Pumping Lemma example

**Pumping Lemma**
If $A$ is a regular language, then there is a number $p$ (the pumping length) where if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

A *non regular language*
Let $C = \{w \mid w$ has an equal number of 0s and 1s$\}$.

**Theorem.**
$C$ is not regular.

*Proof using the pumping lemma.*
Leading to a contradiction, assume $C$ is regular. Then, the pumping lemma applies to $C$. Let $p$ be the pumping length that applies to $C$ from the pumping lemma. Let $s = 0^p1^p$. Since $s$ has length at least $p$ and is in $C$, the pumping lemma applies to $s$, so $s$ can be divided into three strings $s = xyz$ where the conditions 1, 2 and 3 of the pumping lemma hold for $x$, $y$ and $z$. Since conditions 2 and 3 hold, we know $x$ and $y$ contain only 0s and $y$ contains at least one zero. Then $xyyz$ is not in $C$ since it contains more 0s than 1s, contradicting condition 1 of the pumping lemma for $i = 2$.

*Proof using closure properties.*
Leading to a contradiction, assume $C$ is regular. Since regular languages are closed under intersection, then $C \cap 0^*1^*$ is also regular. But $C \cap 0^*1^* = \{0^n1^n \mid n \geq 0\}$, which we already proved is not regular.