Logical Foundations of CS – CS5303

Midterm 1 50 points / 80 minutes

Important note: Please make sure that you justify your answers, and that your answers are readable. All non-justified answers will be graded half the points, all unreadable answers will be graded 0. Besides 4 points will be given for the clarity of your answers, and their presentation.

1 Sets (Total: 20 points)

Exercise 1 (10 points) *Let A, B, C be three subsets of E. Show that:*

$$(A \cup B \subseteq A \cup C \text{ and } A \cap B \subseteq A \cap C) \Rightarrow B \subseteq C$$

Exercise 2 (10 points) *Let A, B, C be three subsets of E. What is the inclusion relation between:*

$$A\Delta(B \cup C) \text{ and } (A\Delta B) \cup (A\Delta C)$$

Prove it.

2 Functions (Total: 10 points)

Exercise 3 (10 points) *Let $f : E \to F$ be an application. Determine a necessary and sufficient condition such that the image of any partition of E by f is a partition of F. Justify your answer.*

To do this exercise, we admit that if f is injective, then $f(A \cap B) = f(A) \cap f(B)$. Let us also note that the image of a partition $(A_i)_{i \in I}$ of E is defined by $f((A_i)_{i \in I}) = (f(A_i)_{i \in I})$.

3 Cardinality (Total: 5 points)

Exercise 4 (5 points) *Pigeon-hole principle. The principle is that if there are p pigeons and only h holes ($h < p$) then there is at least one hole with more than one pigeon. More specifically, prove that there is a hole with at least $n \geq \frac{p}{h}$ pigeons, where n is an integer.*

4 Relations (Total: 6 points)

Exercise 5 (6 points) *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be binary relations over E. Show that:*

- $(\mathcal{R}_1 \cap \mathcal{R}_2)^{-1} = \mathcal{R}_1^{-1} \cap \mathcal{R}_2^{-1}$ **(3 points)**
- $(\overline{\mathcal{R}})^{-1} = \overline{\mathcal{R}^{-1}}$ **(3 points)**

5 Induction (Total: 5 points)

Exercise 6 (5 points) *Let BT be the set of binary trees. Let h, and l denote respectively the height of the tree, and its number of leaves. Using the inductive definition of BT, show that:*

$$\forall x \in BT,\ l(x) \leq 2^{h(x)-1}$$