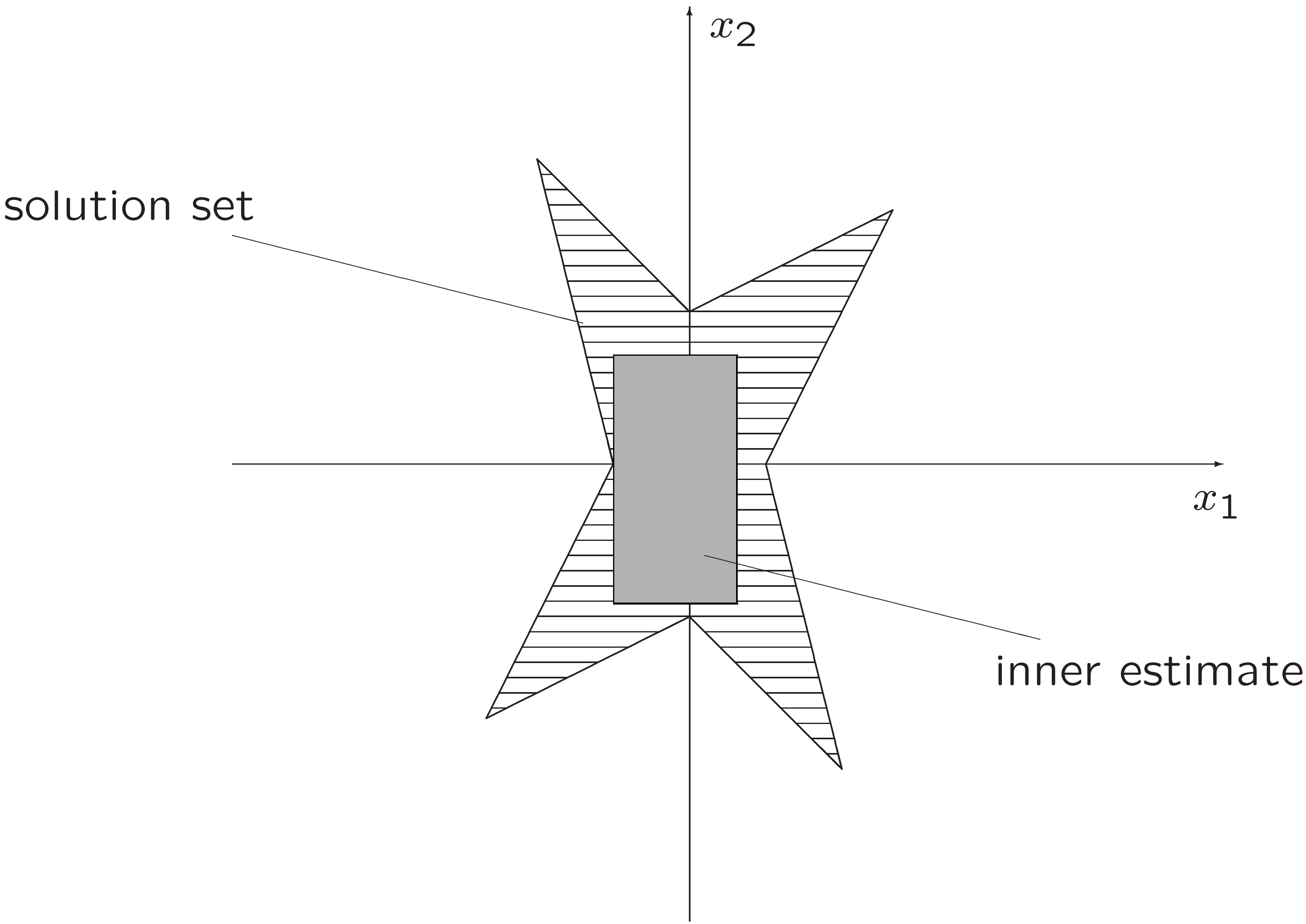
Problem statement

$$\boldsymbol{A}x = \boldsymbol{b}$$

the interval matrix $\boldsymbol{A}$ is supposed to be regular

Find (as tight as possible) interval box
that contains the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$
to interval linear system $\boldsymbol{A}x = \boldsymbol{b}$

“Inner problem”



Problem statement

$$\boldsymbol{A}x = \boldsymbol{b}$$

the interval matrix $\boldsymbol{A}$ need not be square,

need not be regular in square case

Find (as wide as possible) interval box
contained in the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$
of interval linear system $\boldsymbol{A}x = \boldsymbol{b}$

— decision making, identification under interval uncertainty, . . .

Practically, *inclusion maximal* inner estimates are most valuable.

O. Perron — 1907

G. Frobenius — 1908–1912

— theory of nonnegative point matrices

O. Perron — 1907

G. Frobenius — 1908–1912

— theory of nonnegative point matrices

Does there exists something

equally elegant

for nonegative interval matrices?! . . .

Observation

If, in the interval linear equations system $\boldsymbol{A}x = \boldsymbol{b}$,
all the entries of the matrix $\boldsymbol{A}$ are nonnegative,
the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$ has *monotonic shape*

Theoretical basis

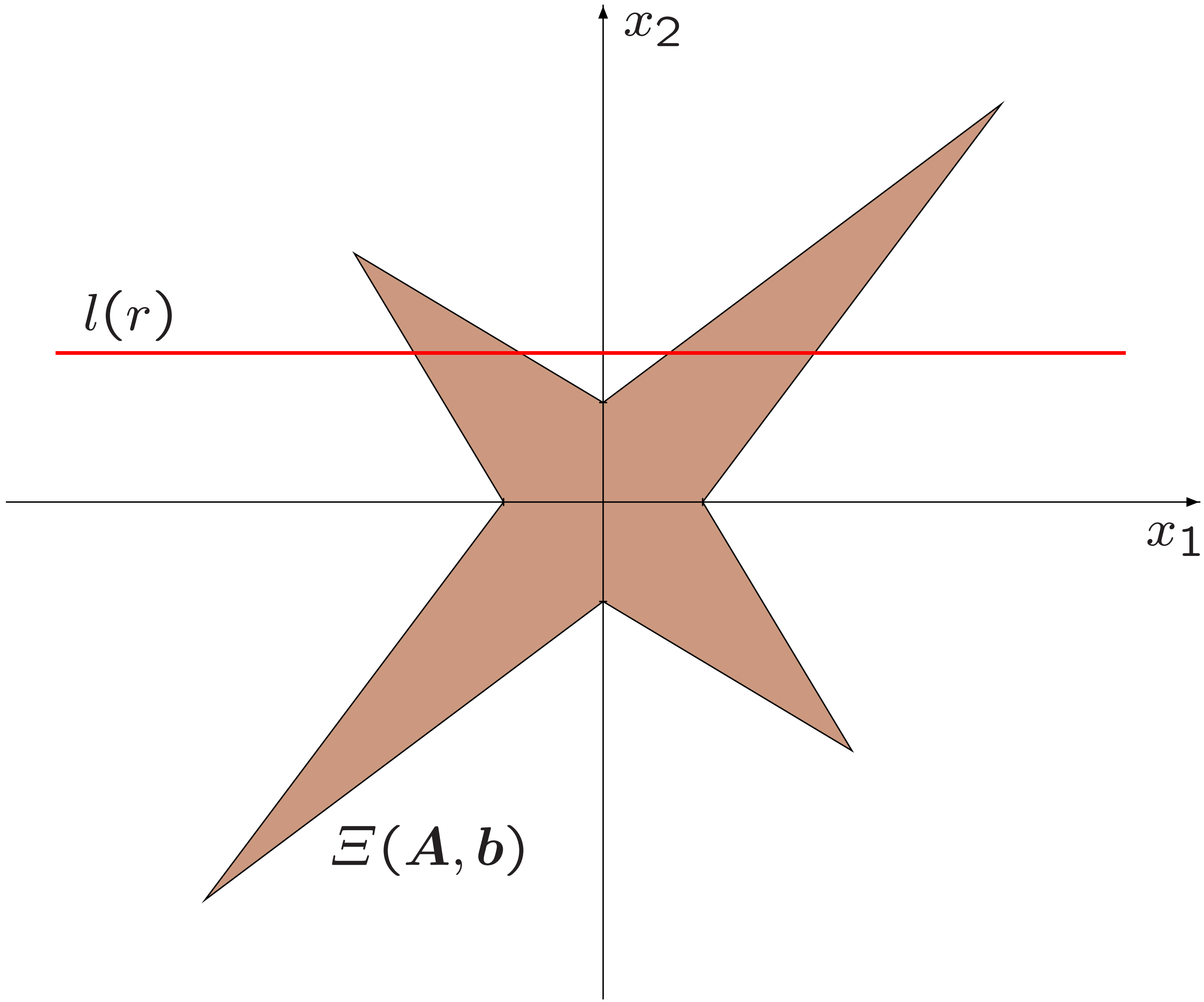
We fix an index $\nu \in \{1, 2, \ldots, n\}$ and consider in $\mathbb{R}^n$ a straignt line l with the equation

$$\left\{ \begin{array}{ccc} x_1 & = & r_1, \\ & \vdots & \\ x_{\nu-1} & = & r_{\nu-1}, \\ x_\nu & = & t\,, \\ x_{\nu+1} & = & r_{\nu+1}, \\ & \vdots & \\ x_n & = & r_n \end{array} \right. \qquad (t \in \mathbb{R} \text{ is a parameter}),$$

parallel to the ν-th coordinate axis.

Every such line is determined by a vector $r \in \mathbb{R}^{n-1}$,

$r = (r_1, \ldots, r_{\nu-1}, r_{\nu+1}, \ldots, r_n)^\top$, and we can denote it as $l(r)$.



"axial cut" of the solution set

Theoretical basis

We define

$$\underline{\Omega}_\nu(r) \;=\; \min\Bigl\{\, x_\nu \mid x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \,\Bigr\},$$

$$\overline{\Omega}_\nu(r) \;=\; \max\Bigl\{\, x_\nu \mid x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \,\Bigr\},$$

— minimum amd maximum values of the ν-th coordinate of the points from the intersection of $l(r)$ with the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$.

Main auxiliary result

Proposition

If the matrix $\boldsymbol{A}$ *of the interval linear system* $\boldsymbol{A}x = \boldsymbol{b}$ *is nonnegative, then the functions* $\underline{\Omega}_\nu(r)$ *and* $\overline{\Omega}_\nu(r)$, $\nu = 1, 2, \ldots, n$, *are nonincreasing with respect to every variable on their effective domains.*

How can we compute the values of $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$?

Let us "substitute" the equation of the line $l(r)$ into the system:

$$\begin{cases} \boldsymbol{a}_{1\nu}\, t & + & \sum_{j\neq\nu} \boldsymbol{a}_{1j}\, r_j & = & \boldsymbol{b}_1, \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{a}_{m\nu}\, t & + & \sum_{j\neq\nu} \boldsymbol{a}_{mj}\, r_j & = & \boldsymbol{b}_m. \end{cases} \qquad (\star)$$

If $\boldsymbol{A}$ is nonnegative, then the solution set of the i-th equation is

$$\left(\boldsymbol{b}_i - \sum_{j\neq\nu} \boldsymbol{a}_{ij} r_j \right) \Big/ \, \boldsymbol{a}_{i\nu}.$$

We can solve each of the one-dimensional equations comprising the system $(\star)$ separately, and then intersect the resulting solution sets.

The set $\mathcal{S}$ thus obtained, as the result of separate solution of one-dimensional equations and intersection of their solution sets, is the set of values of the ν-th coordinate of points from $\Xi \cap l(r)$.

It may be empty if the system $(\star)$ is incompatible, but in any case

$$\underline{\Omega}_\nu(r) \;=\; \min \mathcal{S} \qquad\text{and}\qquad \overline{\Omega}_\nu(r) \;=\; \max \mathcal{S}.$$

If the intervals $\boldsymbol{a}_{i\nu}$, $i = 1, 2, \ldots, m$, do not contain zero in the interior, then all the solution sets to one-dimensional equations are *connected* intervals of the form

$$[p, q] \quad\text{or}\quad (-\infty, p] \quad\text{or}\quad [q, +\infty) \quad\text{or}\quad (-\infty, +\infty).$$

In the points of the effective domain of $\underline{\Omega}_\nu(r)$, there holds

$$\underline{\Omega}_\nu(r) \;=\; \max_{1\le i\le m}\left\{ \underline{\left(\boldsymbol{b}_i - \sum_{j\neq\nu} \boldsymbol{a}_{ij} r_j\right) \Big/ \boldsymbol{a}_{i\nu}} \right\}.$$

In the points of the effective domain of $\overline{\Omega}_\nu(r)$, there holds

$$\overline{\Omega}_\nu(r) \;=\; \min_{1\le i\le m}\left\{ \overline{\left(\boldsymbol{b}_i - \sum_{j\neq\nu} \boldsymbol{a}_{ij} r_j\right) \Big/ \boldsymbol{a}_{i\nu}} \right\}.$$

Proof of Proposition

Both lower and upper envelopes of any family of nonincreasing (nondecreasing) functions is nonincreasing (nondecreasing) too.

If $\boldsymbol{a}_{ij} \geq 0$ and $\boldsymbol{a}_{i\nu} \geq 0$, then for all i, j and ν the expressions

$$\frac{\Big(\text{endpoint of } \boldsymbol{b}_i\Big) - \sum_{j \neq \nu} \Big(\text{endpoint of } \boldsymbol{a}_{ij}\Big)\ r_j}{\text{endpoint of } \boldsymbol{a}_{i\nu}}$$

are monotonically nonincreasing with respect to every argument r_j (providing that the rest arguments are fixed).

Therefore, the functions

$$\underline{\omega}_{i\nu}(r) \;=\; \underline{\left(\boldsymbol{b}_i - \sum_{j \neq \nu} \boldsymbol{a}_{ij} r_j \right) / \; \boldsymbol{a}_{i\nu}}, \qquad i = 1, 2, \ldots, m,$$

being the lower envelopes of the above functions, and the functions

$$\overline{\omega}_{i\nu}(r) \;=\; \overline{\left(\boldsymbol{b}_i - \sum_{j \neq \nu} \boldsymbol{a}_{ij} r_j \right) / \; \boldsymbol{a}_{i\nu}}, \qquad i = 1, 2, \ldots, m,$$

being their upper envelopes, are nonincreasing with respect to r_k.

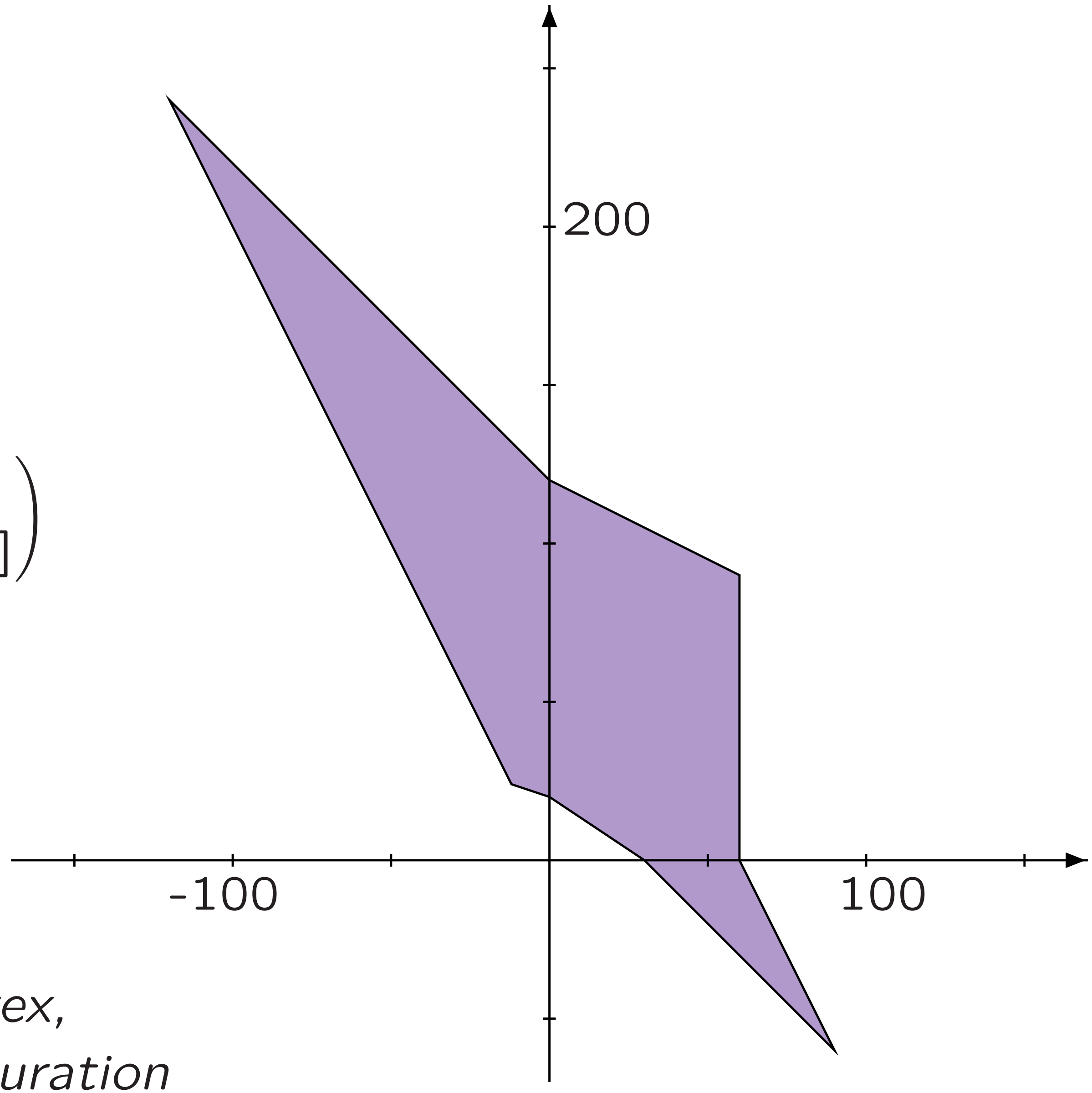
Since

$$\underline{\Omega}_\nu(r) = \max_i \underline{\omega}_{i\nu}(r) \qquad \text{and} \qquad \overline{\Omega}_\nu(r) = \min_i \overline{\omega}_{i\nu}(r),$$

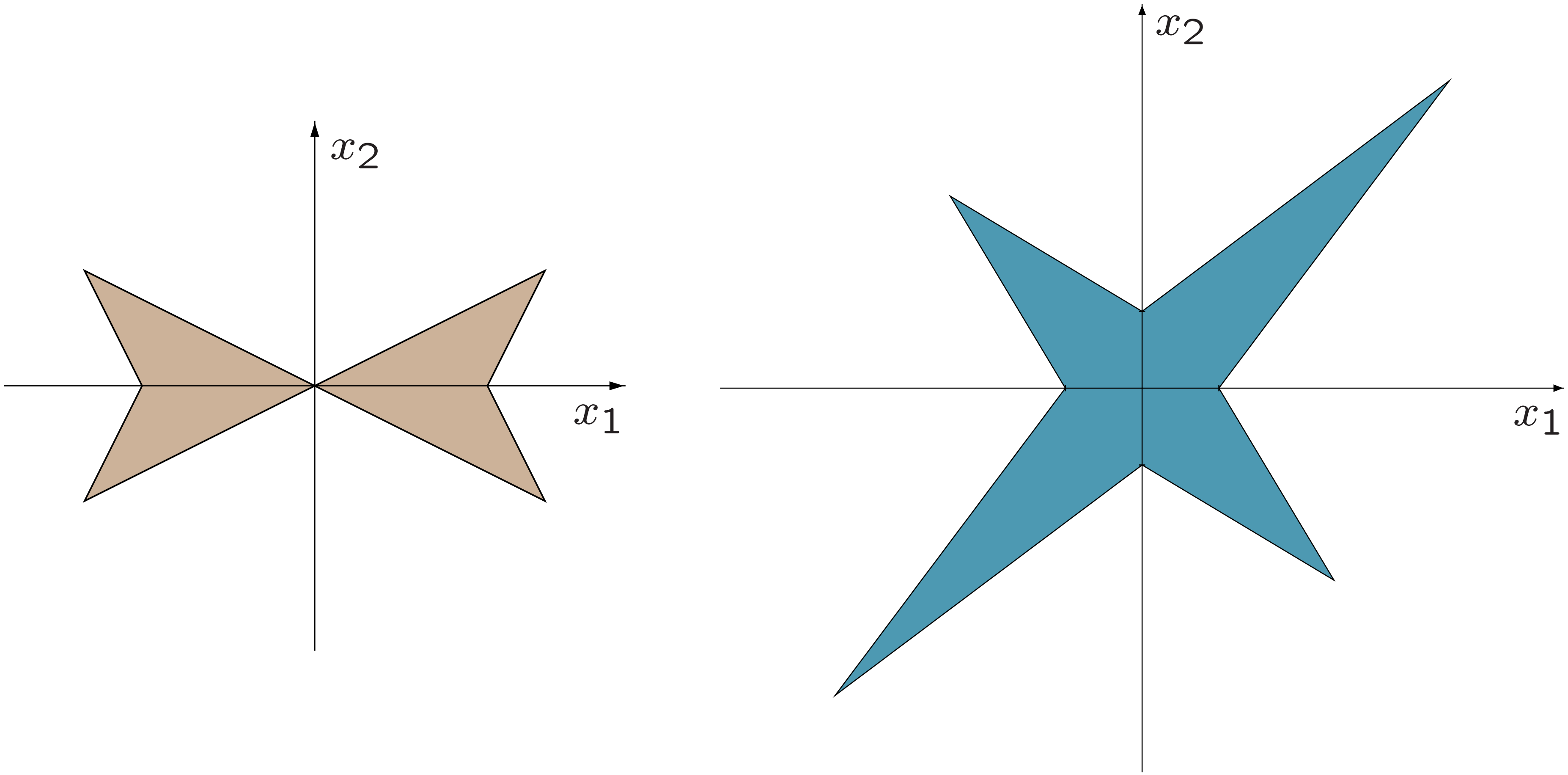
the proposition follows.

Example — Hansen system

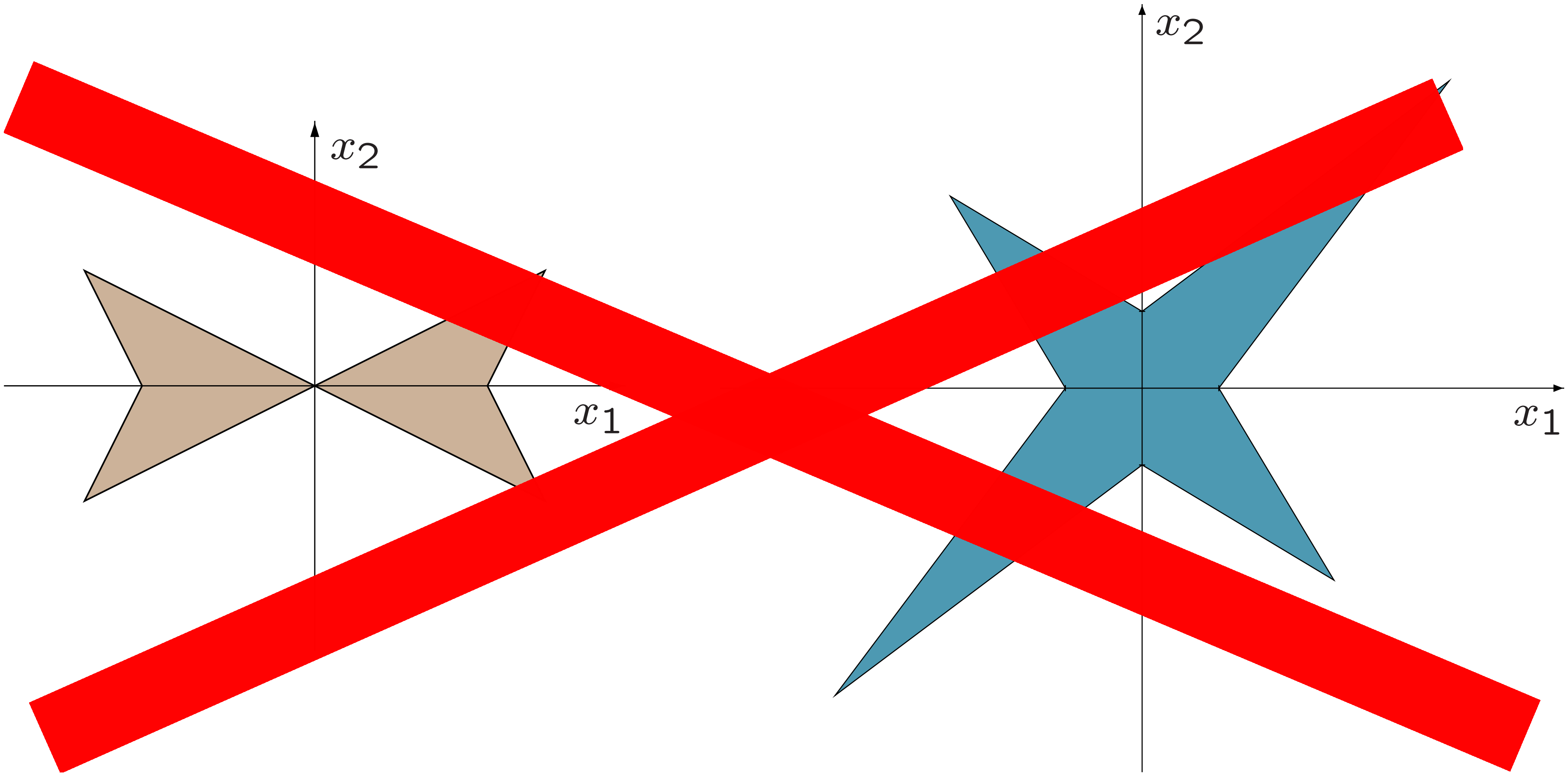
$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$



the solution set is not convex,
but has a monotonic configuration

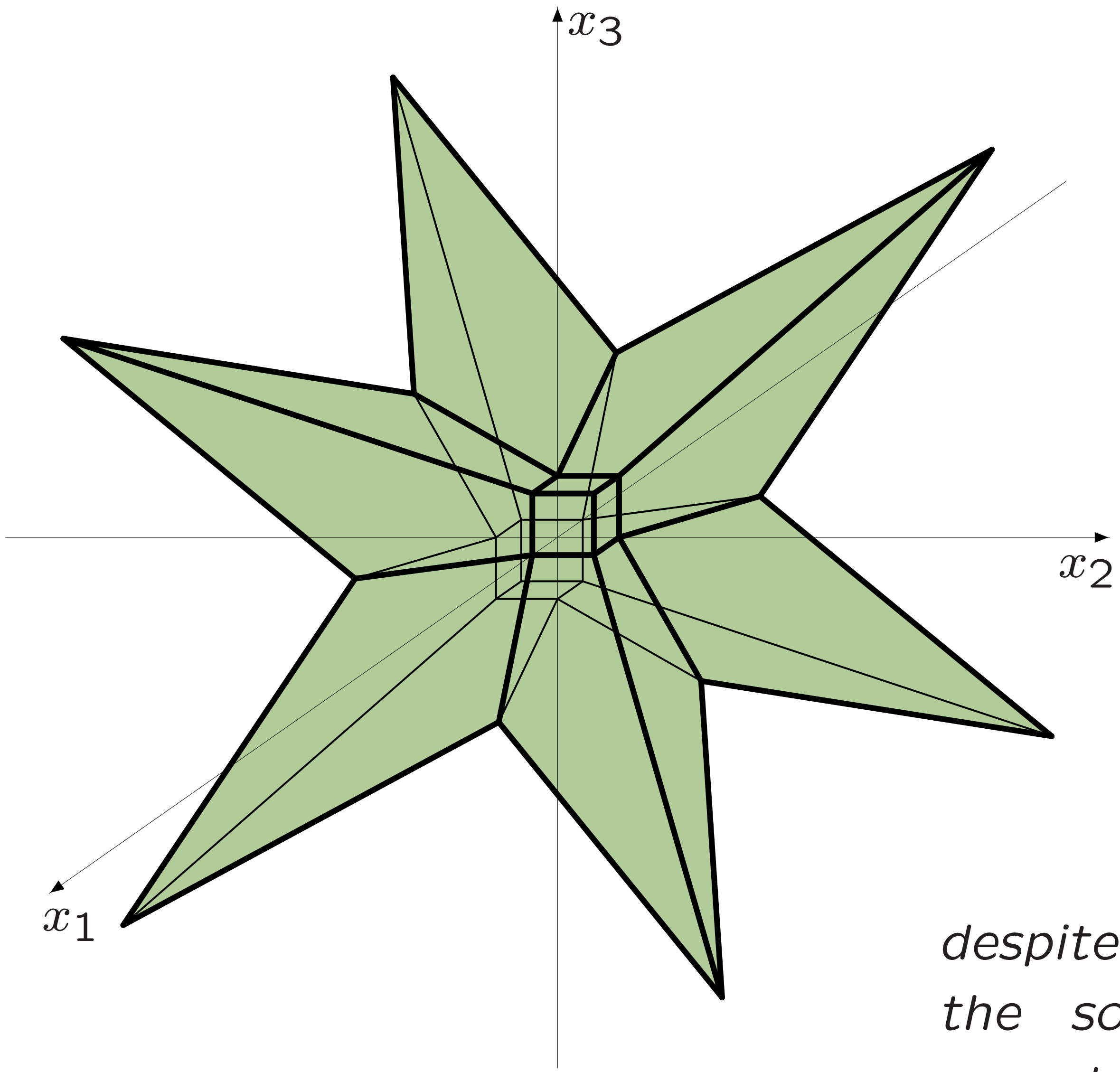


Bulging "corners" that spoil monotonicity are impossible
for the solution sets of 2D interval linear systems
with nonegative matrices.



Bulging “corners” that spoil monotonicity are impossible for the solution sets of 2D interval linear systems with nonegative matrices.

Example — Neumaier system



$$\begin{pmatrix} 3.5 & [0,2] & [0,2] \\ [0,2] & 3.5 & [0,2] \\ [0,2] & [0,2] & 3.5 \end{pmatrix} x = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}$$

despite the seemingly chaotic shape, the solution set is bounded by monotonic surfaces

A remark

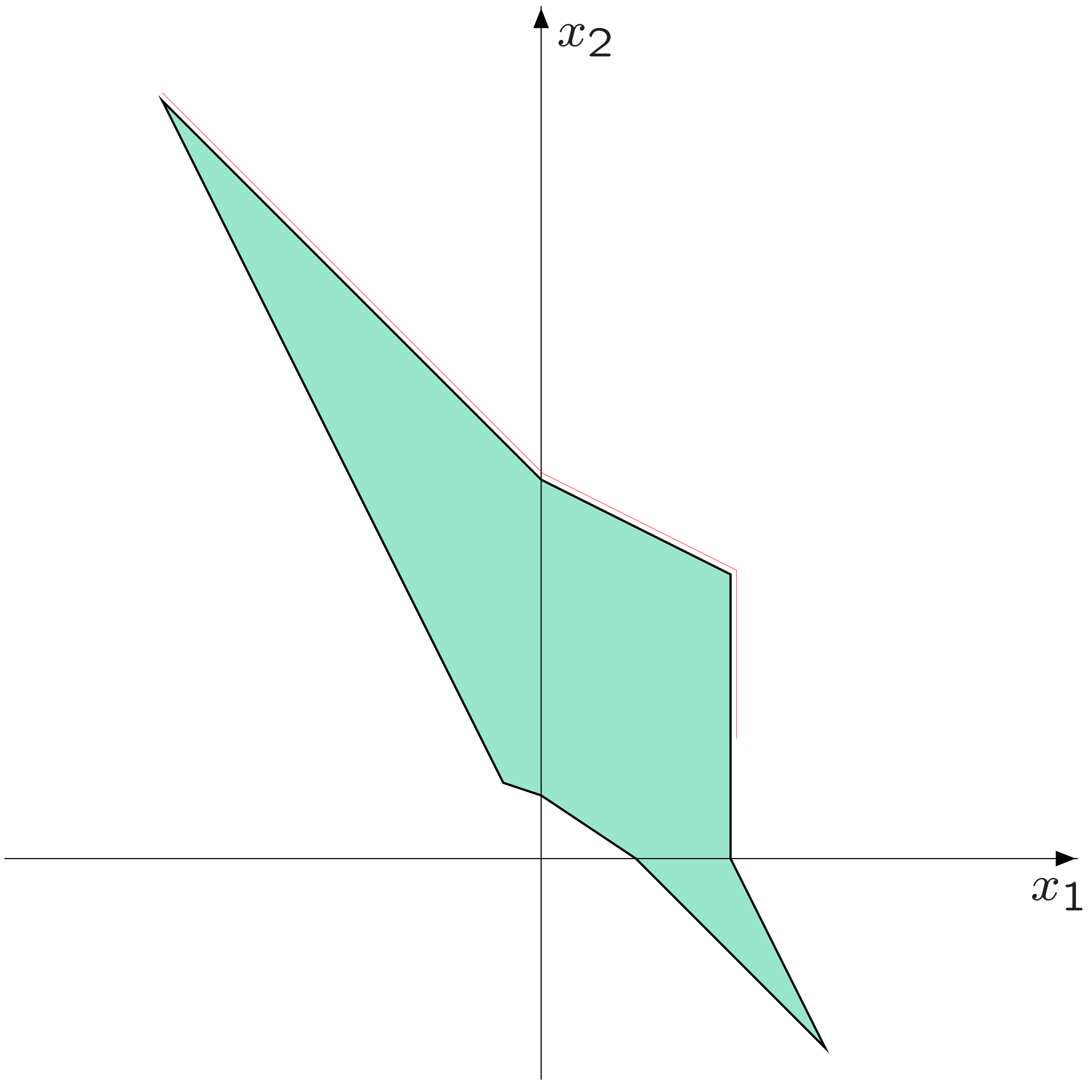
The functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ may be discontinuous, which is due to zero endpoints of some interval entries in the matrix of the system.

However,

if the matrix of the system is positive, i.e. $\boldsymbol{a}_{ij} > 0$ for every i and j, then the functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$, $\nu = 1, 2, \ldots, n$, are continuous.

Several unsuccessful attempts

?. . .



Complexity result

Lakeyev A.V. and Kreinovich V.

NP-hard classes of linear algebraic systems with uncertainties

Reliable Computing. – 1997. – Vol. 3, No. 1. – P. 51–81.

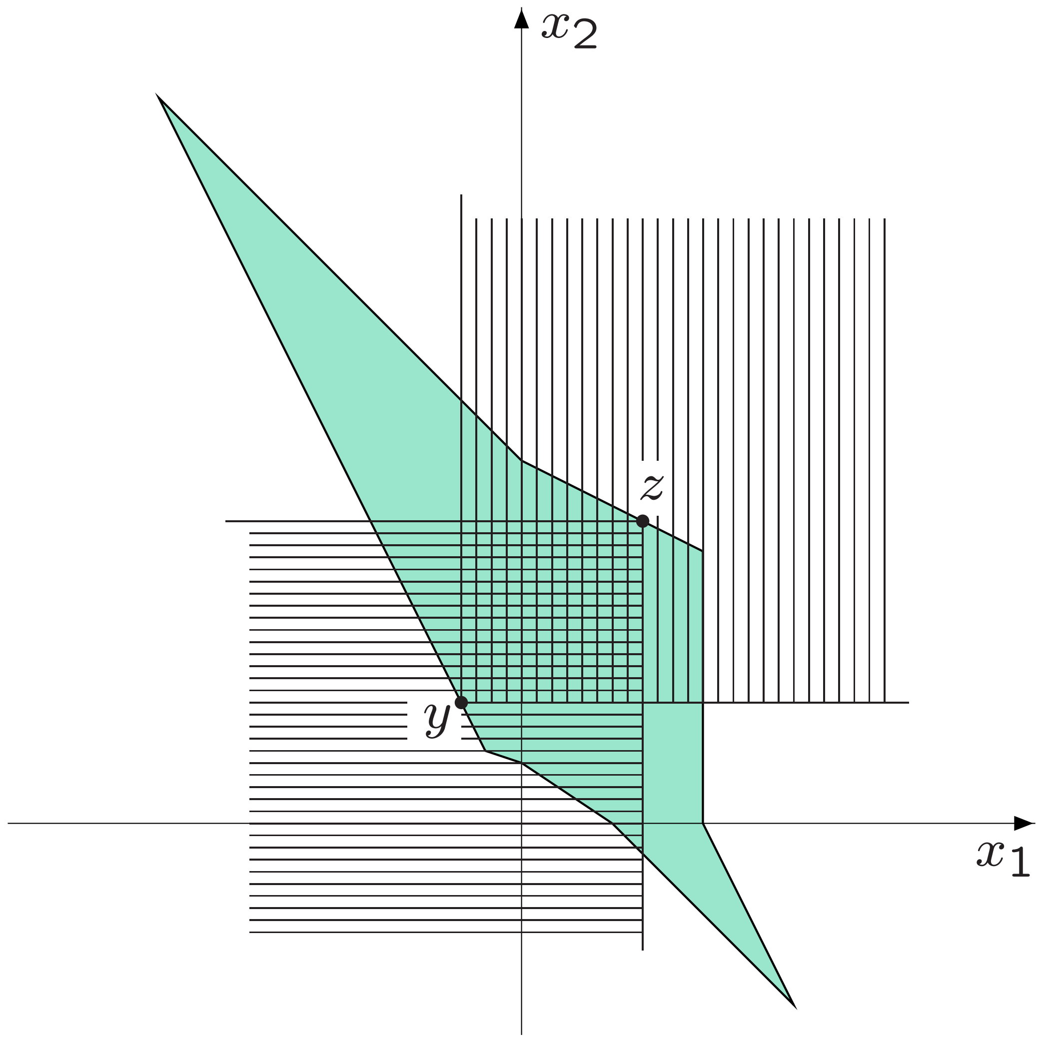
— outer estimation of the solution sets
to interval linear systems is NP-hard
even if matrices of the systems are positive

Outer estimation failed . . .

Maybe, inner estimation will be more successful?

Theorem

If, in the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$*, the matrix* $\boldsymbol{A}$ *is nonnegative, then for any two points* y*,* $z \in \Xi(\boldsymbol{A}, \boldsymbol{b})$*, such that* $y \leq z$*, the interval box* $[y, z]$ *is a subset of the solution set* $\Xi(\boldsymbol{A}, \boldsymbol{b})$*.*



Proof

It follows from the definition of the functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ that, for any $r \in \mathbb{R}^{n-1}$ and every $\nu \in \{1, 2, \ldots, n\}$, there holds

$$\underline{\Omega}_\nu(r) \;\leq\; \Big\{ x_\nu \mid x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \Big\} \;\leq\; \overline{\Omega}_\nu(r).$$

If the matrix $\boldsymbol{A}$ is nonnegative, then

$$\Big\{ x_\nu \mid x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \Big\} \;=\; \Big[\underline{\Omega}_\nu(r), \overline{\Omega}_\nu(r) \Big],$$

since the set $\{ x_\nu \mid x \in \varXi(\boldsymbol{A}, \boldsymbol{b}) \cap l(r) \}$ is connected. Therefore, the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ is the intersection of the epigraph of $\underline{\Omega}_\nu(r)$ and hypergraph of $\overline{\Omega}_\nu(r)$.

The theorem stems from the fact
that the functions $\underline{\Omega}_\nu(r)$ and $\overline{\Omega}_\nu(r)$ are nonincreasing.

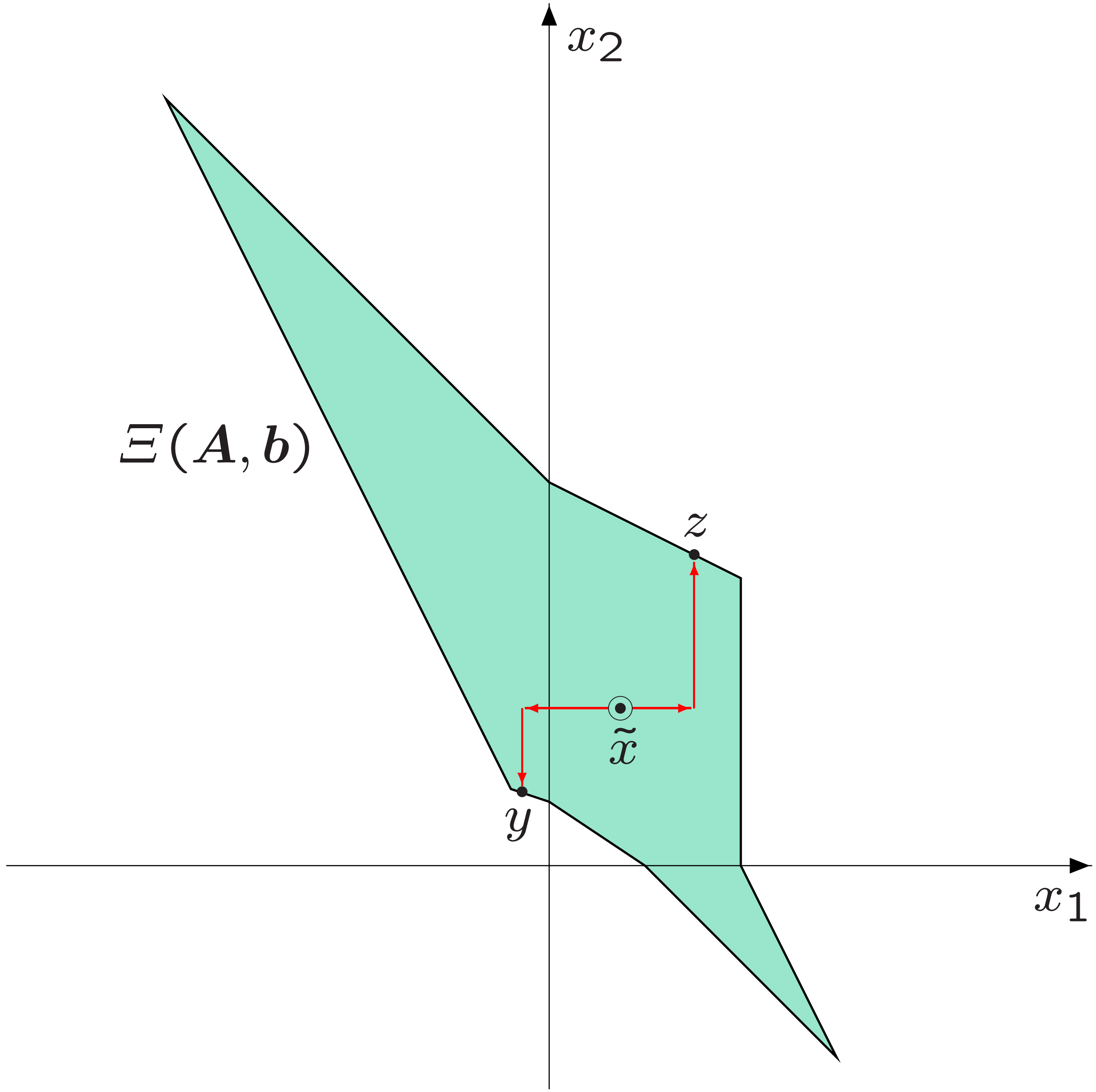
Algorithm for inner estimation

— constructs the lower y and upper z bounds of the box $[y, z] \subseteq \Xi(\boldsymbol{A}, \boldsymbol{b})$, starting from a point $\tilde{x} \in \Xi(\boldsymbol{A}, \boldsymbol{b})$.

Initially, we assign

$$y \leftarrow \tilde{x}, \qquad z \leftarrow \tilde{x},$$

and then the i-th, $i = 1, 2, \dots, n$, step of the algorithm moves the points y and z apart along the i-th coordinate direction


x_2
$\Xi(\boldsymbol{A}, \boldsymbol{b})$
z
$\tilde{x}$
y
x_1

Algorithm INonNeg for inner estimation of solution sets to interval linear systems

Input

Interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ with nonnegative matrix.

A point $\tilde{x}$ from the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$ under estimation.

Parameters λ, $\mu \in]0, 1]$.

Output

Lower y and upper z bounds of the interval vector $[y, z]$ contained in the solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$.

Auxiliary scalar parameters λ and μ, $0 < \lambda, \mu \leq 1$, help adjusting the form of the interval estimate $[y, z]$ and its location within the solution set $\Xi(\boldsymbol{A}, \boldsymbol{b})$.

These parameters control the relative values of the shifts of y_i and z_i with respect to $\tilde{x}_i$ during the i-th algorithm step with respect to $\tilde{x}_i$.

Algorithm `INonNeg`

```
y ← x̃ ;    z ← x̃ ;
DO FOR  k = 1  TO  n
    Y ← (−∞, ∞) ;    Z ← (−∞, ∞) ;
    DO FOR  i = 1  TO  m
```

$$\boldsymbol{Y} \leftarrow \boldsymbol{Y} \cap \left(\left(\boldsymbol{b}_i - \sum_{j=1, j\neq k}^{n} \boldsymbol{a}_{ij}\, y_j \right) \Big/ \boldsymbol{a}_{ik} \right);$$

$$\boldsymbol{Z} \leftarrow \boldsymbol{Z} \cap \left(\left(\boldsymbol{b}_i - \sum_{j=1, j\neq k}^{n} \boldsymbol{a}_{ij}\, z_j \right) \Big/ \boldsymbol{a}_{ik} \right);$$

```
    END DO
    IF ( k < n ) THEN
```

$$y_k \leftarrow \lambda\, \underline{\boldsymbol{Y}} + (1-\lambda)\, \tilde{x}_k\,; \qquad z_k \leftarrow (1-\mu)\, \tilde{x}_k + \mu\, \overline{\boldsymbol{Z}}\,;$$

```
    ELSE
```

$$y_k \leftarrow \underline{\boldsymbol{Y}}\,; \qquad z_k \leftarrow \overline{\boldsymbol{Z}}\,;$$

```
    END IF
END DO
```

Numerical example

For Hansen system

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix},$$

using the algorithm `INonNeg` with the parameters $\lambda = \mu = 1$ results in

$$\begin{pmatrix} [-25.909, 60] \\ [51.818, 90] \end{pmatrix},$$

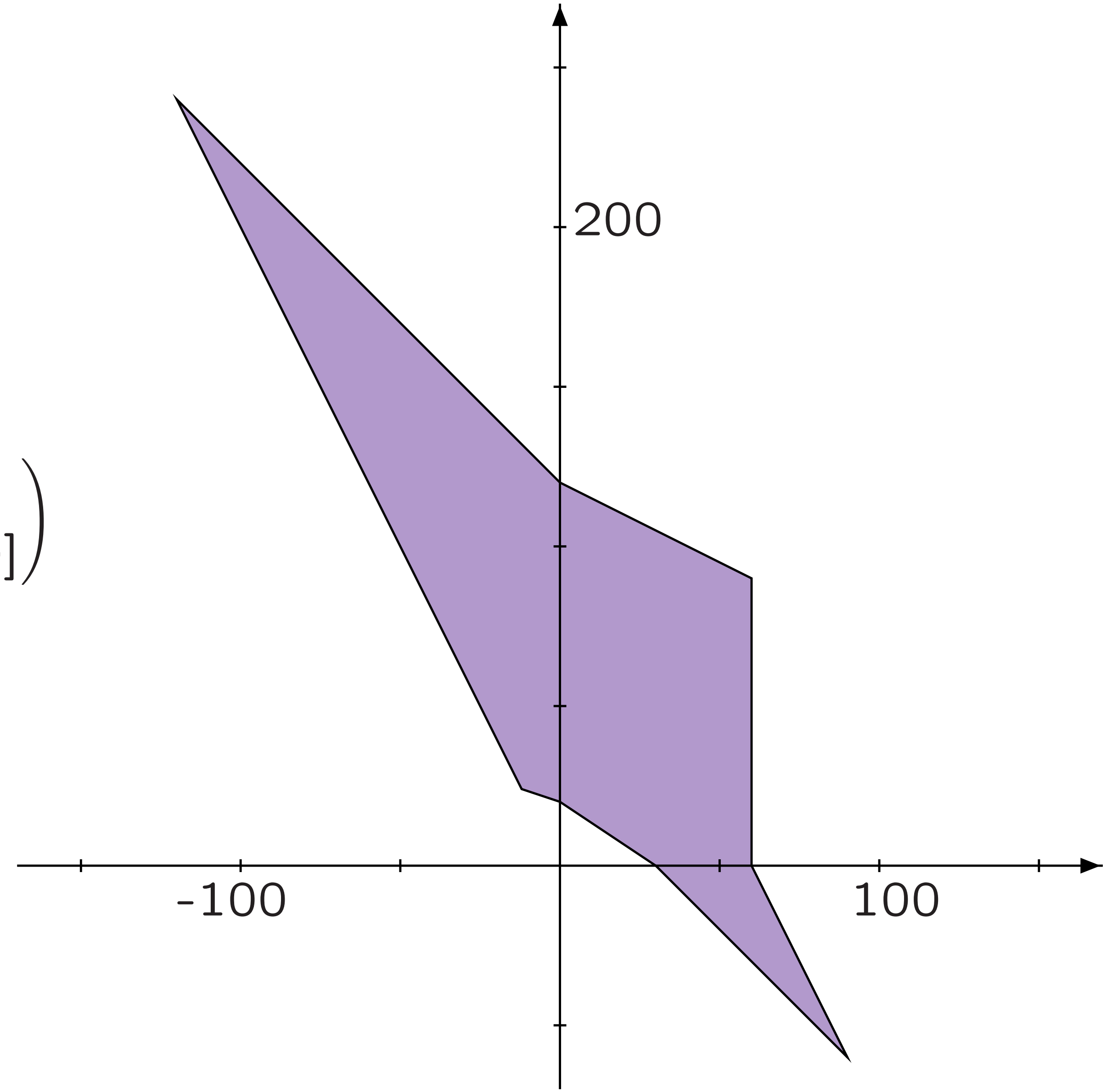
while the parameters $\lambda = \mu = 0.7$ produce the inner estimate

$$\begin{pmatrix} [-13.022, 47.114] \\ [26.045, 96.443] \end{pmatrix}.$$

Solution to the “midpoint system” $(\mathrm{mid}\,\boldsymbol{A})\,x = \mathrm{mid}\,\boldsymbol{b}$ is taken as the starting point $\tilde{x}$.

Numerical example — Hansen system

$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$



Generalizations?. . .

Generalizations?. . .

For generalized solution sets!

Generalized solution sets

— originate from the observation that

interval uncertainty has *dual character*

Usually, we use an interval $\boldsymbol{v}$ only in connection with a property $P(v)$ that may be fulfilled or not for their point members $v \in \boldsymbol{v}$, and

- either the property $P(v)$ holds *for all* $v \in \boldsymbol{v}$,
- or the property $P(v)$ holds *for some* $v \in \boldsymbol{v}$, not necessarily all, maybe, even for only one.

The above distinction is rendered by logical quantifiers —

- in the first case, we write "$(\forall v \in \boldsymbol{v})\ P(v)$"

  speaking of *interval uncertainty of A-type*,

- in the second case, we write "$(\exists v \in \boldsymbol{v})\ P(v)$"

  speaking of *interval uncertainty of E-type*.

Generalized solution sets

For an interval system of equations $F(\boldsymbol{a}, x) = \boldsymbol{b}$ the most general definition of the solution set looks like

$$\Big\{\, x \in \mathbb{R}^n \mid (\, Q_1 v_{\pi_1} \in \boldsymbol{v}_{\pi_1}) \cdots (\, Q_{l+m} v_{\pi_{l+m}} \in \boldsymbol{v}_{\pi_{l+m}})(\, F(a, x) = b \,) \,\Big\},$$

where

$Q_1, Q_2, \ldots, Q_{l+m}$ are logical quantifiers "$\forall$" or "$\exists$",

$(\, v_1, v_2, \ldots, v_{l+m}) := (\, a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_m) \in \mathbb{R}^{l+m}$
is aggregated vector of the parameters of the system,

$(\, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{l+m}) := (\, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_l, \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \in \mathbb{IR}^{l+m}$ —
is aggregated vector of intervals of their values,

$(\, \pi_1, \pi_2, \ldots, \pi_{l+m})$ is a permutation of the integers $1, 2, \ldots, l + m$.

Generalized solution sets

Definition

The above solution sets are called generalized solution sets to the interval system of equations $F(\boldsymbol{a}, x) = \boldsymbol{b}$.

Example

For an interval linear 2×2-system

$$\begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} \end{pmatrix} x = \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix},$$

we can arrange the solution set

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; (\exists a_{21} \in \boldsymbol{a}_{21})(\exists a_{11} \in \boldsymbol{a}_{11})(\forall a_{22} \in \boldsymbol{a}_{22})(\forall b_1 \in \boldsymbol{b}_1) \right.$$
$$\left. (\exists b_2 \in \boldsymbol{b}_2)(\forall a_{12} \in \boldsymbol{a}_{12})\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) \right\}.$$

Generalized solution sets

Extremely general definition!

We confine ourselves only to the solution sets for which, in the selecting predicate, *all occurrences of the universal quantifier* $\forall$ *precede those of the existential quantifier* $\exists$.

Definition

Generalized solution sets to interval equations systems for which the predicate that selects point from the solution set has AE-form will be referred to as *AE-solution sets* (or *sets of AE-solutions*).

AE-solution sets

Let, for an interval linear $m \times n$-system $\boldsymbol{A}x = \boldsymbol{b}$, quantifier $m \times n$-matrix α and m-vector β be given as well as associated decompositions of the index sets of the matrix and vector of the same size to nonintersecting subsets $\hat{\Gamma} = \{\hat{\gamma}_1, \ldots, \hat{\gamma}_p\}$ and $\check{\Gamma} = \{\check{\gamma}_1, \ldots, \check{\gamma}_q\}$, $p + q = mn$, $\hat{\Delta} = \{\hat{\delta}_1, \ldots, \hat{\delta}_r\}$ and $\check{\Delta} = \{\check{\delta}_1, \ldots, \check{\delta}_s\}$, $r + s = m$.

The set

$$
\begin{aligned}
&\Xi_{\alpha\beta}(\boldsymbol{A}, \boldsymbol{b}) \;:= \\
&\Big\{\, x \in \mathbb{R}^n \mid \\
&\quad (\forall a_{\hat{\gamma}_1} \in \boldsymbol{a}_{\hat{\gamma}_1}) \cdots (\forall a_{\hat{\gamma}_p} \in \boldsymbol{a}_{\hat{\gamma}_p}) \; (\forall b_{\hat{\delta}_1} \in \boldsymbol{b}_{\hat{\delta}_1}) \cdots (\forall b_{\hat{\delta}_r} \in \boldsymbol{b}_{\hat{\delta}_r}) \\
&\quad (\exists a_{\check{\gamma}_1} \in \boldsymbol{a}_{\check{\gamma}_1}) \cdots (\exists a_{\check{\gamma}_q} \in \boldsymbol{a}_{\check{\gamma}_q}) \; (\exists b_{\check{\delta}_1} \in \boldsymbol{b}_{\check{\delta}_1}) \cdots (\exists b_{\check{\delta}_s} \in \boldsymbol{b}_{\check{\delta}_s}) \\
&\qquad (\, Ax = b \,) \,\Big\}
\end{aligned}
$$

will be referred to as *set of AE-solutions of the type* $\alpha\beta$ to the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$.

Equivalently, AE-solutions sets can be defined as

$$\Xi_{\alpha\beta}(\boldsymbol{A}, \boldsymbol{b}) := \Big\{ x \in \mathbb{R}^n \mid (\forall \widehat{A} \in \boldsymbol{A}^{\forall})(\forall \widehat{b} \in \boldsymbol{b}^{\forall}) \\ (\exists \check{A} \in \boldsymbol{A}^{\exists})(\exists \check{b} \in \boldsymbol{b}^{\exists})(\,(\,\widehat{A} + \check{A})\,x = \widehat{b} + \check{b}\,)\Big\},$$

where $\boldsymbol{A} = \boldsymbol{A}^{\forall} + \boldsymbol{A}^{\exists}$ и $\boldsymbol{b} = \boldsymbol{b}^{\forall} + \boldsymbol{b}^{\exists}$ are corresponding disjunct decompositions of the matrix and right-hand side of the system.

Theorem

A point $x \in \mathbb{R}^n$ belongs to AE-solution set $\Xi_{\alpha\beta}(\boldsymbol{A}, \boldsymbol{b})$ if and only if

$$\boldsymbol{A}^{\forall} x - \boldsymbol{b}^{\forall} \ \subseteq \ \boldsymbol{b}^{\exists} - \boldsymbol{A}^{\exists} x.$$

AE-solution sets

United solution set to the interval systems $\boldsymbol{A}x = \boldsymbol{b}$ —

$$\Xi_{uni}(\boldsymbol{A}, \boldsymbol{b}) = \Big\{ x \in \mathbb{R}^n \mid (\exists a_{11} \in \boldsymbol{a}_{11}) \cdots (\exists a_{nn} \in \boldsymbol{a}_{nn}) \\ (\exists b_1 \in \boldsymbol{b}_1) \cdots (\exists b_n \in \boldsymbol{b}_n)(\, Ax = b \,) \Big\}$$

$$= \Big\{ x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(\, Ax = b \,) \Big\},$$

— is the set of solutions to all the point systems $Ax = b$ with the parameters $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

Tolerable solution set to the interval system $\boldsymbol{A}x = \boldsymbol{b}$ —

$$\Xi_{tol}(\boldsymbol{A}, \boldsymbol{b}) = \Big\{ x \in \mathbb{R}^n \mid (\forall A \in \boldsymbol{A})(\exists\, b \in \mathbf{b})(\, Ax = b \,) \Big\},$$

— formed by all such point vectors x that the image Ax falls into $\boldsymbol{b}$ for any $A \in \boldsymbol{A}$.

A proposition

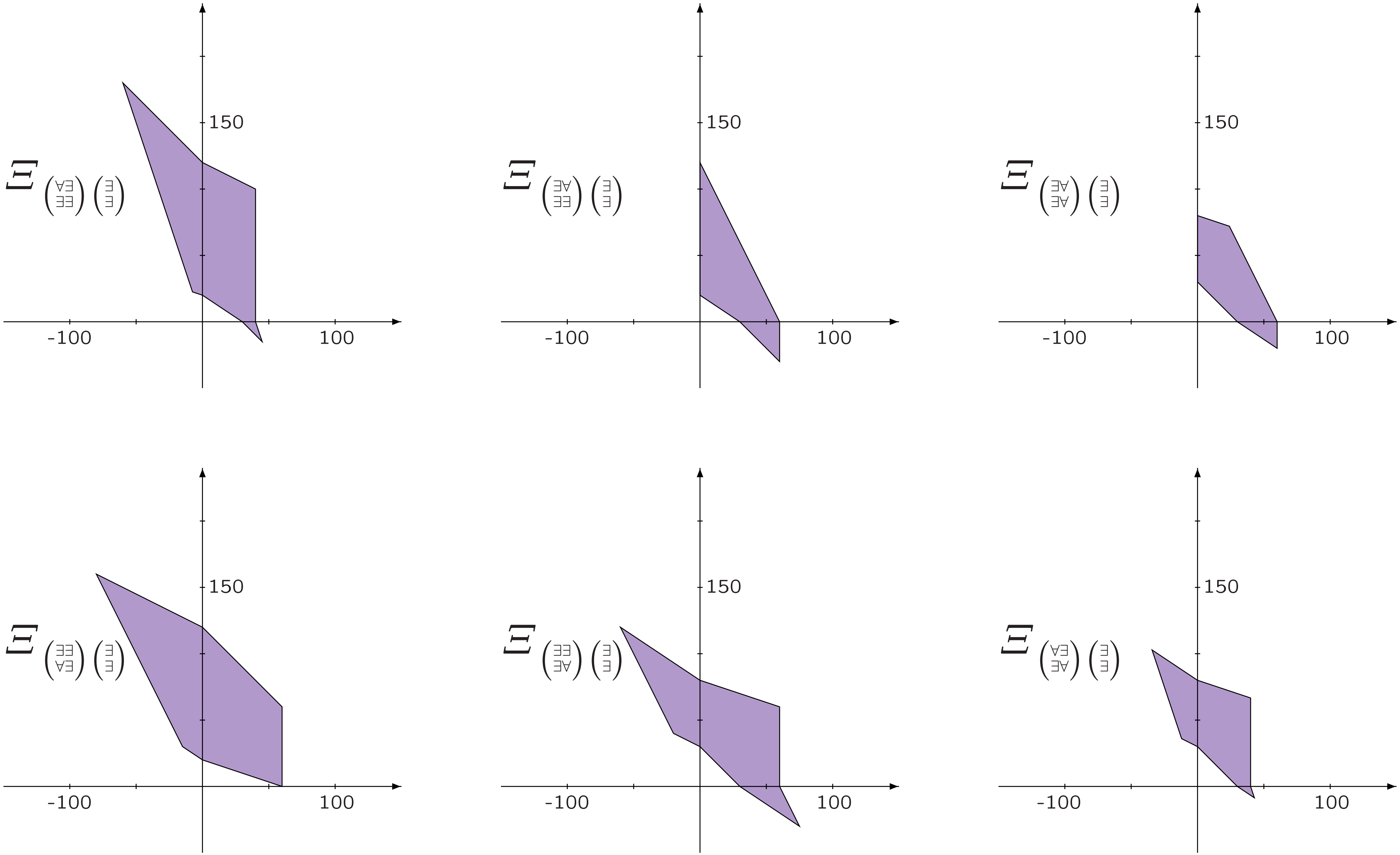
If, in the interval linear equations system $\boldsymbol{A}x = \boldsymbol{b}$, all the entries of the matrix $\boldsymbol{A}$ are nonnegative, AE-solution sets $\Xi_{\alpha\beta}(\boldsymbol{A}, \boldsymbol{b})$ have *monotonic shape*

Main auxiliary result

Proposition

If the matrix $\boldsymbol{A}$ *of the interval linear system* $\boldsymbol{A}x = \boldsymbol{b}$ *is nonnegative, then the functions* $\underline{\Omega}_\nu(r)$ *and* $\overline{\Omega}_\nu(r)$, $\nu = 1, 2, \ldots, n$, *are nonincreasing with respect to every variable on their effective domains.*

Some AE-solution sets for Hansen system



Complexity result

Lakeyev A.V.

Computational complexity of estimation of generalized solution sets for interval linear systems // *Вычислительные Технологии*. – 2003. – Т. 8, No. 1. – С. 12–23.

— in case of “sufficiently many” ∃-quantifiers
outer estimation of AE-solution sets
to interval linear systems is NP-hard
even if matrices of the systems are positive

Outer estimation failed again . . .

Maybe, inner estimation will be more successful?

Outer estimation failed again . . .

Maybe, inner estimation will be more successful?

Basically, **YES**.

Algorithm `INonNeg` is readily applicable for inner estimation of AE-solution sets to nonnegative interval linear systems.

Unfortunately, chosing an initial point $\tilde{x}$

is not an easy problem . . .

Chosing a starting point

For the united solution set $\varXi(\boldsymbol{A}, \boldsymbol{b})$, recognition of whether $\varXi(\boldsymbol{A}, \boldsymbol{b}) \neq \varnothing$ and finding a point $\tilde{x} \in \varXi(\boldsymbol{A}, \boldsymbol{b})$ is NP-hard in general.

Still, there exist special particular cases

when the problem can be solved easily.

E.g., the system is known to have regular interval matrix.

For generalized solution sets,
we do not know of such simple cases so far . . .

Tolerable solution set

$$\Xi_{tol}(\boldsymbol{A}, \boldsymbol{b}) = \Big\{ x \in \mathbb{R}^n \mid (\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b) \Big\}$$

Introduced in

E. Nuding and W. Wilhelm

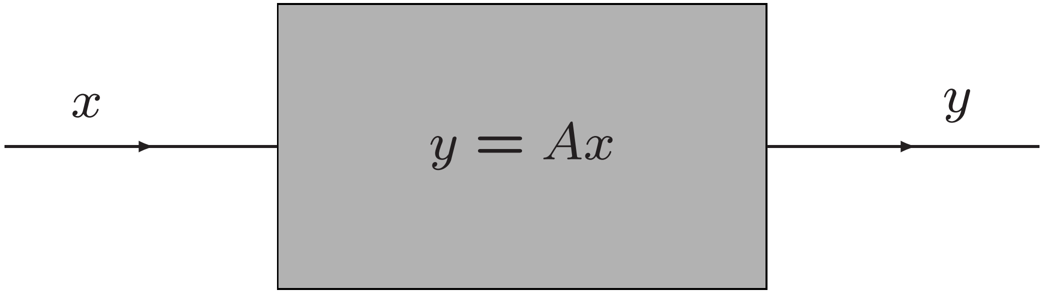
Über Gleichungen und über Lösungen

ZAMM. – 1972. – Bd. 52. – S. T188–T190.

Initially named *set of inner solutions*

Tolerable solution set

We are given a “black box” with the input $x \in \mathbb{R}^n$ and output $y \in \mathbb{R}^m$, while the “input-output” function is linear:



Parameters of the “black box” are not known exactly, available are only intervals $\boldsymbol{a}_{ij} \ni a_{ij}$, $(\boldsymbol{a}_{ij}) = \boldsymbol{A}$.

Tolerable solution set

It makes sense to specify outputs of the ”black box” intervally, as an interval vector $\boldsymbol{y}$, so as to ensure the hit $y \in \boldsymbol{y}$ no matter what the values of a_{ij} from $\boldsymbol{a}_{ij}$ are:

> *Does there exist such input actions $\tilde{x}$ that for any values of parameters $a_{ij} \in \boldsymbol{a}_{ij}$ we still get the output response y within the prescribed tolerance $\boldsymbol{y}$?*

— *the set of all such $\tilde{x}$'s is exactly $\Xi_{tol}(\boldsymbol{A}, \boldsymbol{y})$*

Tolerable solution set

Further researches

J. Rohn — 1978, 1986

N. Khlebalin — 1982, 1983, 1988

A. Deif — 1986

A. Neumaier — 1986

S. Shary — 1988, 1989, 1994, 1995, 1996, 2008

B. Kelling and D. Oelschlägel — 1991, 1994

Ye. Smagina — 1997, 2002

I. Sharaya — 2001, 2005, 2006, 2008

Tolerable solution set

Rohn's theorem

A point $x \in \mathbb{R}^n$ belongs to the tolerable solution set of interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ if and only if $x = x' - x''$ for vectors x', $x'' \in \mathbb{R}^n$ that satisfy the linear inequalities system

$$\begin{cases} \overline{\boldsymbol{A}}x' - \underline{\boldsymbol{A}}x'' & \leq & \overline{\boldsymbol{b}}, \\ -\underline{\boldsymbol{A}}x' + \overline{\boldsymbol{A}}x'' & \leq & -\underline{\boldsymbol{b}}, \\ x',\, x'' & \geq & 0. \end{cases}$$

Inner estimation for tolerable solution sets

. . . there exists efficient algorithms for finding
a point from the tolerable solution set

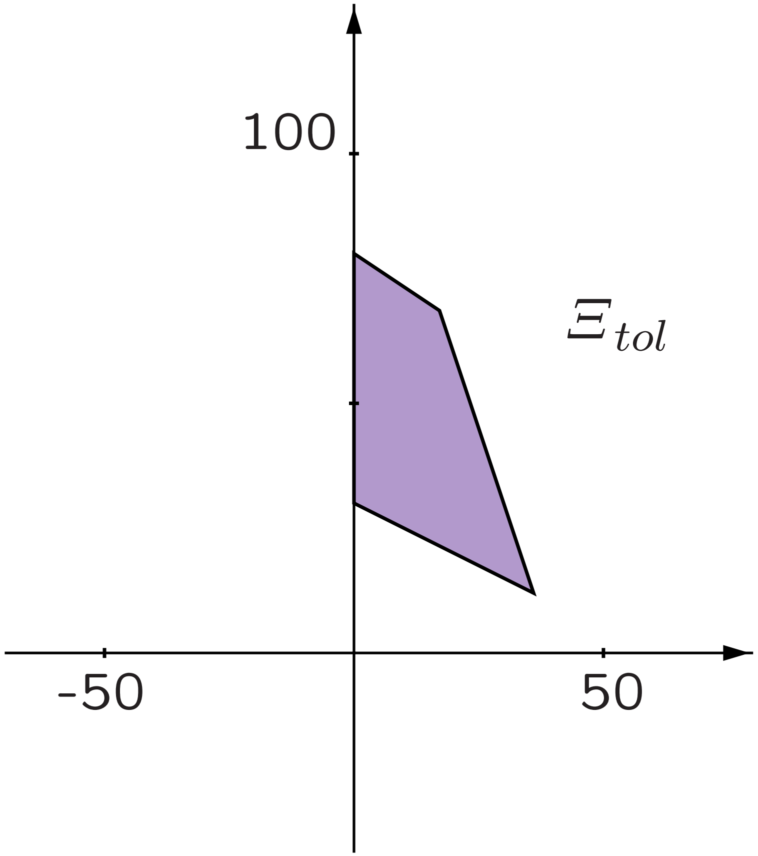
We can compute inclusion maximal inner interval estimates of the tolerable solution set to an interval linear system with nonnegative matrix by algorithm `INonNeg` for polynomial time . . .

Outer estimation for tolerable solution sets

$$\Xi_{tol}(\mathbf{A},\mathbf{b}) = \left\{\, x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b) \,\right\}$$

— tolerable solution set is globally convex

+ has monotonic shape



$$\begin{pmatrix} [2,3] & [0,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$

Outer estimation for tolerable solution sets

$$\Xi_{tol}(\boldsymbol{A}, \boldsymbol{b}) = \left\{ x \in \mathbb{R}^n \mid (\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b) \right\}$$

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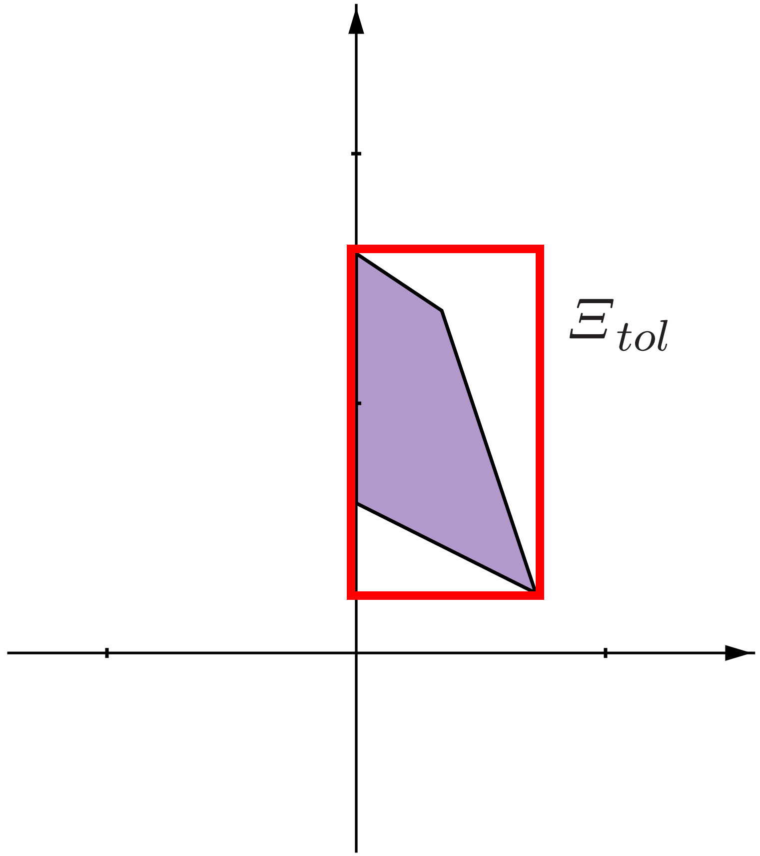

Outer estimation for tolerable solution sets

$$\Xi_{tol}(\boldsymbol{A},\boldsymbol{b}) = \left\{\, x \in \mathbb{R}^n \mid (\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b) \,\right\}$$

— tolerable solution set is globally convex

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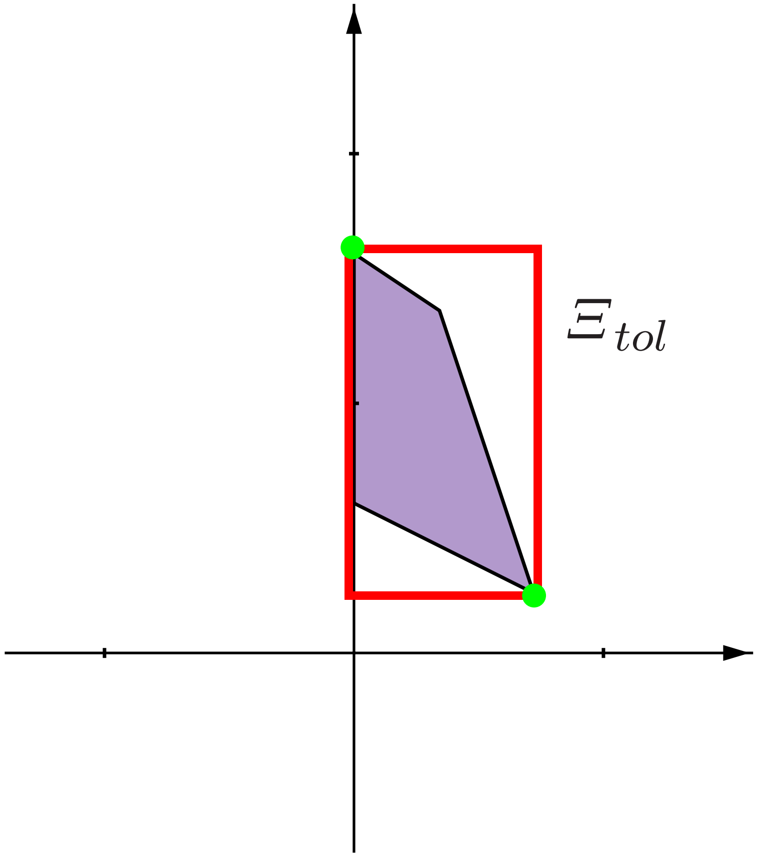
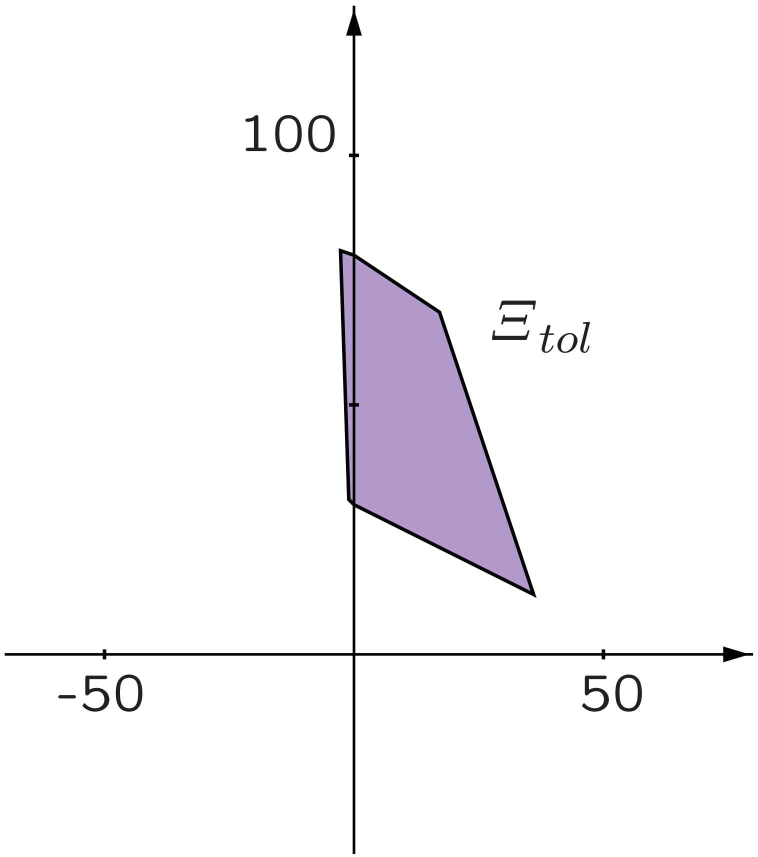


only 2 LP problems should be solved

Outer estimation for tolerable solution sets

How to fight stagnation of the process in discontinuity points?



small perturbations may help!

$$\begin{pmatrix} [2,3] & [0.1,1] \\ [1,2] & [2,3] \end{pmatrix} x = \begin{pmatrix} [0,120] \\ [60,240] \end{pmatrix}$$

Conclusions

For interval linear systems with nonegative matrices

- Solution sets have "monotonic shape", i.e. are bounded by surfaces that represent graphs of monotonic functions.
- Outer estimation of solution sets is NP-hard if "sufficiently many" matrix entries have interval E-uncertainty.
- Inclusion maximal inner interval estimates of solution sets can be computed in polynomial time (by algorithm `INonNeg`) provided that a point from the solution set is known.
- For tolerable solution set, both inner and outer estimation can be performed efficiently and results in the best possible estimates.

I appreciate your attention!

Publications

С.П. Шарый

Интервальные алгебраические задачи и их численное решение

Диссертация . . . доктора физ.-мат. наук. – Новосибирск:
Институт вычислительных технологий СО РАН, 2000. – 327 с.

С.П. Шарый

Внутреннее оценивание множеств решений
неотрицательных интервальных линейных систем

Сибирский Журнал Вычислительной Математики.
– 2006. – Том 9, №2. – С. 189–206.

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S.P. Shary

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Doctor of Science dissertation (physics & math). Novosibirsk, Institute of computational technologies SD RAS, 2000. – 327 p. (in Russian)

S.P. Shary

Inner estimation of solution sets
to nonnegative interval linear systems

Siberian Journal of Computational Mathematics, vol. 9 (2006), No. 2, pp. 189–206. (in Russian)