Facets of multiple *all_different* predicates

Serge Kruk
(joint work with Susan Toma and Mark Wallace)

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Abstract

We explore the properties of the comb structure of multiple `alldifferent` predicates by presenting two classes of facet-inducing inequalities of the convex hull of integer solutions. We present experimental results based on these inequalities in a hybrid Constraint Programming - Integer Programming solver.

Outline

- Timetabling Example
- The `alldifferent` Constraint
- Mathematical Formulation
- New Facets
- Separation Algorithm
- Experimental results

A timetabling problem

- Students choose a number of courses
- Each course may have many sections
- Instructors teach a number of courses
- Each section meets sometime
- Each section meets in some room

Assign students, instructors, room to meeting times

NP-hard if . . .

- . . . only one student
- . . . only only one section per course
- . . . only one instructor per course
- . . . only one meeting per section
- . . . only one room per section

And multiple combinations of the above

In practice, even worse

- Must satisfy student requests
- Students are busy 6 periods per day
- Students must have lunch period
- Some course must be paired
- Some courses must avoid certain periods
- Meetings must be spread out during week
- Instructors have limits on meetings per day
- Instructors have limits on adjacent meetings

Of course, there is no feasible solution!

Elements of a Model

Data

- John: MTH105, MTH204, MTH218
- Mary: MTH105, MTH206, MTH210, MTH219
- Prof. Cheng: MTH206, MTH218

Model

- x_i contains timeslots of courses of i
- alldifferent$\{x_{John}\}$
- alldifferent$\{x_{Mary}\}$
- alldifferent$\{y_{Cheng}\}$
- Moreover, meeting assignments must be coherent

alldifferent is the workhorse of such a model

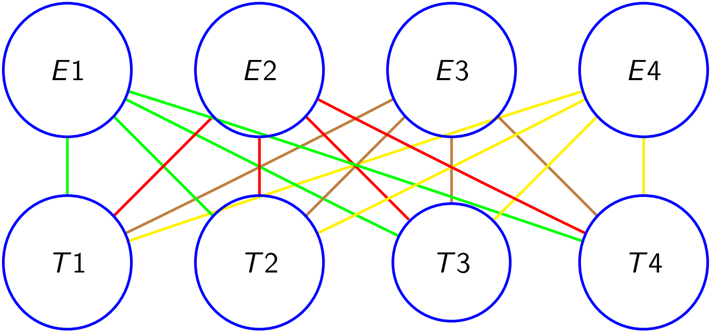
alldifferent Constraint

Definition

The alldifferent constraint on the collection of variables $x_1, x_2, \ldots, x_n$ with respective finite domains $D_1, D_2, \ldots, D_n$ is defined as:

$$\texttt{alldifferent}\{x_1, \ldots, x_n\}$$
$$=$$
$$\{(d_1, \ldots, d_n) | d_i \in D_i, i \neq j \Rightarrow d_i \neq d_j\}$$

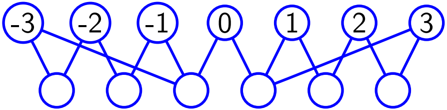
Graph of *one* alldifferent



Algorithms for one alldifferent

- Bounds consistency
- Range consistency
- Hyper-arc consistency

HAC of alldifferent is not enough



$\{\{0,1\},\{1,2\},\{2,3\},\{0,3\},\{0,-1\},\{-1,-2\},\{-2,-3\},\{0,-3\}\}$

Simultaneous Matching problem

Comb structures

Let us restrict ourselves to combs

CP Formulation of Comb Structure

- $D = \{0, 1, ..., k\} \subset \mathbb{Z}$
- $x_i, x_i^j \in D$

$\texttt{alldifferent}\{x_1, x_2, ..., x_n\}$	Handle
$\texttt{alldifferent}\{x_1, x_1^j : j \in T_1\}$	Tooth 1
$\texttt{alldifferent}\{x_2, x_2^j : j \in T_2\}$	Tooth 2
$\vdots$	$\vdots$
$\texttt{alldifferent}\{x_n, x_n^j : j \in T_n\}$	Tooth n

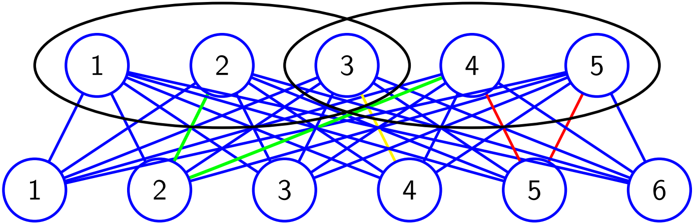
IP Formulation of Comb Structure

- P_y is polytope of the simultaneous matchings of the comb structure of multiple `alldifferent`:

$$
\begin{array}{c}
x_i = \sum_r r y_{ir} \quad \forall i \in H \\
x_i^j = \sum_r r y_{ir}^j \quad \forall j \in T_i \\
\sum_r y_{ir} = 1 \quad \forall i \in H \\
\sum_i y_{ir} \leq 1 \quad \forall r \in D \\
\sum_r y_{ir}^j = 1 \quad \forall j \in T_i \\
\sum_i y_{ir}^j \leq 1 \quad \forall r \in D \\
y_{ir}, y_{ir}^j \in \{0,1\} \quad \forall i \in H, \quad \forall j \in T_i, \quad \forall r \in D
\end{array}
$$

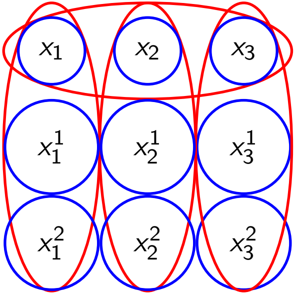
- The integral polytope P_I is the projection of P_y onto the x-space

As a graph



{1,2,3},{3,4,5}

Comb Structure of Timetabling Example

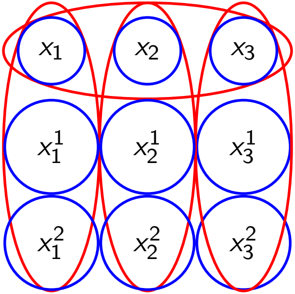


alldifferent$\{x_1, x_2, x_3\}$	Handle
alldifferent$\{x_1, x_1^1, x_1^2\}$	Tooth 1
alldifferent$\{x_2, x_2^1, x_2^2\}$	Tooth 2
alldifferent$\{x_3, x_3^1, x_3^2\}$	Tooth 3

New predicate?

alldifferent$\{\{x_1, x_2, x_3\},\{x_1, x_1^1, x_1^2\},\{x_2, x_2^1, x_2^2\},\{x_3, x_3^1, x_3^2\}\}$

Timetabling Example Solutions



Some solutions include:

$$
\begin{aligned}
&(x_1, \; x_2, \; x_3, \quad x_1^1, \; x_1^2, \quad x_2^1, \; x_2^2, \quad x_3^1, \; x_3^2) \\
=&(1, \; 2, \; 3, \quad 4, \; 5, \quad 4, \; 5, \quad 4, \; 5)
\end{aligned}
$$

and

$$
\begin{aligned}
&(x_1, \; x_2, \; x_3, \quad x_1^1, \; x_1^2, \quad x_2^1, \; x_2^2, \quad x_3^1, \; x_3^2) \\
=&(2, \; 5, \; 3, \quad 0, \; 1, \quad 0, \; 4, \quad 0, \; 1)
\end{aligned}
$$

Extended Work

Our facet-inducing inequalities for the multiple `alldifferent` comb structure extend the work done by previous researchers including:

- H.P. Williams and H. Yan who provided facets for the 1-`alldifferent` constraint in their paper titled "Representations of the `alldifferent` Predicate of Constraint Satisfaction in Integer Programming"
- G. Appa, D. Magos, and I. Mourtos who provided facets for the 2-`alldifferent` constraint in their paper titled "On the system of two `alldifferent` predicates"

New Facets of P_I

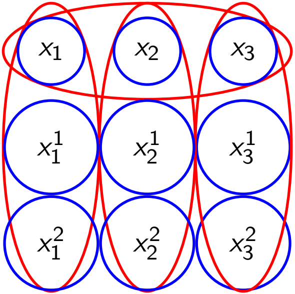
The following are facet-inducing inequalities for the comb structure whenever we exceed the minimum number of domain values.

$$\sum_{i \in S}(x_i + |S| \sum_{j \in S_i} x_i^j) \geq \sum_{m=|S_i|}^{|S_i|+|S|-1} m + |S|^2 \sum_{m=0}^{|S_i|-1} m,$$
$$\forall S \subseteq H : |S| > 1, \forall S_i \subseteq T_i : i \in S, |S_i| = s \qquad (1)$$

$$\sum_{i \in S}(x_i + |S| \sum_{j \in S_i} x_i^j) \leq \sum_{m=k-|S_i|-|S|+1}^{k-|S_i|} m + |S|^2 \sum_{m=k-|S_i|+1}^{k} m$$
$$\forall S \subseteq H : |S| > 1, \forall S_i \subseteq T_i : i \in S, |S_i| = s \qquad (2)$$

Facets of Timetabling Example

- D$=\{0, 1, \ldots, 5\}$
- $\text{LB} \leq \sum_{i\in S}(x_i + |S| \sum_{j\in S_i} x_i^j) \leq \text{UB}$

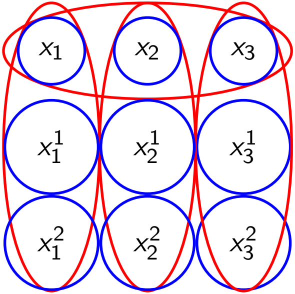


- $S = \{x_1, x_3\}, S_1 = \{x_1^1\}, S_3 = \{x_3^2\}$
  - $3 \leq x_1 + x_3 + 2x_1^1 + 2x_3^2 \leq 27$
- $S = \{x_1, x_3\}, S_1 = \{x_1^1, x_1^2\}, S_3 = \{x_3^1, x_3^2\}$
  - $9 \leq x_1 + x_3 + 2x_1^1 + 2x_1^2 + 2x_3^1 + 2x_3^2 \leq 41$

Outline of Proof of Facets of P_I

To show that inequalities (1) and (2) are facet-inducing, we prove two things:

- They are valid inequalities
  - Assign domain values appropriately to show that the lower and upper bounds are satisfied.



$$9 \leq x_1 + x_3 + 2x_1^1 + 2x_1^2 + 2x_3^1 + 2x_3^2 \leq 41$$

Outline of Proof of Facets of P_I

- Next, we show that any other valid inequalities can be obtained by taking nonnegative linear combinations of (1) and (2).
  - We define the set F_1 where:

$$F_1 = \{x \in P_I : \sum_{i \in S}(x_i + |S| \sum_{j \in S_i} x_i^j) = \sum_{m=|S_i|}^{|S_i|+|S|-1} m + |S|^2 \sum_{m=0}^{|S_i|-1} m\}$$

  - Let A denote the set indexing all the variables in the system.

New Facets Proof

Theorem

For $a \in \Re^{|A|}, a_0 \in \Re$, *if* $ay = a_0$ *holds* $\forall y \in F_1$, *there exists a scalar* π *such that:*

$$a_i = \begin{cases} \pi, & i \in S \\ 0, & i \in H - S \end{cases} \tag{3}$$

$$a_i^j = \begin{cases} |S|\pi, & j \in S_i \\ 0, & j \in T_i - S_i \end{cases} \tag{4}$$

and

$$a_0 = \pi[\sum_{m=|S_i|}^{|S_i|+|S|-1} m + |S|^2 \sum_{m=0}^{|S_i|-1} m]. \tag{5}$$

Separation Algorithm

$\pi = \text{sort}(\bar{x}_1, \dots, \bar{x}_n)$, $\pi_i = \text{sort}(\bar{x}_i^1, \dots, \bar{x}_i^{s_i})$

for $i = 2 \dots n$ **do**

$L = \sum_{l=n-i+1}^{n} \bar{x}(\pi(l))$; $U = \sum_{l=1}^{i} \bar{x}(\pi(l))$

for $j = 1 \dots \min\{s_1, \dots, s_n\}$ **do**

for $m = 1, \dots, i$ **do**

$L = L + i * \bar{x}_{\pi(n-m+1)}^{\pi_{\pi(n-m+1)}(s_{\pi(n-m+1)}-j+1)}$; $U = U + i * \bar{x}_{\pi(m)}^{\pi_{\pi(m)}(j)}$

end for

if $L < \sum_{m=j}^{j+i-1} m + i^2 \sum_{m=0}^{j-1} m =: L_{ij}$ **then**

$\sum_{l=n-i+1}^{n} x(\pi(l)) + i \sum_{m=1}^{i} x_{\pi(n-m+1)}^{\pi_{\pi(n-m+1)}(s_{\pi(n-m+1)}-j+1)} \geq L_{ij}$

end if

if $U > \sum_{m=k-j-i+1}^{k-j} m + i^2 \sum_{m=k-j+1}^{k} m =: U_{ij}$ **then**

$\sum_{l=1}^{i} x(\pi(l)) + i \sum_{m=1}^{i} x_{\pi(m)}^{\pi_{\pi(m)}(j)} \leq U_{ij}$

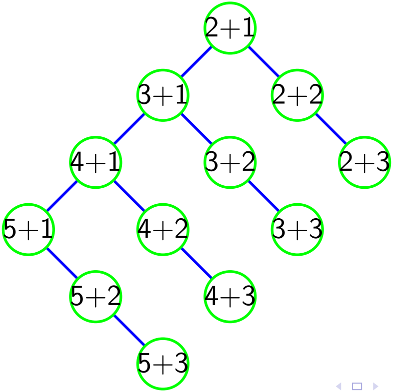
end if

end for

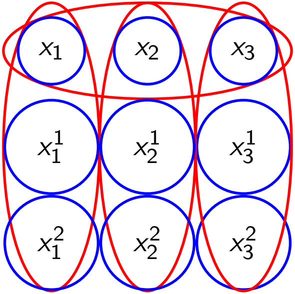
end for

Separation Algorithm

- Add variables from the handle one by one
- Add corresponding teeth variable one by one
- Compare value to bound
- Complexity: $|H| * max_i |T_i|$
- Our algorithm implicitly constructs the tree:



Applying our Separation Algorithm to Timetabling Example



- Optimizing $x_1 + x_2 + x_3 + 3x_1^2 + 3x_2^2 + 3x_3^1 + 3x_1^1 + 3x_2^1 + 3x_3^2$ over the `alldifferent` facet-inducing inequalities known up to now gives an objective function value of 88.5 and the point:

$$\begin{aligned} &(x_1, \quad x_2, \quad x_3, \qquad x_1^1, \quad x_1^2, \qquad x_2^1, \quad x_2^2, \qquad x_3^1, \quad x_3^2) \\ =&(2.5, \quad 2.5, \quad 2.5, \qquad 4, \quad 5, \qquad 4, \quad 5, \qquad 5, \quad 4) \end{aligned}$$

- Not a solution

How the Separation Algorithm Works

Obtained point:

$$\begin{array}{l} (x_1, \; x_2, \; x_3, \quad x_1^1, \; x_1^2, \quad x_2^1, \; x_2^2, \quad x_3^1, \; x_3^2) \\ =(2.5, \; 2.5, \; 2.5, \quad 4, \; 5, \quad 4, \; 5, \quad 5, \; 4) \end{array}$$

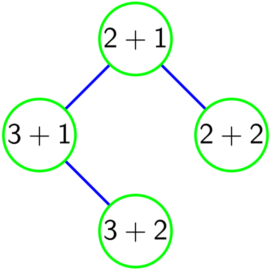
- Sort the variables in descending order within (Handle, Tooth 1, Tooth 2, Tooth 3)

$$\begin{array}{l} (x_1, \; x_2, \; x_3, \quad x_1^2, \; x_1^1, \quad x_2^2, \; x_2^1, \quad x_3^1, \; x_3^2) \\ =(2.5, \; 2.5, \; 2.5, \quad 5, \; 4, \quad 5, \; 4, \quad 5, \; 4) \end{array}$$

How the Separation Algorithm Works

- Check for violated inequalities in this order:
  - **Facet $2+1$**: $x_1 + x_2 + 2x_1^2 + 2x_2^2 = 25 \leq 27$
  - **Facet $2+2$**: $x_1 + x_2 + 2x_1^2 + 2x_2^2 + 2x_1^1 + 2x_2^1 = 41 \leq 41$
  - **Facet $3+1$**: $x_1 + x_2 + x_3 + 3x_1^2 + 3x_2^2 + 3x_3^1 = 52.5 \leq 54$
  - **Facet $3+2$** :
    $\mathbf{x_1 + x_2 + x_3 + 3x_1^2 + 3x_2^2 + 3x_3^1 + 3x_1^1 + 3x_2^1 + 3x_3^2 = 88.5 \not\leq 87}$

The following tree is implicitly constructed:



Are these any good?

Experimental setup:

- CP solver : ECLiPSe
- LP solver : Coin-OR Clp

Yan-William & Toma-Kruk

TK-YW	YW
1.000	0.921
1.000	0.984
0.991	0.951
1.000	0.982
0.995	0.970
0.992	0.982
0.984	0.967
0.972	0.963
0.955	0.945
0.938	0.931

Table: Ratio of LP relaxation and best CP solution after 10 seconds

Random Timetabling Instances

Experimental data pending . . .

Current work

- More general facets
- Separation algorithm based on Network Flow subproblem
- Variations on `alldifferent`

Open questions (partial answers)

- New predicate `alldifferent` $\{H, T0, T1, \ldots, Tn\}$?
  - Not for CP formulations; maybe for IP
- Do better than bounding the objective?
  - Yes, we can use dual values to prune search tree
- Recognize 'good' combs amongst multiple alldifferent?
  - Maybe use the cuts on different structures