Logical Foundations of CS – CS5303
Ordered sets

These notes were written using Maths for Computer Science, by André Arnold and Irène Guessarian.

1 Order and pre-order relations

Definition 1 *An order relation is a reflexive, anti-symmetric and transitive relation. A strict order is an irreflexive and transitive relation.*

Note: if $\mathcal{R}$ is a strict order over E, then $\mathcal{R} \cup Id_E$ is an order over E. Reversely, if $\mathcal{R}$ is an order relation over E, then $\mathcal{R} \setminus Id_E$ is a strict order over E.

Notations: in general, order relations are denoted by $\leq$ or $\preceq$, while strict orders are denoted by $<$ or $\prec$.

These relations are very important in Computer Science for they give a mean to compare objects.

1.1 Total and partial orders

Definition 2 *An order relation is a total order if:*

$$\forall e,\ e',\quad \textit{such that}\ \ e \neq e',\quad e\mathcal{R}e' \quad \textit{or}\ e'\mathcal{R}e$$

Otherwise, $\mathcal{R}$ is called a partial order.

Examples:

- the usual order on $\mathbb{R}$ is a total order.
- the following relation:
$$a \leq_{div} b \ \text{ iff }\ \exists c,\ b = ac$$
is a partial order over integers.
- the inclusion relation over $\mathcal{P}(E)$ is a partial order if $|E| > 1$, and a total order if $|E| \leq 1$.

1.2 Pre-orders

Definition 3 *A pre-order is a transitive relation.*

Example: Let $E = \mathcal{P}_f(\mathbb{N})$ the set of finite subsets of $\mathbb{N}$. All element $X \in E$ contains a smaller element, inf(X), and a larger element, max(X). $\mathcal{R}$ is defined as follows: $X\mathcal{R}X'$ iff $inf(X) \leq inf(X')$ and $max(X) \leq max(X')$. This relation is reflexive and transitive. But it is not anti-symmetric because: $X\mathcal{R}X'$ and $X'\mathcal{R}X$ implies that $inf(X) = inf(X')$ and $max(X) = max(X')$, but not that $X = X'$.

Exercise: Let $\mathcal{R}$ be a pre-order over E. Show that $Id_E \cup (\mathcal{R} \cap \mathcal{R}^{-1})$ is an equivalence relation.

Exercise: Considering again the relation $\mathcal{R}$ defined over $E = \mathcal{P}_f(\mathbb{N})$ in the previous example, what is the relation defined by $Id_E \cup (\mathcal{R} \cap \mathcal{R}^{-1})$.

Let $\mathcal{R}$ be a pre-order over E, and let $\mathcal{E}$ be the associated equivalence. Over $E/\mathcal{E}$, we define the relation $\mathcal{R}'$ as follows:

$$[e]_\mathcal{E}\mathcal{R}'[e']_\mathcal{E} \quad \text{iff} \quad e\mathcal{R}e'$$

$\mathcal{R}'$ is antisymmetric and transitive. $\mathcal{R}'$ is an order if $\mathcal{R}$ is reflexive, and $\mathcal{R}'$ is a strict order if $\mathcal{R}$ is irreflexive. $\mathcal{R}'$ is called the quotient order of pre-order $\mathcal{R}$.

2 Ordered sets

Definition 4 *An ordered set, denoted $(E, \leq)$ is a set endowed with an order $\leq$.*

A given order can have different orders: they are just different ordered sets.

Definition 5 *Let $(E_1, \leq_1)$ and $(E_2, \leq_2)$ be two ordered sets. A function $f : E_1 \rightarrow E_2$ is a monotonic application if:*

$$\forall x, y \in E_1, \quad x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$$

Definition 6 *A set $(E, \leq)$ is totally ordered is $\leq$ is a total order. It is partially ordered if $\leq$ is a partial order.*

Definition 7 *Let $(E, \leq)$ be an ordered set. An ordered subset of $(E, \leq)$ is an ordered set $(E', \leq')$ such that $E' \subseteq E$ and $\leq' = \leq \cap (E' \times E')$, i.e.,*

$$\forall x, y \in E', \quad x \leq' y \textit{ iff } x \leq y$$

Definition 8 *A chain of E is totally ordered subset of E. A chain is maximal if it is contained in no other chain of E.*

3 Min and Max

Definition 9 *Let E' be a subset of an ordered set $(E, \leq)$. $x \in E$ is a majoring element of E' (resp. a minoring) if $\forall y \in E'$, $y \leq x$ (resp. $x \leq y$). The set of majoring (resp. minoring) elements of E' is denoted $maj(E')$ (resp. $min(E')$).*

Exercise: Show that $maj(E') \cap E'$ contains at most one element. Same thing for $min(E') \cap E'$.

4 Lattices and fix-points