Logical Foundations of CS – CS5303

Quiz 1 50 points / 50 minutes

Important note: Please make sure that you justify your answers, and that your answers are readable. All non-justified answers will be graded half the points, all unreadable answers will be graded 0. Besides 3 points will be given for the clarity of your answers, and their presentation.

1 Sets (Total: 9 points)

Exercise 1 (5 points) *Let A, B, C be three subsets of E. Show that:*

$$A \cap \overline{B} = A \cap \overline{C} \Leftrightarrow A \cap B = A \cap C$$

Solution: given in class.

Exercise 2 (4 points) *Let A be a subset of E, and let $(B_i)_{i \in I}$ a family of subsets of E. Show that:*

$$A \cup (\bigcap_{i \in I} B_i) = \bigcap_{i \in I} (A \cup B_i)$$

Solution: given in class.

2 Functions (Total: 12 points)

Exercise 3 (4 points) *Let $f : A \to B$ be an application. Show that:*

$$f \text{ injective} \Rightarrow \forall X \subseteq A,\ f^{-1}(f(X)) = X$$

Solution: Suppose that $f : A \to B$ is an injective function. This means, by definition, that:

$$\forall x_1,\ x_2\ \in\ A,\ (f(x_1) = f(x_2))\ \Rightarrow\ x_1 = x_2$$

Let us show that $\forall X \subseteq A,\ f^{-1}(f(X)) = X$.

To prove this, we are going to consider such an X, subset of A, and prove successively that: $f^{-1}(f(X)) \subseteq X$ and $X \subseteq f^{-1}(f(X))$. And in order to prove each inclusion $Y \subseteq Z$, we will each time consider an element of Y and show that it is in Z.

1. $\boldsymbol{f^{-1}(f(X)) \subseteq X}$? Let $x \in \boldsymbol{f^{-1}(f(X))}$. By definition of $f^{-1}(Y)$, $y \in f^{-1}(Y)$ means that $\exists z \in Y$ such that $z = f(y)$. As a result:

$$x \in \boldsymbol{f^{-1}(f(X))} \overset{def}{\Leftrightarrow} \exists y \in f(X), f(x) = y$$

At this point we know that $f(x) \in f(X)$. Now we need to finish the proof by showing that if $f(x) \in f(X)$ then necessarily $x \in X$.

In order to prove this, we use the contrapositive rule[1], and we assume that $x \notin X$.

If $x \notin X$, and $f(x) = y \in f(X)$, then it means that $\exists x' \in X$, such that $f(x') = y$. As a result, $\exists x,\ x'$, such that $f(x) = f(x') = y$ and $x \neq x'$. This contradicts the property of injectivity. As a result, since f is injective, it means that $x = x'$ or $f(x) \notin f(X)$.

If $x = x'$, then $x \in X$, and we are done with the proof. If $f(x) \notin f(X)$, it contradicts our intermediate deduction that $f(x) \in f(X)$, and therefore, by using the contrapositive rule, the condition is contradicted: i.e., $\neg(x \notin X)$ or $\neg(f$ is injective$)$. Since f is injective, it means that $\neg(x \notin X)$ holds, i.e., $x \in X$, and we are done again with the proof.

2. $X \subseteq f^{-1}(f(X))$?

Let $x \in X$. Let us prove that $x \in f^{-1}(f(X))$. Since $x \in X$, there exists $y \in f(X)$ such that $y = f(x)$. As a result, $x = f^{-1}(y)$ and therefore $x \in f^{-1}(\{y\})$, and more generally, $x \in f^{-1}(f(X))$.

Exercise 4 (8 points) *Let us consider two applications $f : A \to B$ and $g : B \to C$. Show that:*

1. *$g \circ f$ injective $\Rightarrow$ f injective*
2. *$g \circ f$ surjective $\Rightarrow$ g surjective*

Solution:

1. *Suppose that $g \circ f$ is injective, and that f is not injective.*

Then it means that there are at least x_1 and $x_2 \in A$, such that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$. As a result, $g(f(x_1)) = g(f(x_2)) = g(y)$. This contradicts the fact that $g \circ f$ is injective.

2. *Suppose that $g \circ f$ is surjective and that g is not surjective.*

Then it means that there exists at least one element of C, say z^*, such that $\forall y \in B$, $f(y) \neq z$. Now since $g \circ f$ is supposed to be surjective too, it means that $\forall z \in C$, $\exists x \in A$, such that $z = g(f(x))$. In particular, for $z^* \in C$, $\exists x^* \in A$, such that $z^* = g(f(x^*))$. Let $y^* = f(x^*)$. By definition of f, $y^* \in B$. Therefore, $\exists y^* \in B$, such that $z^* = f(y^*)$, which contradicts g not being surjective. We conclude that under the assumption that $g \circ f$ is surjective, g is necessarily surjective too.

3 Cardinality (Total: 6 points)

Exercise 5 (6 points) *Show that the set $\mathbb{N} \times \mathbb{N}$ is countable.*

[1] The objective here is to prove that ($x \notin X$ and f injective) $\Rightarrow f(x) \notin X$. And then, since we know (we just proved it), that $f(x) \in f(X)$, then it contradicts our assumption that ($x \notin X$ and f is injective). As soon as our assumption is contradicted, it means that $\neg(x \notin X)$ or $\neg(f$ is injective$)$. But since we know, by hypothesis in this problem, that f is injective, then we deduce that $\neg(x \notin X)$, which is $x \in X$.

Solution: To show that a set E is countable, you just have to show that there exist a bijective function between E and $\mathbb{N}$. So here, we are going to try and define a bijective function between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

Let $f:\ \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$\forall (x,y) \in \mathbb{N} \times \mathbb{N}, \quad (x,y) \ \mapsto \ y + (0+1+2+3+...+(x+y)) = y + \sum_{i=0}^{x+y} i$$

Now, we have to prove that this function is a bijective function. There are several ways to prove that f is a bijective function, among which:

- show that it is injective and surjective
- show that $\forall n \in \mathbb{N}$, $|f^{-1}(\{n\})| = 1$.

We choose the second option, and we prove it by showing that $\forall n \in \mathbb{N}$, there exists a unique element (x,y) of $\mathbb{N} \times \mathbb{N}$ such that $n = f(x,y)$.

$$y = n - (0+1+\cdots+a) \quad \text{and} \quad x = a - y$$

where a is the unique element of $\mathbb{N}$ such that:

$$(0+1+\cdots+a) \leq n < (0+1+\cdots+a+(a+1))$$

making thus y and x unique.

4 Relations (Total: 20 points)

Exercise 6 (20 points) *Let E be a finite set: $E = \{e_1, \ldots, e_n\}$, et let $\mathcal{R}$ be a binary relation over E. We can represent $\mathcal{R}$ using a $n \times n$ matrix, say $\mathcal{M}_\mathcal{R}$, whose elements belong to $\{0,1\}$, and are defined as follows:*

$$m_{i,j} = \begin{cases} 1 & \textit{if } e_i \ \mathcal{R} \ e_j \\ 0 & \textit{otherwise} \end{cases}$$

1. *Describe $\mathcal{M}_\mathcal{R}$ in case:*
   (a) *$\mathcal{R}$ is a symmetrical relation*
   (b) *$\mathcal{R}$ is a reflexive relation*
2. *Suppose you have two relations over E, say $\mathcal{R}$ and $\mathcal{R}'$, how can you compute: $\mathcal{M}_{\mathcal{R}^{-1}}$, $\mathcal{M}_{\overline{\mathcal{R}}}$.*

Solution:

1.a. If $\mathcal{R}$ is a symmetric relation, it means that as soon as $(e,e')in\mathcal{R}$, then $(e',e) \in \mathcal{R}$ too. Let $\mathcal{M}_\mathcal{R}(i,j)$ be the coefficient of the $\mathcal{M}_\mathcal{R}$ representing whether $(e,e')in\mathcal{R}$ or not.

Then, if $\mathcal{M}_\mathcal{R}(i,j) = 1$, it means that $(e,e') \in \mathcal{R}$, and therefore $(e',e) \in \mathcal{R}$ which translate in the matrix as $\mathcal{M}_\mathcal{R}(j,i) = 1$. Reversely, if $\mathcal{M}_\mathcal{R}(i,j) = 0$, it means that $(e,e') \notin \mathcal{R}$, and therefore $(e',e) \notin \mathcal{R}$ (otherwise we would have $(e,e') \in \mathcal{R}$ too, by symmetry), which translate in the matrix as $\mathcal{M}_\mathcal{R}(j,i) = 0$.

As a result, $\forall i,j$, $\mathcal{M}_\mathcal{R}(i,j) = \mathcal{M}_\mathcal{R}(j,i)$, which means that $\mathcal{M}_\mathcal{R}$ is symmetric.

1.b. If $\mathcal{R}$ is a reflexive relation, it means that $\forall e \in E$, $(e,e) \in \mathcal{R}$. This translates in the matrix as $\forall i$, $\mathcal{M}_\mathcal{R}(i,i) = 1$. Therefore $\mathcal{M}_\mathcal{R}$ is a matrix whose diagonal contains only 1s.

2. Given $\mathcal{M}_\mathcal{R}$, we need to define a transformation to get $\mathcal{M}_{\mathcal{R}^{-1}}$ and $\mathcal{M}_{\overline{\mathcal{R}}}$. Let us go back to the definitions of $\mathcal{R}^{-1}$ and $\overline{\mathcal{R}}$.

$\boldsymbol{\mathcal{R}^{-1}}$: $(e,e') \in \mathcal{R}$ iff $(e',e) \in \mathcal{R}^{-1}$. As a result, $\mathcal{M}_{\mathcal{R}^{-1}} = \mathcal{M}_\mathcal{R}^T$.

$\boldsymbol{\overline{\mathcal{R}}}$: $(e,e') \in \mathcal{R}$ iff $(e,e') \notin \overline{\mathcal{R}}$. As a result, $\mathcal{M}_{\overline{\mathcal{R}}} = One - \mathcal{M}_\mathcal{R}$, where One is a matrix of the same size as $\mathcal{M}_\mathcal{R}$, but containing only 1s.