

Convex Envelopes for Bounded Multilinear Functions

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A generic global optimization problem

$$\begin{aligned} \min f(x) \\ \text{s.t. } g_i(x) \leq 0, \quad i = 1, \dots, m \\ l \leq x \leq u \\ x_j \in \mathbb{Z}, \quad j \in J \end{aligned}$$

- Functions f and g_i may be nonconvex
- J may be empty
- We seek a global optimal solution

Spatial branch-and-bound

General solution approach

- Construct a convex relaxation
 - Yields computable lower bound on optimal value
 - Requires assumptions on functions
- Improve the relaxation by **spatial branching**
 - Either $x_j \leq m$ or $x_j \geq m$
 - Update convex relaxation
 - Eliminate regions from consideration using the lower bound
- Key to success: **good lower bounds**

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Approach for obtaining good lower bounds

General approach

Study particular problem structure – forms for $f(x)$ and $g_i(x)$ – to find “tight” convex relaxations

Multilinear functional form

$$\phi(x) = \prod_{i \in I} x_i$$

Appears in many applications

- Chemical blending processes
- Electricity transmission
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Agenda

- 1 Multilinear terms and related sets
- 2 $n = 2$
- 3 Bounded Multilinear with $n > 2$
- 4 Unbounded Multilinear with $n > 2$

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$$w_j = f_j(\mathbf{x})$$

Multilinear set

Consider the set:

 M_n

$$M_n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = \prod_{i=1}^n x_i, l_i \leq x_i \leq u_i, i = 1, 2, \dots, n+1\}$$

Where $0 < l_i < u_i, i = 1, 2, \dots, n+1$.

Note:

- M_n is nonconvex.
- We may assume $\bar{l}_{n+1} \leq l_{n+1} < u_{n+1} \leq \bar{u}_{n+1}$ where $\bar{l}_{n+1} = \prod_{i=1}^n l_i$ and $\bar{u}_{n+1} = \prod_{i=1}^n u_i$.

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$$x_3 = x_1 x_2 \quad (\text{no bounds on } x_3)$$

Let's consider the case $n = 2$ and there is **no explicit bounds** on x_3 . The following linear relaxation gives the convex envelope of M_2 :

[McCormick83] and [Al-Khayyal and Falk83]

$$x_3 \geq l_2 x_1 + l_1 x_2 - l_1 l_2$$

$$x_3 \geq u_2 x_1 + u_1 x_2 - u_1 u_2$$

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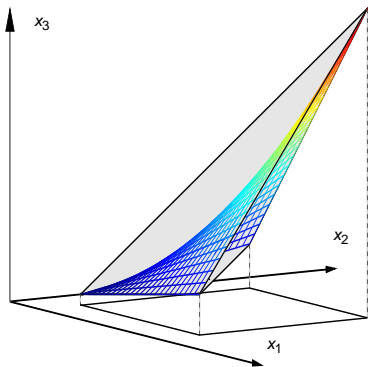
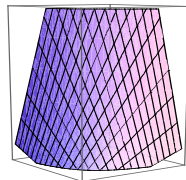
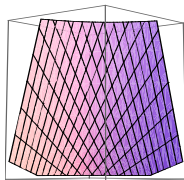
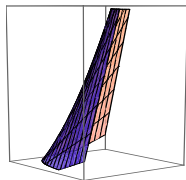


Figure: McCormick envelope of M_2 with no bound on x_3 .

$$x_3 = x_1 x_2 \quad (\text{non-trivial bounds on } x_3)$$

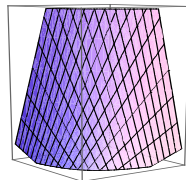
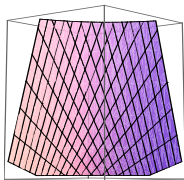
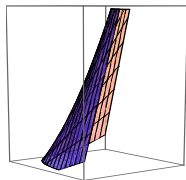
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We are interested in finding the convex hull of this set.

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$x_3 = x_1 x_2$ (non-trivial bounds on x_3)

Assumptions:

- $l_3 \leq l_1 u_2$; otherwise $l_3/u_2 > l_1$ would be a valid lower bound on x_1
- $u_3 \geq l_1 u_2$; otherwise $u_3/l_1 < u_2$ would be a valid upper bound on x_2
- $l_3 \leq u_1 l_2; \dots$
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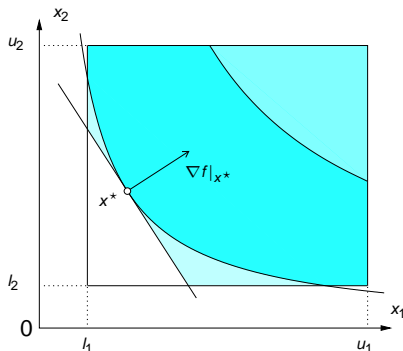
As a result, the projection of M_2 onto x_1 and x_2 space will look like this:

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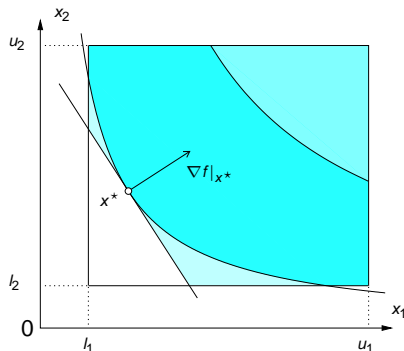


Note

The tangent to the curve $\pi : x_1 x_2 = l_3$ at $x^* = (x_1^*, x_2^*)$ gives the linear inequality:

$$a_1(x_1 - x_1^*) + a_2(x_2 - x_2^*) \geq 0; \quad a_1 = \frac{\partial \pi}{\partial x_1} \Big|_{x^*} = x_2^* \quad \text{and} \quad a_2 = \frac{\partial \pi}{\partial x_2} \Big|_{x^*} = x_1^*$$

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Lifting $x_2^*(x_1 - x_1^*) + x_1^*(x_2 - x_2^*) \geq 0$

Fact

The lifted inequality

$$x_2^*(x_1 - x_1^*) + x_1^*(x_2 - x_2^*) + a(x_3 - l_3) \geq 0$$

will be valid and tight for M_2 for an appropriate value for a .

- For all $a \geq 0$ the above inequality is valid for M_2 .
- For small enough $a < 0$ the above inequality is valid for M_2 .
- To strengthen the inequality as much as possible, the goal is to find a value of a that is as small as possible for which the inequality remains valid.

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Intuition

For the above inequality to be tight, the plane

$$x_2^*(x_1 - x_1^*) + x_1^*(x_2 - x_2^*) + a(x_3 - l_3) = 0$$

should touch the curve $x_1 x_2 = u_3$ at exactly **one** point.

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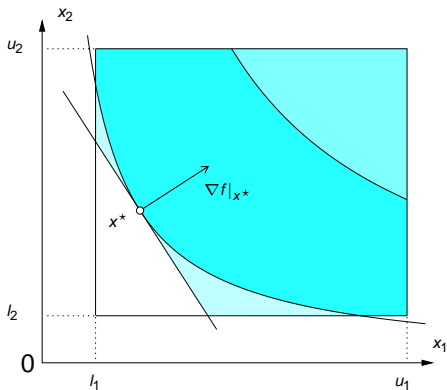
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Special cases

If the plane $x_2^*(x_1 - x_1^*) + x_1^*(x_2 - x_2^*) + a(x_3 - l_3) = 0$ touches the curve $x_1 x_2 = u_3$ at one point, say $y^* = (y_1^*, y_2^*)$, and is not tangent to the curve, considering its corresponding inequality is valid for M_2 , then the rplane touches the curve at one of its endpoints.

Doing some algebra

Case 1: If the plane is tangent to the curve $a = \frac{2(1 - \sqrt{\frac{u_3}{l_3}})l_3}{u_3 - l_3}$.

Case 2: If the plane touches the curve at $(\frac{u_3}{u_2}, u_2)$ then $a = \frac{x_2^*(\frac{u_3}{u_2} - x_1^*) + x_1^*(u_2 - x_2^*)}{l_3 - u_3}$.

Case 3: If the plane touches the curve at $(u_1, \frac{u_3}{u_1})$ then $a = \frac{x_2^*(u_1 - x_1^*) + x_1^*(\frac{u_3}{u_1} - x_2^*)}{l_3 - u_3}$.

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Lemma 1

If $\alpha > 1$ and $\mathbf{x}_1^* \mathbf{x}_2^* = l_3$ then the function $x_3 = x_1 x_2$ along the line segment connecting $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ and $(\alpha \mathbf{x}_1^*, \alpha \mathbf{x}_2^*)$ is strongly convex.

Proof idea

It's easy to show that for any point $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ on the line segment connecting $(\mathbf{x}_1^*, \mathbf{x}_2^*, l_3)$ and $(\alpha \mathbf{x}_1^*, \alpha \mathbf{x}_2^*, u_3)$, excluding the endpoints, where $\mathbf{x}_1^* \mathbf{x}_2^* = l_3$ and $u_3 = \alpha^2 l_3$, we have $\tilde{x}_3 > \tilde{x}_1 \tilde{x}_2$.

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If $\frac{u_3}{\alpha u_2} \leq x_1^* \leq \frac{u_1}{\alpha}$, $\frac{u_3}{\alpha u_1} \leq x_2^* \leq \frac{u_2}{\alpha}$, $x_1^* x_2^* = l_3$, and the plane

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touches M_2 at just two points and this inequality is valid for M_2 .

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Proof by contradiction and using Lemma 1.

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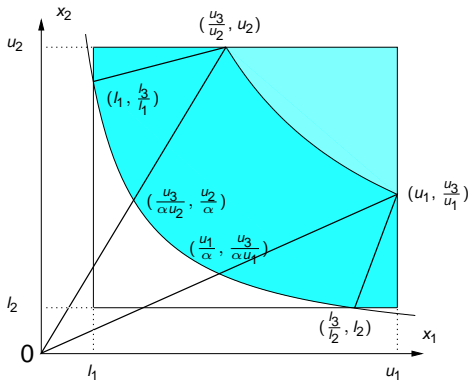
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is not dominated by any valid inequality for M_2 .

Proof idea

Proof by contradiction.

Case 1: $\frac{u_3}{\alpha u_2} \leq \mathbf{x}_1^* \leq \frac{u_1}{\alpha}$, $\frac{u_3}{\alpha u_1} \leq \mathbf{x}_2^* \leq \frac{u_2}{\alpha}$, $\mathbf{x}_1^* \mathbf{x}_2^* = l_3$



Case 2: $l_2 < x_2^* \leq \frac{\sqrt{l_3 u_3}}{u_1}$

Lemma 4

If $l_2 < x_2^* \leq \frac{\sqrt{l_3 u_3}}{u_1}$ and $x_1^* x_2^* = l_3$, then the function $x_3 = x_1 x_2$ along the line segment connecting (x_1^*, x_2^*) and $(u_1, \frac{u_3}{u_1})$ is strongly convex.

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If $l_2 < x_2^* \leq \frac{\sqrt{l_3 u_3}}{u_1}$, $x_1^* x_2^* = l_3$, and the plane

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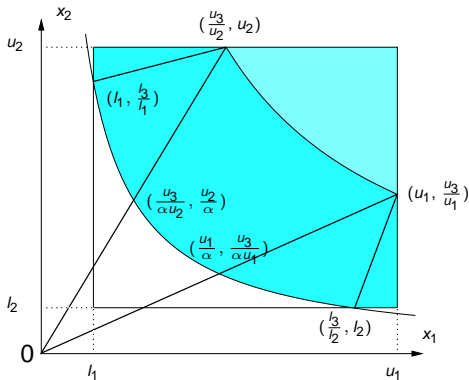
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Validity of McCormick Inequalities

Proposition 10

McCormick inequalities are valid for M_2 and are not dominated by any other inequalities.

$$x_3 \geq u_2 x_1 + u_1 x_2 - u_1 u_2$$

$$x_3 \leq l_2 x_1 + u_1 x_2 - l_1 u_2$$

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Convex Hull of M_2

Theorem

The inequalities

$$x_2^*(x_1 - x_1^*) + x_1^*(x_2 - x_2^*) + a(x_3 - l_3) \geq 0$$

where

$$a = \begin{cases} \frac{2(1 - \sqrt{\frac{u_3}{l_3}})l_3}{u_3 - l_3} & \text{if } \frac{u_3}{\alpha u_2} \leq x_1^* \leq \frac{u_1}{\alpha}, \frac{u_3}{\alpha u_1} \leq x_2^* \leq \frac{u_2}{\alpha} \\ \frac{x_2^*(\frac{u_3}{u_2} - x_1^*) + x_1^*(u_2 - x_2^*)}{l_3 - u_3} & \text{if } l_2 < x_2^* \leq \frac{\sqrt{l_3 u_3}}{u_1} \\ \frac{x_2^*(u_1 - x_1^*) + x_1^*(\frac{u_3}{u_1} - x_2^*)}{l_3 - u_3} & \text{if } l_1 < x_1^* \leq \frac{\sqrt{l_3 u_3}}{u_2} \end{cases}$$

along with $x_3 \geq l_3$ and $x_3 \leq u_3$ and the McCormick inequalities

$$x_3 \geq u_2 x_1 + u_1 x_2 - u_1 u_2$$

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give the convex hull of M_2

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Idea: the approach we took for $n = 2$ can be extended to $n > 2$.

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Extension to $n > 2$

Consider the convex set $M' = \{x \in \mathbb{R}_+^n : \prod_{i=1}^n x_i \geq l_{n+1}\}$.

Given a point x^* on the boundary of M' , i.e. such that $\prod_{i=1}^n x_i^* = l_{n+1}$, a valid inequality is the supporting hyperplane to M' :

$$\sum_{i=1}^n a_i (x_i - x_i^*) \geq 0,$$

where again the coefficient a_i is the i -th component of the gradient of the function $\xi_n(x) = \prod_{i=1}^n x_i$:

$$a_i = \frac{\partial \xi_n}{\partial x_i} = \prod_{j \neq i} x_j^*.$$

We need to lift this inequality to get

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Convex hull approximation

Convex hull approximation

It might be very expensive or inefficient to compute the convex hull of M_n for general n ; but there might be reasonable ways to approximate the this convex hull.

Convex Hull Approximation Idea

- Define

$$L_n = \left\{ (x_1, x_2, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \begin{array}{l} x_{n+1} = \prod_{i=1}^n x_i; \\ \ell_i \leq x_i \leq u_i, i = 1, \dots, n; \\ \ell_{n+1} \leq x_{n+1} \leq \prod_{i=1}^n u_i \end{array} \right\}.$$

Find $\text{conv}(L_n)$.

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How can we use these concepts?

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- Try to get error bounds on how closely $\text{conv}(L_n) \cap \text{conv}(U_n)$ approximates $\text{conv}(M_n)$;
- Try to describe special cases in which we know $\text{conv}(M_n)$. For example
 - We think that if $\ell_j \prod_{i \in \{1, \dots, n\} \setminus j} u_i \leq u_{n+1}, \forall j$, then we can easily enumerate the lifting possibilities for lifting a_{n+1} and thereby compute the convex hull.
 - If $u_j \prod_{i \in \{1, \dots, n\} \setminus j} \ell_i \geq \ell_{n+1}, \forall j$, something analogous is likely true, except that we would be lifting into tangents of the feasible region at $x_{n+1} = u_{n+1}$.

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Unbounded Multilinear Sets

 M_n^*

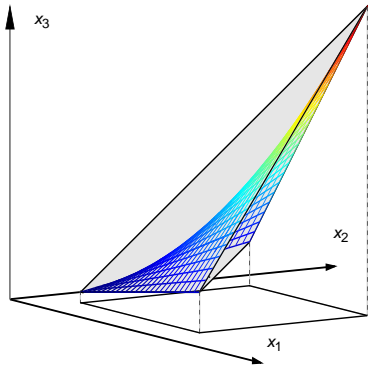
$$M_n^* = \{x \in \mathbb{R}^{n+1} : x_{n+1} = \prod_{i=1}^n x_i, l_i \leq x_i \leq u_i, i = 1, 2, \dots, n\}$$

Rikun 97

$\text{conv}(M_n^*) = \text{conv}(S)$, where

$$S = \{x \in \mathbb{R}^{n+1} : x_i \in \{l_i, u_i\}, i = 1, 2, \dots, n; x_{n+1} = \prod_{i=1}^n x_i\}$$

$$x_3 = x_1 x_2 \quad (\text{no bounds on } x_3)$$



Separation

Given a point $\hat{\mathbf{x}} \in \mathbb{R}^{n+1} : \ell_i \leq \hat{x}_i \leq u_i, i = 1, \dots, n$, it is easy to see that either $\hat{\mathbf{x}} \in \text{conv}(M_n^*)$, or else $s > 0$ in every optimal solution to

$$\begin{aligned} \min \quad & s \\ \text{s.t.} \quad & \sum_{v \in V_n} \bar{x}_i^v \lambda_v + s \geq \hat{x}_i, i = 1, \dots, n+1 \\ & \sum_{v \in V_n} \bar{x}_i^v \lambda_v - s \leq \hat{x}_i, i = 1, \dots, n+1 \\ & \sum_{v \in V_n} \lambda_v = 1 \\ & \lambda_v \geq 0, v \in V_n, \end{aligned}$$

where V_n is the index set of extreme points of $\text{conv}(M_n^*)$.

Valid Inequalities

If $s > 0$, the optimal solution of the dual linear program

$$\begin{aligned} \max \quad & \sum_{i=1}^{n+1} \hat{x}_i \pi_i - \sum_{i=1}^{n+1} \hat{x}_i \rho_i - \eta \\ \text{s.t.} \quad & \sum_{i=1}^{n+1} \bar{x}_i^v \pi_i - \sum_{i=1}^{n+1} x_i^v \rho_i - \eta \leq 0, \forall v \in V_n \\ & \sum_{i=1}^{n+1} \pi_i + \sum_{i=1}^{n+1} \rho_i = 1 \\ & \pi, \rho \geq 0 \end{aligned}$$

yields a violated inequality that cuts \hat{x} off from M_n^*

$$\sum_{i=1}^{n+1} (\pi'_i - \rho'_i) x_i \leq \eta'$$

where (π', ρ', η') is the optimal dual solution in question.

An interesting fact!

Proposition 11

Let (λ', s) be an optimal primal basic solution and (π', ρ', η') be an optimal dual solution. Let $V^+ = \{v \in V_n : \lambda'_v > 0\}$.

- 1 $|V^+| \leq n + 1$;
- 2 For all $x^v : v \in V^+$, x^v lies in the hyperplane defined by the inequality (π', ρ', η')
- 3 If and only if $|V^+| = n + 1$, the inequality (π', ρ', η') defines a facet of $\text{conv}(M_n^*)$.

Proof idea

Can be shown from basic complementary slackness. Number 3 follows from the first two, and from the definition of a facet.

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Important

- In the absence of dual degeneracy, the optimal dual solution not only defines a valid inequality, it defines a facet.
- Here no dual degeneracy means $n + 1$ lambda variables have non-zero values in the primal optimal solution.

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Complications

- Enumerating all the corner points is very easy ($\{l_i, u_i\}$).
- Exponential number of corner points of the convex hull (2^n).
- We are solving LP's
 - We might be still able to handle the exponential number of corner points for small n .
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