

**HOW FAR WE ARE FROM THE COMPLETE KNOWLEDGE:  
COMPLEXITY OF KNOWLEDGE ACQUISITION  
IN DEMPSTER-SHAFER APPROACH.**

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*Abstract:* When a knowledge base represents the experts' uncertainty, then it is reasonable to ask how far we are from the complete knowledge, that is, how many more questions do we have to ask (to these experts, to nature by means of experimenting, etc) in order to attain the complete knowledge. Of course, since we do not know what the real world is, we cannot get the precise number of questions from the very beginning: it is quite possible, for example, that we ask the right question first and thus guess the real state of the world after the first question. So we have to estimate this number and use this estimate as a natural measure of completeness for a given knowledge base.

We give such estimates for Dempster-Shafer formalism. Namely, we show that this average number of questions can be obtained by solving a simple mathematical optimization problem. In principle this characteristic is not always sufficient to express the fact that sometimes we have more knowledge. For example, it has the same value if we have an event with two possible outcomes and nothing else is known, and if there is an additional knowledge that the probability of every outcome is 0.5. We'll show that from the practical viewpoint this is not a problem, because the difference between the necessary number of questions in both cases is practically negligible.

*Keywords:* complexity of knowledge acquisition, Dempster-Shafer formalism.

## **1. BRIEF INTRODUCTION TO THE PROBLEM.**

**Knowledge is usually not complete.** The vast majority of modern knowledge bases include uncertain knowledge, that is, statements about which the experts themselves are not 100% sure that they are absolutely true. This uncertainty leads to uncertainty in the answers to the queries: instead of yes-no answers, we get answers like "probably" and "with probability 0.8". Sometimes the uncertainty is too high, and we cannot get anything definite from the resulting expert system. When a knowledge base represents the experts'

uncertainty, then it is reasonable to ask how far we are from the complete knowledge, that is, how many more questions do we have to ask (to these experts, to nature by means of experimenting, etc) in order to attain the complete knowledge. Of course, since we do not know what the real world is, we cannot get the *precise* number of questions from the very beginning: it is quite possible, e.g., that we ask the right question first and thus guess the real state of the world after the first question. So we can only get *estimates* for the number of necessary questions. These estimates are a natural measure of completeness for a given knowledge base.

**Estimates of incompleteness are useful.** Such estimates can be useful in several cases. For example, suppose that we feel like our knowledge base needs updating and we want to estimate the cost of the update. The main part of updating is the acquisition of the new knowledge from the experts. Since it is desirable to take the best (and therefore highly paid) specialists as experts, the knowledge acquisition cost is an essential part of the total update cost. From our previous experience, we can get the expected per question cost  $c$  by dividing the previous update cost by the number of questions asked. To estimate the total acquisition cost, we multiply  $c$  by the number of necessary questions.

Another situation where these estimates are applicable is when we choose between the existing knowledge bases (for example, when we decide which of them to buy). When choosing we must take into consideration cost, performance time, etc. But the main characteristic of the knowledge base is how much information it contains. It is difficult to estimate this amount of information directly, but we can use the estimates of the number of questions if they are available: Evidently the fewer questions we need to ask in order to obtain the complete knowledge, the more information was there initially. So the knowledge base, for which we have to ask the minimal number of questions, is the one with the greatest amount of information.

**What we are planning to do.** There exist several different formalisms for representing uncertainty (see, e.g., Smets et al, 1988). In the present paper we estimate the necessary number of questions for the case of Dempster-Shafer formalism. Namely, we show that this average number of questions can be obtained by solving a simple mathematical optimization problem.

It turns out that the same techniques can be applied to estimate the complexity of knowledge acquisition for the probabilistic approach to uncertainty (Nilsson, 1986). In principle this characteristic is not always sufficient to express the fact that sometimes we have more knowledge. For example, it has the same value if we have an event with two

possible outcomes and nothing else is known, and if there is an additional knowledge that the probability of every outcome is 0.5. We'll show that from the practical viewpoint this is not a problem, because the difference between the necessary number of questions in both cases is practically negligible.

**The structure of the paper** is as follows: there exists a well-known case, where a formula for the average number of questions is known: the case of probabilistic knowledge, that was considered in the pioneer Shannon papers on information theory. We are planning to use the same methods that were used in its derivation. Since the derivation is not as well known as Shannon's formula itself, we'll briefly describe it in Section 2. In Section 3, we'll formulate a likewise problem for Dempster-Shafer formalism in mathematical terms and present our results. In Section 4, we'll show that this characteristic is sometimes not sufficient, but from practical viewpoint there is no need to worry. In Section 5 we apply the same techniques to the case of a probabilistic knowledge. Proofs are in Section 6.

## 2. SHANNON'S FORMULA REVISITED

**First let's analyze the simplest possible case: formulation.** Before we actually analyze Shannon's formula, let us recall how to compute the complexity of knowledge acquisition in the simplest case: Namely, we consider one event, and we know beforehand that it can result in one of finitely many incompatible outcomes. Let's denote these outcomes by  $A_1, A_2, \dots$ , and their total number by  $n$ . For example, in the coin tossing case  $n$  equals two, and  $A_1$  and  $A_2$  are "heads" and "tails"; in the weather example  $n = 4$ ,  $A_1$  is "raining",  $A_2$  is "snowing", etc. How many binary questions do we have to ask in order to find out which of the outcomes occurred?

**The simplest case: result.** The answer is well known: we must ask  $Q$  questions, where  $Q$  is the smallest integer that is greater than or equal to  $\log_2 n$ . This number is sometimes called the *ceiling* of  $\log_2 n$  and is denoted by  $\lceil \log_2 n \rceil$ . And if we ask less than  $Q$  questions, we will be unable to always find the outcome.

Although the proof of this fact is well-known (see, e.g., Horowitz and Sahni, 1984), we repeat it here, because this result will be used as a basis for all other estimates.

**The simplest case: proof.** First we have to prove that  $Q$  questions are sufficient. Indeed, let's enumerate all the outputs (in arbitrary order) by numbers from 0 to  $n - 1$ , and write these numbers in the binary form. Using  $q$  digits, one gets numbers from 0 to  $2^q - 1$ , that is, totally  $2^q$  numbers. So one digit is sufficient for  $n = 1, 2$ ; two digits for

$n = 1, 2, \dots, 4$ ,  $q$  digits for  $n = 1, 2, \dots, 2^q$ , and in order to represent  $n$  numbers we need to take the minimal  $q$  such that  $2^q \geq n$ . Since this inequality is equivalent to  $q \geq \log_2 n$ , we need  $Q$  digits to represent all these numbers. So we can ask the following  $Q$  questions: "is the first binary digit 0?", "is the second binary digit 0?", etc, up to "is the  $q$ -th digit 0?".

The fact that we cannot use less than  $Q$  questions is also easy to prove. Indeed, suppose we use  $q < Q$  questions. After we ask  $q$  binary questions, we get a sequence of  $q$  0's and 1's ( $q$  bits). If there is one bit, we have 2 possibilities: 0 or 1. We have  $q$  bits, so we have  $2 \cdot 2 \cdot 2 \dots \cdot 2$  ( $q$  times)  $= 2^q$  possible sequences. This sequence is the only thing that we use to distinguish outcomes, so if we need to distinguish between  $n$  outcomes, we need at least  $n$  sequences. So the number of sequences  $2^q$  must be greater than or equal to  $n$ :  $2^q \geq n$ . Since logarithm is a monotonic function, this inequality is equivalent to  $q \geq \log_2 n$ . But  $Q$  is by definition the smallest integer, that is greater than or equal to this logarithm, and  $q$  is smaller, than  $Q$ . Therefore  $q$  cannot be  $\geq \log_2 n$ , and hence  $q < Q$  questions are not sufficient.

**Situations that are covered by Shannon's formula.** The above formula works fine for the case when we have a single event, and we need to find what its outcome is. But in many real-life cases same types of events happen again and again: for example, we can toss the coin again and again, and we must predict weather every day, etc. In such cases there is a potentially infinite sequence of repeating independent events. By the moment when we are asking about the outcome of the current event, we normally already know what outcomes happened before, which of them were more frequent, which were more seldom.

In some cases these frequencies change essentially in course of time: for example, in case of the global warming the frequencies of cold weather days will become smaller and smaller. But in many cases we can safely assume that these frequencies are more or less the same. This means that the outcomes, that were more frequent in the past, will still be more frequent, and vice versa.

Of course, the frequencies with which some output occurs in two long sequences of  $N$  events, are not precisely equal. But it is usually assumed, that the larger  $N$  is, the smaller is the difference between them. In other words, when  $N$  tends to  $\infty$ , the frequencies converge to a number that is called a *limit frequency*, or a *probability*  $p_i$  of an outcome  $i$ . We can also express the same supposition by saying that the frequencies are *estimates* for these probabilities: the bigger sample we take, the better are these estimates.

These frequencies are the additional information, that Shannon (1948) used to diminish the number of necessary questions.

**Why probabilities help to diminish the number of questions: explanation in commonsense terms.** If we have just one event, then probabilities or no probabilities, we still have to ask all  $Q = \lceil \log_2 n \rceil$  questions. However, if we have  $N$  similar events, and we are interested in knowing the outputs of all of them, we do not have to ask  $Q$  questions all  $N$  times: we can sometimes get out with less than  $QN$  questions and still know all the outputs.

Let's give a simple example why it is possible. Suppose we have 2 outputs ( $n = 2$ ), and their probabilities are  $p_1 = 0.99$  and  $p_2 = 0.01$ . If there is just one event, we have to ask  $Q = 1$  question. Let's now consider the case of 10 consequent events. If we knew no probabilities, there would be  $2^{10} = 1024$  possible combinations of outputs, and so we need to ask at least  $10 = \log_2 1024$  questions in order to find all the outputs.

But we do know the probabilities. And due to the fact, that the probability of the second event is very small, it is hardly unprobable, that there will be 2 or more cases out of 10 with the second output. If we neglect these unprobable cases, we conclude that there are not 1024, but only 11 possible combinations: second outcome in first event, first in all the other; second outcome in the second event, first in all the other, ... (10 such combinations), and the eleventh which corresponds to first outcome in all the events. To find a combination out of 11 possible we need only  $\lceil \log_2 11 \rceil = 4$  questions. On average we have 4/10 questions per event.

As a real-world example take technical diagnosis: a system doesn't work, and we must find out which of  $n$  components failed. Here we have two outcomes: good and failed. In case the reliability of these components is sufficiently high, so that  $p_2 \ll 1$ , we can neglect the possibility of multiple failures, and thus simplify the problem.

**Some statistics.** When talking about Shannon's theory one cannot avoid using statistics. However, we'll not copy (Shannon, 1948): instead we reformulate so that it would be easy to obtain a Dempster-Shafer modification.

Suppose that we know the probabilities  $p_i$ , and that we are interested in the outcome of  $N$  events, where  $N$  is given. Let's fix  $i$  and estimate the number of events  $N_i$ , in which the output is  $i$ .

This number  $N_i$  is obtained by adding all the events, in which the output was  $i$ , so  $N_i = n_1 + n_2 + \dots + n_N$ , where  $n_k$  equals to 1 if in  $k$ -th event the output is  $i$  and 0 otherwise. The average  $E(n_k)$  of  $n_k$  equals to  $p_i \cdot 1 + (1 - p_i) \cdot 0 = p_i$ . The mean square deviation  $\sigma[n_k]$

is determined by the formula  $\sigma^2[n_k] = p_i(1 - E(n_k))^2 + (1 - p_i)(0 - E(n_k))^2$ . If we substitute here  $E(n_k) = p_i$ , we get  $\sigma^2[n_k] = p_i(1 - p_i)$ . The outcomes of all these events are considered independent, therefore  $n_k$  are independent random variables. Hence the average value of  $N_i$  equals to the sum of the averages of  $n_k$ :  $E[N_i] = E[n_1] + E[n_2] + \dots + E[n_N] = Np_i$ . The mean square deviation  $\sigma[N_i]$  satisfies a likewise equation  $\sigma^2[N_i] = \sigma^2[n_1] + \sigma^2[n_2] + \dots = Np_i(1 - p_i)$ , so  $\sigma[N_i] = \sqrt{p_i(1 - p_i)N}$ .

For big  $N$  the sum of equally distributed independent random variables tends to a Gaussian distribution (the well-known *central limit theorem*), therefore for big  $N$  we can assume that  $N_i$  is a random variable with a Gaussian distribution. Theoretically a random Gaussian variable with the average  $a$  and a standard deviation  $\sigma$  can take any value. However, in practice, if, e.g., one buys a measuring instrument with guaranteed 0.1V standard deviation, and it gives an error 1V, it means that something is wrong with this instrument. Therefore it is assumed that only some values are practically possible. Usually a "k-sigma" rule is accepted that the real value can only take values from  $a - k\sigma$  to  $a + k\sigma$ , where  $k$  is 2, 3 or 4. So in our case we can conclude that  $N_i$  lies between  $Np_i - k\sqrt{p_i(1 - p_i)N}$  and  $Np_i + k\sqrt{p_i(1 - p_i)N}$ . Now we are ready for the formulation of Shannon's result.

*Comment.* In this quality control example the choice of  $k$  matters, but, as we'll see, in our case the results do not depend on  $k$  at all.

### Formulation of Shannon's results.

**Definitions.** Suppose that a real number  $k > 0$  and a positive integer  $n$  are given.  $n$  is called *the number of outcomes*. By a *probabilistic knowledge* we mean a set  $\{p_i\}$  of  $n$  real numbers,  $p_i \geq 0$ ,  $\sum p_i = 1$ .  $p_i$  is called a *probability* of  $i$ -th event.

Suppose that an integer  $N$  is given; it is called *the number of events*. By a *result of  $N$  events* we mean a sequence  $r_k$ ,  $1 \leq k \leq N$  of integers from 1 to  $n$ .  $r_k$  is called the *result of  $k$ -th event*. The number of events, that resulted in  $i$ -th outcome, will be denoted by  $N_i$ . We say that the result of  $N$  events is *consistent* with the probabilistic knowledge  $\{p_i\}$  if for every  $i$  the following inequality is true:  $Np_i - k\sqrt{p_i(1 - p_i)N} \leq N_i \leq Np_i + k\sqrt{p_i(1 - p_i)N}$ .

Let's denote the number of all consistent results by  $N_{cons}(N)$ . The number  $\lceil \log_2(N_{cons}(N)) \rceil$  will be called *the number of questions, necessary to determine the results of  $N$  events* and denoted by  $Q(N)$ . The fraction  $Q(N)/N$  will be called the *average number of questions*. The limit of the average number of questions will be called the *complexity of knowledge acquisition*.

**THEOREM** (Shannon). *When the number of events  $N$  tends to infinity, the average number of questions tends to  $\sum -p_i \log_2(p_i)$ .*

*Comments.* 1. This sum is known as an *entropy* of a probabilistic distribution  $\{p_i\}$  and denoted by  $S$  or  $S(\{p_i\})$ . So Shannon's theorem says that if we know the probabilities of all the outputs, then the average number of questions that we have to ask in order to get a complete knowledge equals to the entropy of this probabilistic distribution. In other words: in case we know all the probabilities, the complexity of knowledge acquisition equals to the entropy of this probabilistic distribution.

2. As promised, the result does not depend on  $k$ .

3. Since we modified Shannon's definitions, we cannot use the original proof. Our proof is given in Section 6.

### 3. DEMPSTER-SHAFER CASE

**Dempster-Shafer (DS) formalism in brief** (Smets et al, 1988). The basic element of knowledge in this formalism is as follows: an expert gives several hypotheses  $E_1, \dots, E_p$  about the real world (these hypotheses are not necessarily incompatible), and describes his degrees of belief  $m(E_1), m(E_2), \dots, m(E_p)$  in each of these hypotheses. These values are called *masses*, and their sum is supposed to be equal to 1.

There are also combination rules that allow us to combine the knowledge of several experts; as a result we again get a set of hypotheses (that combine the hypotheses of several experts), and their masses (degrees of belief).

So in general the knowledge consists of a finite set of statements  $E_1, E_2, \dots, E_p$  about the real world, and a set of real numbers  $m(E_i)$  such that  $\sum m(E_i) = 1$ .

**What "complete knowledge" means in DS.** This knowledge is incomplete: first of all, because we do not know which of the hypotheses  $E_i$  is true. But even if we manage to figure that out, the uncertainty can still remain, because this hypothesis  $E_i$  does not necessarily determine uniquely the state of our system. Therefore, if we want to estimate how far we are from the complete knowledge, we must know what is meant by a complete knowledge. In other words, we need to know the set  $W$  of possible states of the analyzed system (these states are sometimes called *possible worlds*). Of course, there are infinitely many states of any real objects, but usually we are interested only in finitely many properties  $P_1, P_2, \dots, P_m$ . It means that if for some pair of states  $s_1, s_2$  each of these properties is true

in  $s_1$  if and only if it is true in  $s_2$ , then we consider them as one state. In this sense a state is uniquely determined by the  $m$ -dimensional Boolean vector, that consists of truth values  $P_i(s)$ . So the set of all possible worlds consists of all such vectors, for which a state  $s$  with these properties is possible at all.

**Where do we take the masses from?** In order to use this formalism to describe actual knowledge we must somehow assign the masses to the experts' beliefs. The fact that the sum of these masses equals to 1 prompts the interpretation of masses as probabilities. And, indeed, the very formalism stemmed from probabilities, therefore probabilistic way is one of the possible ways to estimate masses.

For example, we can ask several experts what statement better describes their knowledge, take all these statements for  $E_i$  and for  $m(E_i)$  take the fraction  $N(E_i)/N$ , where  $N$  is the total number of experts, and  $N(E_i)$  is the number of experts whose knowledge is described by the statement  $E_i$ . Or, alternatively, we can ask one expert, and by analyzing the similar situations he can say that in the part  $m(E_i)$  of all these cases a hypothesis  $E_i$  was true. It is also possible that the expert does not know so many cases, but he tries to make a guess, based on his experience of likewise cases.

There exist other methods to determine masses, that are not of probabilistic origin, but we'll consider only probabilistic ones for 3 reasons (more detailed explanations of the pro-probabilistic viewpoint can be found in Pearl, 1989, Dubois and Prade, 1989, Halpern and Fagin, 1990, Shafer and Pearl, 1990):

**We'll consider only probabilistic methods to determine masses; why?**

1) There are arguments (starting from Savage, 1954, 1962) that if an expert assigns the degrees of belief in a rational manner, then they automatically satisfy all the properties of probabilities (they are called *subjective probabilities*);

2) Several non-probabilistic methods of assigning degrees of belief that we successfully applied, turned out to have probabilistic origin; for example, for the rules of MYCIN, the famous successful expert system (Shortliffe, 1976, Buchanan and Shortliffe, 1984), it was proved in (Heckerman, 1986);

3) Finally, in case we interpret masses as probabilities, we know precisely what we mean by saying that we believe in  $E_i$  with the degree of belief  $m(E_i)$ : namely, as we'll show right now, this knowledge can be easily reformulated in terms of the future behavior of the system. Therefore we can understand in precise terms, what is meant by this knowledge,

and what knowledge do we need in addition so that we would be able to narrow our predictions to one actual outcome and thus get a complete knowledge. In case we do not use a probabilistic interpretation, what restrictions this knowledge imposes on future outcomes is difficult to figure out.

**What does a DS knowledge mean?** In case we accept a probabilistic interpretation, then the knowledge that the hypothesis  $E_i$  is true with mass  $m(E_i)$ , can be interpreted as follows: if we have  $N$  similar events, then among these  $N$  cases there are approximately  $Nm(E_1)$  in which the outcomes satisfy the statement  $E_1$ ; among the remaining ones there are approximately  $Nm(E_2)$  cases in which  $E_2$  is true, etc.

*Warning.* This does not mean that  $E_1$  is true only in  $Nm(E_1)$  cases: since the statements  $E_i$  are not necessarily incompatible, it is possible that some of the outcomes that satisfy  $E_2$  will satisfy  $E_1$  as well. *Example.* Suppose that the whole knowledge of an expert is that to some extent he believes in some statement  $E$ . If we denote the corresponding degree of belief by  $m$ , we can express this knowledge in DS terms as follows: he believes in  $E_1 = E$  with degree of belief  $m(E_1) = m$ , and with the remaining degree of belief  $m(E_2) = 1 - m$  he knows nothing, i.e.,  $E_2$  is a statement that is always true. In our terms this knowledge means that out of  $N$  events there are  $\approx Nm$ , in which  $E$  is true, and  $\approx N(1 - m)$ , in which  $E_2$  is true. But  $E_2$  is always true, so the only conclusion is that in at least  $\approx Nm$  events  $E$  is true. It is possible that  $E$  is always true (if it is also true for the remaining  $N(1 - m)$  events), and it is also possible that  $E$  is true only in  $Nm$  cases (if  $E$  is false for the outcomes of the remaining events).

We are almost ready to formalize this idea; the only problem is how to formalize "approximately". But since we interpret masses as probabilities, we can apply the same statistical estimates as in the previous section. So we arrive at the following definitions.

### **Definitions and the main result.**

**Denotations.** For any finite set  $X$ , we'll denote by  $|X|$  the number of its elements.

**Definitions.** Suppose that a real number  $k > 0$  is given. Suppose also that a finite set  $W$  is given. Its elements will be called *outcomes*, or *possible worlds*.

*Comment.* In the following text we'll suppose that the possible worlds are ordered, so that instead of talking about a world we can talk about its number  $i = 1, \dots, n = |W|$ . In these terms  $W$  is equal to the set  $\{1, 2, \dots, n\}$ .

By a *Dempster-Shafer knowledge* or *DS knowledge* for short we mean a finite set of pairs  $\langle E_i, m_i \rangle$ ,  $1 \leq i \leq p$ , where  $E_i$  are subsets of  $W$  (called *statements*) and  $m_i$  are real numbers (called *masses* or *degrees of belief*) such that  $m_i \geq 0$  and  $\sum m_i = 1$ .

If an outcome  $r$  belongs to the set  $E_i$ , we'll say that  $r$  *satisfies*  $E_i$ . Suppose that an integer  $N$  is given; it is called *the number of events*. By a *result of  $N$  events* we mean a sequence  $r_k$ ,  $1 \leq k \leq N$  of integers from 1 to  $n$ .  $r_k$  is called the *outcome of  $k$ -th event*. We say that the result of  $N$  events is *consistent* with the DS knowledge  $\langle E_i, m_i \rangle$ , if the set  $\{1, 2, \dots, N\}$  can be divided into  $p$  subsets  $H_1, H_2, \dots, H_p$  with no common elements in such a way that:

- 1) if  $k$  belongs to  $H_j$ , then the outcome  $r_k$  of  $k$ -th event satisfies  $E_j$ ;
- 2) the number  $|H_i|$  of elements in  $H_i$  satisfies the inequality  $Nm_i - k\sqrt{m_i(1 - m_i)N} \leq |H_i| \leq Nm_i + k\sqrt{m_i(1 - m_i)N}$ .

Let's denote the number of all results, that are consistent with a given DS-knowledge, by  $N_{cons}(N)$ . The number  $\lceil \log_2(N_{cons}(N)) \rceil$  will be called *the number of questions, necessary to determine the results of  $N$  events* and denoted by  $Q(N)$ . The fraction  $Q(N)/N$  will be called the *average number of questions*. The limit of average number of questions, when  $N \rightarrow \infty$ , will be called the *complexity of knowledge acquisition*.

To formulate our estimate we need some additional definitions.

**Definitions.** By a *probabilistic distribution* we mean an array of  $n$  non-negative numbers  $p_1, \dots, p_n$  such that  $\sum p_j = 1$ . We say that a probabilistic distribution is *consistent* with the DS knowledge  $\langle E_i, m_i \rangle$ ,  $i = 1, \dots, p$ , if and only if there exist non-negative numbers  $z_{ij}$  such that  $\sum_i z_{ij} = p_j$ ,  $\sum_j z_{ij} = m_i$  and  $z_{ij} = 0$  if  $j$  does not belong to  $E_i$ .

*Comments.* 1. Informally, we want to divide the whole fraction  $m_i$  of events, about which the expert predicted that  $E_i$  is true, into the groups with fractions  $z_{ij}$  for all  $j \in E_i$ , so that the outcomes in a group  $z_{ij}$  is  $j$ .

2. This definition is not explicitly constructive, but if we fix a probabilistic distribution and a DS knowledge, the question whether they are consistent or not is a linear programming problem, so we can use the known algorithms to solve it (simplex method or the algorithm of Karmarkar (1984)).

By an *entropy* of a DS knowledge we mean a maximum entropy of all probabilistic distributions that are consistent with it.

In other words, this entropy is a solution to a following mathematical problem:  $-\sum p_j \log_2 p_j \rightarrow \max$  under the conditions that  $\sum_i z_{ij} = p_j$ ,  $\sum_j z_{ij} = m_i$ ,  $z_{ij} \geq 0$  and  $z_{ij} = 0$  for  $j$  not in  $E_i$ , where  $i$  runs from 1 to  $p$ , and  $j$  from 1 to  $n$ .

If we substitute  $p_j = \sum_i z_{ij}$ , we can reformulate it without using  $p_j$ : Entropy is a solution of the following mathematical optimization problem:

$$-\sum_i \left( \sum_j z_{ij} \right) \log_2 \left( \sum_j z_{ij} \right) \rightarrow \max,$$

under the conditions that  $\sum_j z_{ij} = m_i$ ,  $z_{ij} \geq 0$  and  $z_{ij} = 0$  for  $j$  not in  $E_i$ .

*Comments.* 1. Entropy is a smooth convex function, all the restrictions are linear in  $z_{ij}$ , so in order to compute the entropy of a given DS knowledge we must maximize a smooth convex function on a convex domain. In numerical mathematics there exist sufficiently efficient methods for doing that.

2. For the degenerate case, when a DS knowledge is a probabilistic one, i.e., when  $n = p$  and  $E_i = \{i\}$ , there is precisely one probabilistic distribution that is consistent with this DS knowledge: this very  $p_j$ , and therefore the entropy of a DS knowledge in this case coincides with Shannon's entropy.

**MAIN THEOREM.** *The complexity of knowledge acquisition for a DS knowledge  $\langle E_i, m_i \rangle$  is equal to the entropy of this knowledge.*

*Comments.* 1. Our definition of entropy is thus a natural generalization of Shannon's entropy to a DS case. This not mean, of course, that this is *the* generalization. The notion of entropy is used not only to compute the average number of questions, but in several other applications: in communication theory, in pattern recognition, etc. Several different generalizations of entropy to DS formalism have been proposed and turned out to be efficient in these other problems (see, e.g., Yager, 1983, Pal and Datta Majumer, 1986, Dubois and Prade, 1987, Nguyen, 1987, Klir and Folger, 1988, Dubois and Prade, 1989, Pal, 1991, Kosko, 1992).

2. That the complexity of knowledge acquisition must be greater or equal that the entropy of a DS knowledge is rather easy to prove. Indeed, if a probabilistic distribution  $p_j$  is consistent with a DS knowledge, and a result of  $N$  events is consistent with this distribution, then it is consistent with a DS-knowledge as well. Therefore there are at least as many results consistent with DS knowledge as there are results consistent with  $p_j$ . Therefore the average number of questions in a DS case must be not smaller than the

average number of questions (entropy) for every probabilistic distribution that is consistent with this knowledge. So it must be greater than or equal to the maximum of all such probabilistic entropies; and we have called this maximum an entropy of a DS knowledge. The fact that it is precisely equal, and not greater, is more difficult to prove, and demands combinatorics (see Section 6).

#### 4. THE ABOVE COMPLEXITY CHARACTERISTIC IS NOT SUFFICIENT, BUT WE NEED NOT WORRY ABOUT THAT

**Example.** The above characteristic describes the average number of questions that we need to ask in order to attain the complete knowledge. However, we'll now show that it is sometimes possible that we add the new information, and this characteristic remains the same. The simplest of such situations is as follows: suppose that there are only two possible outcomes. If we know nothing about them, this can be expressed in DS terms as follows: there is only one statement ( $p = 1$ ), and this statement  $E_1$  is identically true (i.e.,  $E_1 = W = \{1, 2\}$ ). In this case the above mathematical optimization problem is easy to solve, and yields 1. This result is intuitively very reasonable: if we know nothing, and there are two alternatives, we have to ask one binary question in order to figure out, which of the outcomes actually occurred.

Suppose now that we analyzed the previous cases and came to a conclusion that on average in half of these cases the first outcome occurred, in half of them the second one. In other words, we add the new information that the probability of both outcomes is equal to  $1/2$ . This is really a new information, because it diminishes the number of possibilities: For example, if we observed 100 events, in case we knew nothing it was quite possible that in all the cases we would observe the first outcome. In case we know that the probability is  $1/2$ , then the possible number  $N_1$  of cases, in which the first outcome occurs, is restricted by the inequalities  $1/2 \cdot 100 - k\sqrt{1/2(1 - 1/2)100} \leq N_1 \leq 1/2 \cdot 100 + k\sqrt{1/2(1 - 1/2)100}$ , or  $50 - 5k \leq N_1 \leq 50 + 5k$ . Even for  $k = 4$  the value  $N_1 = 100$  does not satisfy this inequality and is therefore impossible (therefore for  $k < 4$  it also cannot be equal to 100).

In other words, we added a new information. But if we compute the uncertainty (entropy) of the resulting probabilistic distribution, we get  $-1/2 \log_2(1/2) - 1/2 \log_2(1/2) = -1 \cdot \log_2(1/2) = 1$ , i.e., again 1! We added the new information, but the uncertainty did not diminish. We still have to ask in average one question in order to get a complete knowledge.

**Isn't it a paradox?** No, because we were estimating the *average* amount of questions  $\lim Q(N)/N$ . We have two cases, in which the necessary number of questions  $Q_1(N)$  in the first case is evidently bigger than in the second one ( $Q_1(N) > Q_2(N)$ ), but this difference disappears in the limit. In order to show that it is really so, let us compute  $Q(N)$  in both cases.

If we know nothing, then all sequences of 1 and 2 are possible as the results, i.e., in this case  $N_{cons}$  is equal to  $2^N$ . Therefore  $\log_2 N_{cons} = N$ , and  $Q_1(N) = \lceil \log_2 N_{cons} \rceil = N$ .

In the second case computations are more complicated (so we moved them to Section 6), and the result for big  $N$  is  $Q_2(N) = N - c$ , where  $c$  is a constant depending on  $k$ . Since  $c/N \rightarrow 0$ , in the limit this difference disappears and so it looks like in these two cases the uncertainty is the same.

**Do we need to worry about that?** To answer this question let's give a numeric estimate of the difference between  $Q_1(N)$  and  $Q_2(N)$ ; this difference occurs only when the inequality  $N/2 - kN/2 \leq N_1 \leq N/2 + kN/2$  really restricts the possible values of  $N$ . If  $k = 2$ , then for  $N \leq 4$  all possible values of  $N_1$  from 0 to  $N$  satisfy it, so  $Q_1 = Q_2$ . Therefore the difference starts only with  $N = 5$ . The bigger  $k$ , the bigger is the  $N$ , from which the difference appears.

The value of this difference  $c = Q_1(N) - Q_2(N)$  depends on  $k$  (see the proof in Section 6). The smaller the  $k$ , the bigger is  $c$ . For the smallest possible value  $k = 2$  this value is  $\approx 0.1$ . In comparison with 5 it is 2%. For bigger  $N$  or bigger  $k$  it is even smaller.

So this difference makes practical sense, if we can somehow estimate  $Q(N)$  with a similar (or better) precision. But  $Q(N)$  is computed from the initial degrees of belief (masses)  $m_i$ . There is already a tiny difference between, say, 70% and 80% degree of belief, and hardly anyone can tell claim that in some cases he is 72% sure, and in some other cases 73%, and that he feels the difference. There are certainly not so many subjective degrees of belief. In view of that the degrees of belief are defined initially with at best 5 – 10% precision. Therefore the values of  $Q(N)$  are known with that precision only, and in comparison to that adding  $\leq 2\%$  of  $c$  is, so to say, under the noise level.

So the answer to the question in the title is: no, we don't need to worry.

## 5. PROBABILISTIC KNOWLEDGE

Let's analyze the case of an probabilistic knowledge as described in (Nilsson, 1986), when we know the probabilities of several statements.

**Definitions.** Suppose that a real number  $k > 0$  is given. Suppose also that a finite set  $W = \{1, 2, \dots, n\}$  is given. Its elements will be called *outcomes*, or *possible worlds*. By a *probabilistic knowledge* we mean a finite set of pairs  $\langle E_i, p(E_i) \rangle$ ,  $1 \leq i \leq p$ , where  $E_i$  are subsets of  $W$  and  $0 \leq p(E_i) \leq 1$ . Subsets  $E_i$  are called *statements*, and the number  $p(E_i)$  is called a *probability* of  $i$ -th statement.

If an outcome  $r$  belongs to the set  $E_i$ , we'll say that  $r$  *satisfies*  $E_i$ .

Suppose that an integer  $N$  is given; it is called *the number of events*. By a *result of  $N$  events* we mean a sequence  $r_k$ ,  $1 \leq k \leq N$  of integers from 1 to  $n$ .  $r_k$  is called the *outcome of  $k$ -th event*. We say that the result of  $N$  events is *consistent* with the probabilistic knowledge  $\langle E_i, p(E_i) \rangle$ , if for all  $i$  from 1 to  $p$  the number  $N_i$  of all  $r_k$  that belong to  $E_i$  satisfies the inequality  $Np(E_i) - k\sqrt{p(E_i)(1-p(E_i))N} \leq N_i \leq Np(E_i) + k\sqrt{p(E_i)(1-p(E_i))N}$ .

Let's denote the number of all results, that are consistent with a given probabilistic knowledge, by  $N_{cons}(N)$ . The number  $\lceil \log_2(N_{cons}(N)) \rceil$  will be called *the number of questions, necessary to determine the results of  $N$  events* and denoted by  $Q(N)$ . The fraction  $Q(N)/N$  will be called the *average number of questions*. The limit of average number of questions, when  $N \rightarrow \infty$ , will be called the *complexity of knowledge acquisition*.

By a *probabilistic distribution* we mean an array of  $n$  non- negative numbers  $p_1, \dots, p_n$  such that  $\sum p_j = 1$ . We say that a probabilistic distribution is *consistent* with a probabilistic knowledge  $\langle E_i, p(E_i) \rangle$ ,  $i = 1, \dots, p$ , if and only if for every  $i$ :  $\sum_{j \in E_i} p_j = p_i$ . By an *entropy* of a probabilistic knowledge we mean a maximum entropy of all probabilistic distributions that are consistent with it, i.e., the solution to a following mathematical optimization problem:  $-\sum p_j \log_2 p_j \rightarrow \max$  under the conditions  $\sum_{j \in E_i} p_j = p(E_i)$ ,  $p_j \geq 0$  and  $\sum_{j=1}^n p_j = 1$ .

*Comment.* This is also a convex optimization problem.

**THEOREM.** *The complexity of knowledge acquisition for a probabilistic knowledge is equal to the entropy of this knowledge.*

*Comments.* 1. Main Theorem and this result can be combined as follows: if our knowledge is not sufficient to determine all the probabilities uniquely, so that several different probabilistic distributions are compatible with it, then the uncertainty of this knowledge is equal to the uncertainty of the distribution with the maximal entropy. It is worth mentioning that the distribution with maximal entropy has many other good properties, and is therefore often used as a most "reasonable" one when processing incomplete data in science

(for a survey see Jaynes, 1979, and references therein; see also Kosheleva and Kreinovich (1979) and Cheeseman (1985)).

2. Similar maximum entropy result can be proved for the case when part of the knowledge is given in a DS form, and part in a probabilistic form. In this case we can also formulate, what we mean by saying that probabilities are consistent with a given knowledge, and prove that the complexity of knowledge acquisition is equal to the maximum entropy of all probabilistic distributions, that are consistent with a given knowledge.

## 6. PROOFS

**Proof of Shannon's Theorem.** As we have mentioned in the main text, the Theorem that we prove is not the original Shannon's, but its modification: Shannon was interested in data communication, and not in asking questions. So we must modify the proof. The proof that we are using first appeared in (Kreinovich, 1989).

Let's first fix some values  $N_i$ , that are consistent with the given probabilistic distribution. Due to the inequalities that express the consistency demand, the ratio  $f_i = N_i/N$  tends to  $p_i$  as  $N \rightarrow \infty$ . Let's count the total number  $C$  of results, for which for every  $i$  the number of events with outcome  $i$  is equal to this  $N_i$ . If we know  $C$ , we will be able to compute  $N_{cons}$  by adding these  $C$ 's.

Actually we are interested not in  $N_{cons}$  itself, but in  $Q(N) \approx \log_2 N_{cons}$ , and moreover, in  $\lim(Q(N)/N)$ . So we'll try to estimate not only  $C$ , but also  $\log_2 C$  and  $\lim((\log_2 C)/N)$ .

To estimate  $C$  means to count the total number of sequences of length  $N$ , in which there are  $N_1$  elements, equal to 1,  $N_2$  elements, equal to 2, etc. The total number  $C_1$  of ways to choose  $N_1$  elements out of  $N$  is well-known in combinatorics, and is equal to  $\binom{N}{N_1} = N!/((N_1)!(N - N_1)!)$ . When we choose these  $N_1$  elements, we have a problem in choosing  $N_2$  out of the remaining  $N - N_1$  elements, where the outcome is 2; so for every choice of 1's we have  $C_2 = \binom{N - N_1}{N_2}$  possibilities to choose 2's. Therefore in order to get the total number of possibilities to choose 1's and 2's, we must multiply  $C_2$  by  $C_1$ . Adding 3's, 4's, ...,  $n$ 's, we get finally the following formula for  $C$ :

$$C = C_1 C_2 \dots C_{n-1} = \frac{N!}{N_1!(N - N_1)!} \frac{(N - N_1)!}{(N_2!(N - N_1 - N_2)!)} \dots = \frac{N!}{N_1! N_2! \dots N_n!}$$

To simplify computations let's use the well-known Stirling formula, according to which  $k!$  is asymptotically equivalent to  $(k/e)^k \sqrt{2\pi k}$ . If we substitute these expressions into the

above formula for  $C$ , we conclude that

$$C \approx \frac{(N/e)^N \sqrt{2\pi N}}{(N_1/e)^{N_1} \sqrt{2\pi N_1} (N_2/e)^{N_2} \sqrt{2\pi N_2} \dots (N_n/e)^{N_n} \sqrt{2\pi N_n}}$$

Since  $\sum N_i = N$ , terms  $e^N$  and  $e^{N_i}$  annihilate each other.

To get further simplification, we substitute  $N_i = N f_i$ , and correspondingly  $N_i^{N_i}$  as  $(N f_i)^{N f_i} = N^{N f_i} f_i^{N f_i}$ . Terms  $N^N$  is the numerator and  $N^{N f_1} N^{N f_2} \dots N^{N f_n} = N^{N f_1 + N f_2 + \dots + N f_n} = N^N$  in the denominator cancel each other. Terms with  $\sqrt{N}$  lead to a term that depends on  $N$  as  $cN^{-(n-1)/2}$ . Now we are ready to estimate  $\log_2 C$ . Since logarithm of the product is equal to the sum of logarithms, and  $\log a^b = b \log a$ , we conclude that  $\log_2 C \approx -N f_1 \log_2 f_1 - N f_2 \log_2 f_2 - \dots - N f_n \log_2 f_n - 1/2(n-1) \log_2 N - \text{const}$ . When  $N \rightarrow \infty$ , we have  $1/N \rightarrow 0$ ,  $\log_2 N/N \rightarrow 0$  and  $f_i \rightarrow p_i$ , therefore  $\log_2 C/N \rightarrow -p_1 \log_2 p_1 - p_2 \log_2 p_2 - \dots - p_n \log_2 p_n$ , i.e.,  $\log_2 C/N$  tends to the entropy of the probabilistic distribution.

*Comment.* We used the denotation  $A \approx B$  for some expressions  $A$  and  $B$  meaning that the difference between  $A$  and  $B$  is negligible in the limit  $N \rightarrow \infty$  (i.e., the resulting difference in  $(\log_2 C)/N$  tends to 0).

Now, that we have found an asymptotics for  $C$ , let's compute  $N_{cons}$  and  $Q(N)/N$ . For a given probabilistic distribution  $\{p_i\}$  and every  $i$  possible values of  $N_i$  form an interval of length  $L_i = 2k\sqrt{p_i(1-p_i)}\sqrt{N}$ . So there are no more than  $L_i$  possible values of  $N_i$ . The maximum value for  $p_i(1-p_i)$  is attained when  $p_i = 1/2$ , therefore  $p_i(1-p_i) \leq 1/4$ , and hence  $L_i \leq 2k\sqrt{N/4} = k\sqrt{N}/2$ . For every  $i$  from 1 to  $n$  there are at most  $(k/2)\sqrt{N}$  possible values of  $N_i$ , so the total number  $N_{co}$  of possible combinations of  $N_i$  is smaller than  $((k/2)\sqrt{N})^n$ .

The total number  $N_{cons}$  of consistent results is the sum of  $N_{co}$  different values of  $C$  (that correspond to different combinations  $N_1, N_2, \dots, N_n$ ). Let's denote the biggest of these  $C$  by  $C_{max}$ . Since  $N_{cons}$  is the sum of  $N_{co}$  terms, and each of them is not greater than the biggest of them  $C_{max}$ , we conclude, that  $N_{cons} \leq N_{co} C_{max}$ . On the other hand, the sum  $N_{cons}$  is bigger than each of its terms, i.e.,  $C_{max} \leq N_{cons}$ . Combining these two inequalities, we conclude, that  $C_{max} \leq N_{cons} \leq N_{co} C_{max}$ . Since  $N_{co} \leq ((k/2)\sqrt{N})^n$ , we conclude that  $C_{max} \leq N_{cons} \leq ((k/2)\sqrt{N})^n C_{max}$ . Turning to logarithms, we find that  $\log_2(C_{max}) \leq \log_2(N_{cons}) \leq \log_2(C_{max}) + (n/2) \log_2 N + \text{const}$ . Dividing by  $N$ , tending to the limit  $N \rightarrow \infty$  and using the fact that  $\log_2 N/N \rightarrow 0$  and the already proved fact that  $\log_2(C_{max})/N$  tends to the entropy  $S$ , we conclude that  $\lim Q(N)/N = S$ . Q.E.D.

**Proof of the Main Theorem.** Let's denote by  $h_i$  some integer numbers that satisfy the inequalities  $Nm_i - k\sqrt{m_i(1-m_i)N} \leq h_i \leq Nm_i + k\sqrt{m_i(1-m_i)N}$  from Section 3. Let's denote the ratios  $h_i/N$  by  $g_i$ . Due to these inequalities, when  $N \rightarrow \infty$ ,  $g_i \rightarrow m_i$ .

Unlike the previous Theorem, even if we know  $g_i$ , i.e., know how many outcomes belong to  $E_i$  for every  $i$ , we still cannot uniquely determine the frequencies  $f_j$  of different outcomes. If there exists a result of  $N$  events with given frequencies  $g_i$  and  $f_j$ , then we can further subdivide each set  $H_i$  into subsets  $Z_{ij}$  that correspond to different outcomes  $j \in E_i$ . In this case  $\sum_j Z_{ij} = h_i$  and  $\sum_i Z_{ij} = Nf_j$ ; therefore the frequencies  $t_{ij} = Z_{ij}/N$  satisfy the equalities  $\sum_j t_{ij} = g_i$  and  $\sum_i t_{ij} = f_j$ . Vice versa, if there exist values  $t_{ij}$  such that these two equalities are satisfied, and  $Nt_{ij}$  is an integer for all  $i, j$ , then we can divide  $W$  into sets of size  $h_i$ , each of them into sets with  $Nt_{ij}$  elements and thus find a result with given  $g_i$  and  $f_j$ . If such  $t_{ij}$  exist, we'll say that the frequencies  $g_i$  and  $f_j$  are *consistent* (note an evident analogy between this concept and the definition of consistency between a DS knowledge and a probabilistic distribution).

Let's now prove, that if the set of frequencies  $\{f_j\}$  is consistent with the set  $\{g_i\}$ , and we have a result, in which there are  $Nf_1$  outcomes that are equal to 1,  $Nf_2$  outcomes that are equal to 2, etc., then this result is consistent with the original DS knowledge. Indeed, we can subdivide the set of all the outcomes, that are equal to  $j$ , into subsets with  $Nt_{ij}$  elements for all  $i$  such that  $j \in E_i$ . We'll say that the elements that are among these  $Nt_{ij}$  ones are *labeled by  $i$* . Totally there are  $\sum_j Nt_{ij} = N\sum_j t_{ij} = Ng_i = h_i$  elements, that are labelled by  $i$ , and for all of them  $E_i$  is true. Since  $h_i$  was chosen so as to satisfy the inequalities that are necessary for consistency, we conclude that this result is really consistent with a DS knowledge.

The number  $C$  of results with given frequencies  $\{f_j\}$  has already been computed in the proof of Shannon's theorem:  $\lim ((\log_2 C)/N) = -\sum f_j \log_2 f_j$ .

The total number of the results  $N_{cons}$ , that are consistent with a given DS knowledge, is the sum of  $N_{co}$  different values of  $C$ , that correspond to different  $f_j$ . For a given  $N$  there are at most  $N + 1$  different values of  $N_1 = Nf_1$  ( $0, 1, \dots, N$ ), at most  $N + 1$  different values of  $N_2$ , etc., totally at most  $(N + 1)^n$  different sets of  $\{f_j\}$ . So, like in the proof of Shannon's theorem, we get an inequality  $C_{max} \leq N_{cons} \leq (N + 1)^n C_{max}$ , from which we conclude, that  $\lim Q(N)/N = \lim(\log_2 C_{max})/N$ .

When  $N \rightarrow \infty$ , the values  $g_i$  tend to  $m_i$ , and therefore these frequencies  $f_j$  tend to the probabilities  $p_j$ , that are consistent with a DS knowledge. Therefore  $(\log_2 C)/N$

tends to the entropy of the limit probabilistic distribution, and  $(\log_2 C_{\max})/N$  tends to the maximum of such entropies. But this maximum is precisely the entropy of a DS knowledge as we defined it. So  $\lim(Q(N)/N)$  equals to the entropy of a DS knowledge. Q.E.D.

**The estimates for a probabilistic case** are proved likewise.

**Proof of the statement from Section 4.** We have to consider the case, when  $n = 2$  (there are two possible outcomes). In this case the result of  $N$  events is a sequence of 1's and 2's. A result is consistent with our knowledge if and only if the number  $N_1$  of 1's satisfies the inequality  $N/2 - k\sqrt{N}/2 \leq N_1 \leq N/2 + k\sqrt{N}$  (actually we must demand that the likewise inequality is true for  $N_2 = N - N_1$ , but one can easily see that this second inequality is equivalent to the first one). Let's estimate the number  $N_{cons}$  of such results.

In order to get this estimate let's use the following trick. Suppose that we have  $N$  independent equally distributed random variables  $r_k$ , each of which attains two possible values 1 and 2 with equal probability  $1/2$ . Then the probability of each of  $2^N$  possible sequences of 1's and 2's is the same:  $2^{-N}$ . The probability  $P$  that a random sequence satisfies the above inequalities is equal to the sum of the probabilities of all the sequences that satisfy it, i.e., is equal to the sum of  $N_{cons}$  terms, that are equal to  $2^{-N}$ . So  $P = N_{cons}2^{-N}$ . Therefore, if we manage to estimate  $P$ , we'll be able to reconstruct  $N_{cons}$  by using a formula  $N_{cons} = 2^N P$ .

So let us estimate  $P$ . Let's recall the arguments that lead to the inequalities that we are using. The total number  $N_1$  of 1's in a sequence  $\{r_k\}$  is equal to the sum of terms that are equal to 1 if  $r_k = 1$  and to 0 if  $r_k = 2$ . In other words, it is the sum of  $2 - r_k$ . So  $N_1$  is the sum of several equally distributed variables, and therefore for big  $N$  its distribution is close to Gaussian, with the average  $N/2$  and the standard deviation  $\sigma = \sqrt{N}/2$ . Therefore for big  $N$  the probability that  $N_1$  satisfies the above inequalities is equal to the probability that the value of a Gaussian random variable with the average  $a$  and standard deviation  $\sigma$  lies between  $a - k\sigma$  and  $a + k\sigma$ . This probability  $P$  depends only on  $k$  and does not depend on  $N$  at all. For example, for  $k = 2$   $P \approx 0.95$ , and for bigger  $k$   $P$  is bigger. Since  $N_{cons} = P2^N$ , we conclude, that  $Q(N) \approx \log_2(P2^N) = N - c$ , where  $c = -\log_2 P$ . For  $k = 2$  we get  $c = -\log_2 P \approx 0.1$ , and for bigger  $k$  it is even smaller.

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