T, contradictory constraints arise. Since any witness of satisfiability has to assign some value to that node, we infer that it cannot be T. Thus, we may permanently assign mark F to that node. For this DAG, such an optimization does not seem to help. No test of an unmarked node detects a shared mark or a shared contradiction. Our cubic SAT solver fails for this DAG.

1.7 Exercises

Exercises 1.1
1. Use ¬, →, ∧ and ∨ to express the following declarative sentences in propositional logic; in each case state what your respective propositional atoms p, q, etc. mean:

   *(a)* If the sun shines today, then it won’t shine tomorrow.
   *(b)* Robert was jealous of Yvonne, or he was not in a good mood.
   *(c)* If the barometer falls, then either it will rain or it will snow.
   *(d)* If a request occurs, then either it will eventually be acknowledged, or the requesting process won’t ever be able to make progress.
   *(e)* Cancer will not be cured unless its cause is determined and a new drug for cancer is found.
   *(f)* If interest rates go up, share prices go down.
   *(g)* If Smith has installed central heating, then he has sold his car or he has not paid his mortgage.
   *(h)* Today it will rain or shine, but not both.
   *(i)* If Dick met Jane yesterday, they had a cup of coffee together, or they took a walk in the park.
   *(j)* No shoes, no shirt, no service.
   *(k)* My sister wants a black and white cat.

2. The formulas of propositional logic below implicitly assume the binding priorities of the logical connectives put forward in Convention 1.3. Make sure that you fully understand those conventions by reinserting as many brackets as possible. For example, given \( p \land q \to r \), change it to \((p \land q) \to r\) since \( \land \) binds more tightly than \( \to \).

   *(a)* \(\neg p \land q \to r\)
   *(b)* \((p \to q) \land \neg (r \lor p \to q)\)
   *(c)* \((p \to q) \to (r \to s \lor t)\)
   *(d)* \(p \lor (\neg q \to p \land r)\)
   *(e)* \(p \lor q \to \neg p \land r\)
   *(f)* \(p \lor p \to \neg q\)
   *(g)* Why is the expression \( p \lor q \land r \) problematic?

Exercises 1.2
1. Prove the validity of the following sequents:

   *(a)* \((p \land q) \land r, s \land t \vdash q \land s\)
3. Prove the validity of the sequents below:

* (b) \( p \land q \vdash q \land p \)
* (c) \((p \land q) \land r \vdash p \land (q \land r) \)
* (d) \( p \rightarrow (p \rightarrow q), p \vdash q \)
* (e) \( q \rightarrow (p \rightarrow r), \neg r, q \vdash \neg p \)
* (f) \( \vdash (p \land q) \rightarrow p \)
* (g) \( p \vdash q \rightarrow (p \land q) \)
* (h) \( p \vdash (p \rightarrow q) \rightarrow q \)
* (i) \( (p \rightarrow r) \land (q \rightarrow r) \vdash p \land q \rightarrow r \)
* (j) \( q \rightarrow r \vdash (p \rightarrow q) \rightarrow (p \rightarrow r) \)
* (k) \( p \rightarrow (q \rightarrow r), p \rightarrow q \vdash p \rightarrow r \)
* (l) \( p \rightarrow q, r \rightarrow s \vdash p \lor r \rightarrow q \lor s \)
* (m) \( p \lor q \vdash r \rightarrow (p \lor q) \land r \)
* (n) \( (p \lor (q \rightarrow p)) \land q \vdash p \)
* (o) \( p \rightarrow q, r \rightarrow s \vdash p \land r \rightarrow q \land s \)
* (p) \( p \rightarrow q \vdash ((p \land q) \rightarrow p) \land (p \rightarrow (p \land q)) \)
* (q) \( q \vdash (p \rightarrow (p \rightarrow (q \rightarrow p))) \)
* (r) \( p \rightarrow q \land r \vdash (p \rightarrow q) \land (p \rightarrow r) \)
* (s) \( (p \rightarrow q) \land (p \rightarrow r) \vdash p \rightarrow q \land r \)
* (t) \( \vdash (p \rightarrow q) \rightarrow ((r \rightarrow s) \rightarrow (p \land r \rightarrow q \land s)); \) here you might be able to ‘recycle’ and augment a proof from a previous exercise.

* (u) \( p \rightarrow q \vdash \neg q \rightarrow \neg p \)
* (v) \( p \lor (p \land q) \vdash p \)
* (w) \( r, p \rightarrow (r \rightarrow q) \vdash p \rightarrow (q \land r) \)
* (x) \( p \rightarrow (q \land r), q \rightarrow s, r \rightarrow s \vdash p \rightarrow s \)
* (y) \( (p \land q) \lor (p \land r) \vdash p \lor (q \lor r) \).

2. For the sequents below, show which ones are valid and which ones aren’t:

* (a) \( \neg p \rightarrow \neg q \rightarrow q \rightarrow p \)
* (b) \( \neg p \lor \neg q \vdash \neg (p \land q) \)
* (c) \( \neg p \lor (p \land q) \vdash q \)
* (d) \( p \lor q, \neg q \lor r \vdash p \lor r \)
* (e) \( p \rightarrow (q \lor r), \neg q, \neg r \vdash \neg p \) without using the MT rule
* (f) \( \neg p \land \neg q \vdash \neg (p \lor q) \)
* (g) \( p \land \neg p \vdash \neg (r \rightarrow q) \land (r \rightarrow q) \)
* (h) \( p \rightarrow q, s \rightarrow t \vdash p \lor s \rightarrow q \land t \)
* (i) \( \neg (\neg q \lor q) \vdash p \).

3. Prove the validity of the sequents below:

* (a) \( \neg p \rightarrow p \vdash p \)
* (b) \( \neg p \vdash p \rightarrow q \)
* (c) \( p \lor q, \neg q \vdash p \)
* (d) \( \vdash \neg p \rightarrow (p \rightarrow (p \rightarrow q)) \)
* (e) \( \neg (p \rightarrow q) \vdash q \rightarrow p \)
* (f) \( p \rightarrow q \vdash \neg p \lor q \)
* (g) \( \vdash \neg p \lor q \rightarrow (p \rightarrow q) \)
(h) \( p \rightarrow (q \lor r), \neg q, \neg r \vdash \neg p \)

(i) \((c \land n) \rightarrow t, h \land \neg s, h \land (s \lor c) \rightarrow p \vdash (n \land \neg t) \rightarrow p \)

(j) the two sequents implicit in (1.2) on page 20.

(k) \( q \vdash (p \land q) \lor (p \land q) \) using LEM

(l) \((c \land n) \rightarrow (c \land n) \land (s \land t) \rightarrow p \land q \land \neg s \land t \rightarrow \neg p \)

(m) \( p \land q \rightarrow (p \rightarrow \neg q) \land \neg q \rightarrow \neg r \land \neg q \rightarrow \neg r \)

(n) * \( p \land q \vdash \neg (\neg p \lor \neg q) \)

(o) \( \neg (\neg p \lor \neg q) \vdash p \land q \)

(p) \( p \rightarrow q \vdash p \lor (p \land q) \) possibly without using LEM?

(q) * \( (p \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow q) \)

(b) Given a proof for the sequent of the previous item, do you now have a quick argument for \((p \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow q)\)?

(c) \(((p \rightarrow q) \land (q \rightarrow p)) \rightarrow ((p \lor q) \rightarrow (p \land q)) \)

(d) \((p \rightarrow q) \rightarrow ((p \rightarrow q) \rightarrow q) \).

4. Explain why intuitionistic logicians also reject the proof rule PBC.

5. Prove the following theorems of propositional logic:

(a) * \((p \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p) \)

(b) Given a proof for the sequent of the previous item, do you now have a quick argument for \(((q \rightarrow p) \rightarrow p) \rightarrow ((p \rightarrow q) \rightarrow q)\)?

(c) \(((p \rightarrow q) \land (q \rightarrow p)) \rightarrow ((p \lor q) \rightarrow (p \land q)) \)

(d) \((p \rightarrow q) \rightarrow ((p \rightarrow q) \rightarrow q) \).

6. Natural deduction is not the only possible formal framework for proofs in propositional logic. As an abbreviation, we write \(\Gamma\) to denote any finite sequence of formulas \(\phi_1, \phi_2, \ldots, \phi_n\) \((n \geq 0)\). Thus, any sequent may be written as \(\Gamma \vdash \psi\) for an appropriate, possibly empty, \(\Gamma\). In this exercise we propose a different notion of proof, which states rules for transforming valid sequents into valid sequents. For example, if we have already a proof for the sequent \(\Gamma, \phi \vdash \psi\), then we obtain a proof of the sequent \(\Gamma \vdash \phi \rightarrow \psi\) by augmenting this very proof with one application of the rule \(-i\). The new approach expresses this as an inference rule between sequents:

\[
\begin{array}{c}
\Gamma, \phi \vdash \psi \\
\hline 
\Gamma \vdash \phi \rightarrow \psi
\end{array}
\]

The rule ‘assumption’ is written as

\[
\phi \vdash \phi \quad \text{assumption}
\]

i.e. the premise is empty. Such rules are called axioms.

(a) Express all remaining proof rules of Figure 1.2 in such a form. (Hint: some of your rules may have more than one premise.)

(b) Explain why proofs of \(\Gamma \vdash \psi\) in this new system have a tree-like structure with \(\Gamma \vdash \psi\) as root.

(c) Prove \(p \lor (p \land q) \vdash p\) in your new proof system.