From Interval Arithmetic to Constraints

via Inverses of Operations

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Addition: $X + Y = \{x + y \mid x \in X \wedge y \in Y\}$

Inverse of addition: $X - Y = \{x - y \mid x \in X \wedge y \in Y\}$
because $-$ is the inverse of $+$.

Multiplication: $X \times Y = \{x \times y \mid x \in X \wedge y \in Y\}$

Inverse of multiplication: $X/Y = \{x/y \mid x \in X \wedge y \in Y\}$
because / is the inverse of $\times$.

Plausible, but does not work: the set is not defined when $0 \in Y$.

Ratz's solution:

Multiplication:
$X \times Y = \{z \in \mathcal{R} \mid \exists x \in X, y \in Y \ . \ x \times y = z\}$

Inverse of multiplication: X/Y is least interval containing
$\{z \in \mathcal{R} \mid \exists x \in X, y \in Y \ . \ y \times z = x\}$

In general:
With $f(x,y) = z$, find g (as inverse of f)
such that $f(x, g(x,z)) = z$ for all x and z.

When f is $+$, then g is a function.
When f is $\times$, then g is not total. Cured by Ratz.

When f is max, then what?

With $f(x, y) = z$, find g as inverse of f
such that $f(x, g(x, z)) = z$ for all x and z.

Suppose f is max.
If $x = 3$ and $z = 5$, then one value for g.
If $x = 5$ and $z = 3$, then no value for $g \Rightarrow g$ is not total.
If $x = 5$ and $z = 5$, then many values for $g \Rightarrow$
g is not single-valued.

max is a more difficult case; Ratz to the rescue:

Define for all intervals X and Y
$max^{-1}(X, Y)$ as $\{z \in \mathcal{R} \mid \exists x \in X, y \in Y \, . \, max(y, z) = x\}$

We have defined $max^{-1}(X,Y)$ as

$$\{z \in \mathcal{R} \mid \exists x \in X, y \in Y \ . \ max(y,z) = x\}$$

for all intervals X and Y.

Hence (W. Older, $\sim$ 1990):

$$\text{max}^{-1}([c,d],[a,b]) = \begin{cases} \text{if } b < c\text{:} & \emptyset \\ \text{if } b \geq c \text{ and } a < d\text{:} & [-\infty, b] \\ \text{if } b \geq c \text{ and } a \geq d\text{:} & [a,b] \end{cases}$$

Ratz's trick generalizes spectacularly: from relations $r \subset \mathcal{R}^3$ to relations $r \subset T_0 \times \cdots \times T_{n-1}$ where $T_0, \ldots, T_{n-1}$ are arbitrary types with subsets $D_0, \ldots, D_{n-1}$.

The relation r has n *companion functions* of type $r_i : \mathcal{P}(T_0) \times \cdots \times \mathcal{P}(T_{i-1}) \times \mathcal{P}(T_{i+1}) \times \cdots \times \mathcal{P}(T_{n-1}) \rightarrow \mathcal{P}(T_i)$ defined as

$$\begin{aligned} & r_i(D_0, \ldots, D_{i-1}, D_{i+1}, \ldots, D_{n-1}) \\ \stackrel{def}{=} \; & \pi_i(r \cap (D_0 \times \cdots \times D_{i-1} \times T_i \times D_{i+1} \times \cdots \times D_{n-1})) \\ = \; & \{x_i \in T_i \mid \exists_{j \in (n \ominus i)} x_j \in D_j \,.\, r(x_0, \ldots, x_{n-1})\}, \end{aligned}$$

where $n \ominus i = \{0, \ldots, n-1\} \backslash \{i\}$.

Application to special case $mult \subset \mathcal{R}^3$ defined as

$$mult = \{\langle x, y, z\rangle \in \mathcal{R}^3 \mid x \times y = z\}.$$

Companion function $mult_0 : (\mathcal{P}(\mathcal{R}))^2 \to \mathcal{P}(\mathcal{R})$ is defined as

$$\langle I_1, I_2\rangle \mapsto \{x_0 \in \mathcal{R} \mid \exists x_1, x_2 \in \mathcal{R} \,.\, x_0 \times x_1 = x_2\}.$$

The least interval containing $mult_0$ is interval division according to Ratz.

So far only relations defined by functions.
Companion functions are defined for all n-ary relations.

Example from FAQs:
$[0, 1] \leq [2, 3]$? Of course yes.
$[2, 3] \leq [0, 1]$? Of course not.
$[1, 3] \leq [0, 2]$? Hmm ...

Reformulate as constraint satisfaction problem.

For intervals X and Y, $X \leq Y$ is not a fruitful question to ask. Instead, formulate as Constraint Satisfaction Problem: are there $x \in X$ and $y \in Y$ such that $x \leq y$?
Or, algebraically, is

$$\leq \cap ([X] \times [Y])$$

non-empty?

For X, Y equal to $[0, 1], [2, 3]$, yes.
For X, Y equal to $[2, 3], [0, 1]$, no.
For X, Y equal to $[1, 3], [0, 2]$, we now do have an answer: Yes.
Yes, but

$$\begin{aligned} &\leq \cap ([1, 3] \times [0, 2]) \ = \\ &\leq \cap [(1, 2] \times [1, 2]) \end{aligned}$$

Moreover, $[1, 2] \times [1, 2]$ is the least box for which this is true.

We could reduce the original box without losing any values that satisfy the relation. The constraint $\leq$ induces the *domain-reduction operation*

$$\langle X, Y\rangle \mapsto \langle X \cap r_0(Y), Y \cap r_1(X)\rangle$$

associated with the constraint relation of which r_0 and r_1 are the companion functions.

$D_i \mapsto \pi_i(r \cap (D_0 \times \cdots \times D_{i-1} \times T_i \times D_{i+1} \times \cdots \times D_{n-1}))$
for $i = 0, \ldots, n-1$ is the domain reduction operation for r in general.

Special case: $D_0, \ldots, D_{n-1}$ are intervals and
$r = \{\langle x_0, \ldots, x_{n-1}\rangle \mid x_0 = f(x_1, \ldots, x_{n-1})\}$
Then r_0 is the interval extension of f and $\varphi \circ r_1, \ldots, \varphi \circ r_{n-1}$ are the $n-1$ interval inverses, where $\varphi(X)$ is the least interval containing X.

Constraint formulation subsumes interval arithmetic. E.g.
interval arithmetic: $X^4 + X^2 - 1$ with $X = [0.5, 1]$
interval constraints: $x^4 + x^2 - 1 = y$ with $x \in [0.5, 1]$ and $y \in [-\infty, +\infty]$.

Domain reduction operation gives $y \in [-11/16, 1]$, which is the interval arithmetic value.

Interval constraints:
$x^4 + x^2 - 1$ with $x \in [0.5, 1]$ and $y \in [0, 0]$.
Domain reduction operator repeatedly reduces interval for x until it consists of adjacent floating-point numbers.

Summary:

- Ratz introduced the relational version of interval division.

- With the relational version, when an operation fails to have an inverse, its interval version does have one. Applied to division and to max.

- Ratz's relationalization generalizes to produce the n companion functions of an n-ary relation.

- The companion functions lead to the domain reduction operation associated with a relation. This operation leads to interval constraints as generalization of interval arithmetic.