

WHAT PROCEDURE TO CHOOSE WHILE DESIGNING A FUZZY CONTROL? TOWARDS MATHEMATICAL FOUNDATIONS OF FUZZY CONTROL

Vladik Kreinovich, Chris Quintana
Computer Science Dept., University of Texas, El Paso, TX 79968, USA, vladik@cs.ep.utexas.edu

Robert Lea
NASA Johnson Space Center, Houston, TX 77058, USA, rlea@nasamail.nasa.gov

Abstract. Fuzzy control has been successfully applied in industrial systems. However, there is some caution in using it. The reason is that it is based on quite reasonable ideas, but each of these ideas can be implemented in several different ways, and depending on which of the implementations we choose we get different results. Some implementations lead to a high quality control, some of them not. And since there are no theoretical methods for choosing the implementation, the basic way to choose it now is experimental. But if we choose a method that is good for several examples, there is no guarantee that it will work fine in all of them. Hence the caution.

We are going to provide a theoretical basis for choosing the fuzzy control procedures. In order to choose a procedure that transforms a fuzzy knowledge into a control we need, first, to choose a membership function for each of the fuzzy terms that the experts use, second, to choose operations of uncertainty values that corresponds to “and” and “or”, and third, when we obtain a membership function for control, we must defuzzify it, that is, somehow generate a value of the control u that will be actually used. In the paper we describe a general approach that will help to make all these choices: namely, we prove that under reasonable assumptions membership functions should be linear or fractionally linear, defuzzification must be described by a centroid rule and describe all possible “and” and “or” operations. Thus we give a theoretical explanations of the existing semi-heuristic choices and formulate the basis for the further research on optimal fuzzy control.

1. BRIEF INTRODUCTION

Why do we need mathematical foundations of fuzzy control? In order to design a fuzzy control, we must choose fuzzy variables, choose combination rules for uncertainty values and choose a defuzzification procedure. The efficiency of the resulting fuzzy control essentially depends on these choices. For example (see [KFL91] for details) different choices of combination rules can lead to relaxation times that differ twofold (and if we go for stability this is an essential increase). These choices are now made mainly on a semi-empirical basis: if a resulting system works, that's fine. This approach is acceptable for camcorders or dishwashers: even if something goes wrong with a picture for a moment or two, it is not a problem. However, this level of reliability is absolutely unacceptable for such serious applications as Space Shuttle, and that is the main reason why in spite of the brilliant results of computer simulations ([L88, LJB90], etc) fuzzy control techniques are not yet widely applied to space missions. So what we need is an analysis of different possible choices on every stage of choosing a fuzzy control, an analysis that must be done on the mathematical strictness level and either explain what choices to make or at least severely restrict the set of possible choices, so that the best methods could be then chosen by an exhaustive analysis of the few possible candidates.

What are we planning to do? We'll explain how reasonable demands on the choices of a membership function, operations with certainty values and a defuzzification procedure lead to the natural reformulation of these choice problems in terms of *transformation groups*, a formalism that is extremely successful in modern physics. We'll also show how the known results about transformation groups help to solve these choice problems, resulting in the choice of centroid as a defuzzification procedure, linear, fractionally linear and spline membership functions and a list of possible choices for &- and ∨-operations.

2. WHY FUZZY CONTROL? HOW AND WHY IS IT DESIGNED NOW?

Simplest example of a control system. To illustrate the idea of fuzzy control let's consider something very simple, like a thermostat. Suppose that we want to keep a temperature T equal to some fixed value T_0 . In other words, we want the difference $x = T - T_0$ to be equal to 0. The way to control the temperature is to switch on the heater or the cooler, and to control the degree of cooling and heating. In mathematical terms heating increases the temperature and cooling decreases it, so what we control is a variable that determines the rate with which the temperature changes. In other words, this “degree of heating and cooling” is nothing else but a derivative \dot{T} of temperature with respect to time. So the behavior of the thermostat is determined by the equation $\dot{T} = u$, where u is the control we apply. What control u to apply in every moment of time t depends on the current temperature. If it is higher than T_0 , we must cool the room down, i.e., apply $u < 0$, if $T < T_0$, we must heat it, i.e., apply $u > 0$. So u in general depends on T : $u = u(T)$. All these formulas are easier to express in terms of $x = T - T_0$: $\dot{x} = \dot{T}$, hence for x the dynamics is $\dot{x} = u(x)$, where the value of the control variable depends on x . The remaining problem is: how to control? I.e., what $u(x)$ to use?

We have an expert; why cannot we extract $u(x)$ from him? We are talking about the expert system approach; this means that we have an expert who already knows how to control, and what we are planning to do is to somehow extract his knowledge and put it inside the computer. Ideally we would like to extract the whole dependency $u(x)$ from that expert. The natural idea to do that is to ask him lots of questions, like “suppose that x is 5 degrees; what do you do?”, write down the answers to all those questions, and thus plot $u(x)$. Sounds reasonable at first glance, until you try to apply the same idea to the skill in which practically all the adults consider themselves experts: driving a car. If you ask a driver a question like that “you are going at 55 mph, the car in front of you is at the distance of 30 ft, and it slowed down to 47 mph, for how many seconds do you hit the brakes?”, I guess no one will give a precise number. O.K., you can install measuring devices into a car or a simulator, and simulate this situation, but what will happen is that this time will be different for different simulations. The problem is not that the expert has some precise number (like 1.453 sec) in his mind, but cannot express it in words; the problem is that once it will be 1.3, another time it may be 1.5, etc. (depending on whether he is tired or not, etc.).

An expert usually expresses his knowledge in words. An expert cannot say “hit the breaks for 1.453 sec”, what he can say is “hit the brakes for a while”. So the rules that can be extracted from him are not “if the velocity is 47 then hit the brakes for 1.453 sec”, but something like “if the velocity is a little bit smaller than maximum, hit the brakes for a while”. Let’s illustrate the rules on the thermostat example.

Rules: thermostat example. One does not have to be a great expert to control a thermostat, common sense is sufficient here, and common sense prompts the following rules: If the temperature T is close to T_0 , i.e., if the difference $x = T - T_0$ is negligible, then no control is needed, i.e., u is also negligible. If the room is slightly overheated, i.e., if x is positive and small, we must cool it a little bit (i.e., $u = \dot{x}$ must be negative and small). If the temperature is a little lower, then we need to heat the room a little bit. In other terms, if x is small negative, then u must be small positive, etc. Thus way we can formulate our commonsense experience in terms of rules: If x is negligible, then u must be negligible. If x is small positive, then u must be small negative. If x is small negative, then u must be small positive, etc.

Suppose that we know x . What u to choose? Summarizing the rules, we can say that u is an appropriate choice for a control if and only if either (x is negligible and u is negligible), or (x is small positive and u is small negative), etc. If we use the denotations $C(u)$ for “ u is an appropriate control”, $N(x)$ for “ x is negligible”, SP for “small positive”, SN for “small negative” and use the standard mathematical notations $\&$ for “and”, \vee for “or” and \equiv for “if and only if”, we come to the following formula: $C(x) \equiv (N(x)\&N(u)) \vee (SP(x)\&SN(u)) \vee (SN(x)\&SP(u)) \vee \dots$

How to formalize these words: the idea of fuzzy logic. In order to formalize them we need to express in mathematical terms notions like “negligible”, “small positive”, “small negative”, etc. The main difference between these notions and mathematically precise (“crisp”) ones like “greater than 0.5” is that any value is either greater than 0.5 or not, while we cannot say that any value x is negligible or not. Some values are so small that practically everyone would agree that they are negligible, but the bigger is the value, the less experts will say that it is negligible, and less confident he will be in that statement. For example, if someone is performing a complicated experiment that needs fixed temperature, then for him 0.1 degree is negligible, but 1 degree is not. For another expert ± 5 degrees is negligible.

This degree of confidence is usually described by a *fuzzy logic* [Z65]. The idea is that to every “fuzzy” property like “negligible” or “small positive” and to every real number x we put into correspondence a value $\mu(x)$ from the interval $[0,1]$ that expresses our degree of confidence that this property is true for x : $\mu(x) = 1$ means that we are absolutely sure that this property is true for x . $\mu(x) = 0$ means that we are absolutely sure that this property is false for x . Values between 0 and 1 mean that we are not sure whether it is true or not. This μ is called a *membership function*.

How to determine the membership function for a given property? One of the possibilities is to take several (E) experts, take different values of x , and ask every expert whether he believes that this property is true for each of this values. Suppose that for some x $E(x)$ experts out of E said that x is negligible (or satisfies any other property). Then it is natural to take the ratio of those who said that x is negligible, as a resulting degree of confidence, i.e., take $E(x)/E$ as $\mu(x)$. Another possibility is to ask one expert and express his degree of confidence in terms of the so-called subjective probabilities [S54]. After we’ve got the values $\mu(x)$ for the x ’s that we asked about, we must extrapolate these values to get the expression for $\mu(x)$ for all x .

In fuzzy control we do not fix any specific way to assign membership functions, we just suppose that μ is somehow determined.

What membership functions are actually used. Several different membership functions result from this approach. The simplest ones correspond to the case, when we consider just three values x : one value a , for which we are absolutely sure that this property is true, and two values $a - \Delta, a + \Delta$, for which we are absolutely sure that it is false; and then use linear interpolation in between. The resulting membership function is described by the following expression: $\mu_A(x) = 0$ if $x < a - \Delta$ or $x > a + \Delta$; $\mu_A(x) = 1 + (x - a)/\Delta$ if $a - \Delta \leq x \leq a$ and $\mu_A(x) = 1 - (x - a)/\Delta$ if $a \leq x \leq a + \Delta$. Such functions are called *triangular*.

In case we have several consequent words to describe the same quantity, like “small negative”, “negligible”, “small positive”, etc., it means that every value of x must satisfy one of these properties. If we use ratios of experts or subjective probabilities to get the values of μ , we come to the conclusion, that for every x the sum of the values of $\mu_A(x)$ for all A must be equal to 1. Therefore, where the membership function corresponding to one property starts decreasing from 1 to 0 (in the interval $[a, a + \Delta]$), the membership function that corresponds to the next property must start increasing from 0 to 1. In view of that the value of Δ must be the same for all the properties, and a are equal to 0 for negligible, Δ for “small positive”, 2Δ for the next property, etc.

This is to some extent an oversimplification in comparison with what is actually used: “left Δ ” and “right Δ ” can be different, and there must be infinite intervals corresponding to “very very big” (positive and negative). With these corrections made, these simplest membership functions are efficiently used in fuzzy control [L88], [LTTJ89], [LJB90]. More complicated functions are also used: e.g., fractionally linear functions are used to control trains [MYI87], splines are used, etc.

Returning to rules. If we choose some membership functions, we’ll be able for every x and u to describe our degree of confidence in statements $N(x), N(u)$, etc. In order to compute our degree of confidence in $C(u)$ we must figure out, how to apply $\&$ and \vee to these degrees of confidence. The degrees of confidence are a generalization of truth values (true corresponds to 1, false to 0), so we must somehow extend $\&$ and \vee from the two-valued set $\{0, 1\}$ to the functions $f_{\&}$ and f_{\vee} , that are defined on the whole interval $[0, 1]$. Zadeh originally proposed to use $f_{\&} = \min$ and $f_{\vee} = \max$. Later other functions were proposed, including product for $\&$ and $\min(a + b, 1)$ as $f_{\vee}(a, b)$ (these four functions are most frequently used in fuzzy control). For our thermostat example we get the following expression for the membership function $\mu_C(u)$:

$$\mu_C(u) = f_{\vee}(f_{\&}(\mu_N(x), \mu_N(u)), f_{\&}(\mu_{SP}(x), \mu_{SN}(u)), f_{\&}(\mu_{SN}(x), \mu_{SP}(u)), \dots)$$

In particular, if we use \min and \max , we get

$$\mu_C(u) = \max(\min(\mu_N(x), \mu_N(u)), \min(\mu_{SP}(x), \mu_{SN}(u)), \min(\mu_{SN}(x), \mu_{SP}(u)), \dots)$$

So what control to use? a problem. What we get as a result is a “fuzzy” recommendation: we can use, say $u = 0.5$ with degree of certainty 0.8, $u = 0.4$ with degree of certainty 0.3, etc. In the real expert systems it is all we need: for example, in case of a medical system we give to a doctor the list of all possible illnesses with degrees of certainty, and it is for him to decide: either to believe in the most probable diagnosis, or to analyze the patient more. But in the control case the whole purpose was to automate; so we do not have a person who makes decision, we want a system itself to decide which of the possible controls to use. So we must transform this membership function $\mu_C(u)$ into one value \bar{u} .

Centroid: a solution to this problem. Informally speaking, we want a value that is in average closest to the optimal control. Closest means that the square $(u - \bar{u})^2$ must be minimal. “In average” means that we have to take into account, how often different values of control are appropriate. We do not know the frequencies, what we now are degrees of confidence. But let’s recall that one of the natural interpretations of the degrees of confidence is that they are proportional to the number $N(u)$ of experts who believe that this very value u is the best: $\mu_C(u) = kN(u)$ for some constant k . The more experts say that u is the best, the greater is the probability $p(u)$ that this u will really be the best. In view of that we can estimate that probability as $p(u) = K\mu_C(u)$ for some constant K . Therefore the average deviation of \bar{u} from u equals to $\int p(u)(u - \bar{u})^2 du = K \int \mu_C(u)(u - \bar{u})^2 du$. We must choose \bar{u} so that this deviation is the smallest possible. Differentiating with respect to \bar{u} gives the explicit formula $\bar{u} = (\int u\mu_C(u) du)/(\int \mu_C(u) du)$. This formula is called the *centroid* formula. Now we are ready for a final description.

Fuzzy control: brief description. We extract the rules from the expert (or experts); transform these rules into the and-or statement. Then we find the membership functions for all involved words like “negligible” or “small”. After that we choose functions $f_{\&}$ and f_{\vee} . Now we can compute $\mu_C(u)$ for every x . Using centroid rule, we compute the value \bar{u} . This value is what the fuzzy control algorithm recommends for this case.

This method can be applied also if we control the second derivative, only rules will be longer, like “if x is N and \dot{x} is SP , then u is SN ”, and terms in $\mu_C(x)$ will be longer, like $\min(\mu_N(x), \mu_{SP}(\dot{x}), \mu_{SN}(u))$.

**3. IN ORDER TO CHOOSE A DEFUZZIFICATION
LET'S LIST NATURAL RESTRICTIONS ON THAT CHOICE
AND THUS FORMULATE THE PROBLEM IN MATHEMATICAL TERMS**

Comment. Let's start with the last stage of the fuzzy control design: defuzzification. We'll first analyze the case, when only finitely many values x_i are possible.

Definition. A *fuzzy set* of real numbers (or a *fuzzy real number*) is a function μ from the set R of all real numbers into the interval $[0,1]$. It is called *finite* if the value of μ is different from 0 only for finitely many numbers x_1, x_2, \dots, x_n .

Comment. So in order to describe a finite fuzzy set it is sufficient to give all these values x_i and the corresponding values $\mu_i = \mu(x_i) \in [0, 1]$. A defuzzification procedure must be defined for all the cases when this set is non-empty, i.e., when not all μ_i are equal to 0. Let's list reasonable demands for such a procedure.

D1: The result of defuzzification must lie between x_i . The first natural property of the desired result \bar{x} of a defuzzification procedure is that it must lie between the smallest and the biggest of all possible values x_i of the quantity x . The reason for that is as follows: when we say that $\mu(x_i) > 0$, it means that there is some reason to believe that the actual value of x is equal to x_i (or is at least close to x_i). The fact that $\mu(x)$ is different from 0 only for $x = x_1, \dots, x_n$ means that all possible reasons lead to the values from the interval $[\min x_i, \max x_i]$, and there are no reasons to believe that x is smaller than $\min x_i$ or bigger than $\max x_i$. Hence it seems reasonable to conclude that a single value, chosen by this procedure, must also belong to this same interval.

D2: Symmetry. A finite fuzzy set is a finite set of pairs (x_i, μ_i) . The word "set" means that it does not matter in what order we list these pairs; so evidently the result of defuzzification must not depend on the order in which we list them.

D3: If $\mu_i = 0$ for some i , then the result of defuzzification must not depend on this x_i . This demand is quite natural: if $\mu_i = 0$, this means that x_i is impossible, so we can omit it.

D4: μ_i, x_i and \bar{x} can be interpreted as degrees of uncertainty, and the transformation from μ_i or x_i to \bar{x} is a transformation of degrees of uncertainty. That μ_i is a degree of uncertainty is evident: that's what the values of membership functions describe. Let's show that x_i and μ can also be interpreted as degrees of uncertainty.

To do that let's recall that to describe a finite fuzzy function with n values we must describe $2n$ different parameters $x_1, x_2, \dots, x_n, \mu_1, \dots, \mu_n$. A defuzzification method f takes all these parameters as an input and computes \bar{x} as an output: $\bar{x} = f(x_1, \dots, x_n, \mu_1, \dots, \mu_n)$.

Suppose that we fix somehow the values of all these parameters, except for μ_i for some i . The remaining parameter μ_i can take any value from 0 to 1. The bigger the value of x_i , the more uncertain we are about the actual value of x : if $\mu_i = 0$, then only the values $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are possible. When we increase μ_i , we add one more possibility: that the actual value of x equals to x_i . The bigger is μ_i , the more possible is this additional possibility, and so the value $\mu_i = 1$ corresponds to the greatest possible uncertainty (greatest possible under the condition that the values of all the other parameters are fixed). So in this case μ_i describes our degree of uncertainty.

But these different degrees of uncertainty correspond in general to different values of \bar{x} . So in principle we can express the degree of uncertainty by the value of \bar{x} , and not by the value of μ_i . This is not a purely mathematical trick: for example, in the simplest case, when $n = 2$ and $i = 2$, when $\mu_2 = 0$, it means that with certainty $x = x_1$, so it's natural to conclude that $\bar{x} = x_1$. When μ_2 increase, our degree of believe that x_2 is possible increase as well, and therefore it is natural to "shift" the overall estimate \bar{x} closer to x_2 . In general the same arguments works, so the value \bar{x} really describes our degree of belief in x_i : the closer is \bar{x} to x_i , the more we believe in x_i .

So if all the values of x_j and μ_j are fixed, except for μ_i for some i , we can express our degree of uncertainty (or degree of belief in the possibility of x_i) in two different ways: by the value of μ_i (a bigger μ_i would mean a higher possibility for x_i to be the actual value of x) and by the value of \bar{x} (the closer is \bar{x} to x_i , the bigger is the possibility that the actual value of x is x_i).

In other words, we have two different scales to represent the same degrees of belief: the scale of possible values of μ_i and the scale of possible values of \bar{x} . In these terms the transformation from μ_i to \bar{x} , defined by the formula $\bar{x} = f(x_1, \dots, x_n, \mu_1, \dots, \mu_n)$ with fixed $x_1, \dots, x_n, \mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n$, can be viewed as a transformation that "translates" the value of uncertainty in one scale into the representation of the same

degree of uncertainty in some other scale. In short, *the transformation from μ_i to \bar{x} is a transformation of degrees of certainty.*

Let's now fix the values of all the parameters, except for x_i for some i . In this case x_i can take any value from $-\infty$ to $+\infty$. Let's denote $m = \min(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $M = \max(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Then, in particular, x_i can take any value from M to ∞ . For every choice of x_i the possible values of x lie between the minimum and the maximum of all the values x_j . If $x_i > M$, then the minimum of all the x_j equals to m , and the maximum of all the x_j equals to x_i . Therefore, the bigger is x_i , the bigger is the interval of possible values for x , and the bigger is our uncertainty in x . So in this case x_i can be viewed on as describing our uncertainty in x : the bigger is x_i , the greater is our uncertainty.

On the other hand, if we increase x_i , we keep $n - 1$ possible values at the same place and shift the remaining value x_i to the right. So it is natural to expect that the resulting "overall" value increases as x_i increases. So the bigger \bar{x} , the bigger is our uncertainty. So in this case we also have two different scales to represent the same degrees of belief: the scale of possible values of x_i and the scale of possible values of \bar{x} . In these terms the transformation from x_i to \bar{x} , defined by the formula $\bar{x} = f(x_1, \dots, x_n, \mu_1, \dots, \mu_n)$ with fixed $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, \mu_1, \dots, \mu_n$, can be viewed as a transformation that "translates" the value of uncertainty in one scale into the representation of the same degree of uncertainty in some other scale. In short, *the transformation from x_i to \bar{x} is also a transformation of degrees of uncertainty.*

These considerations may sound non-mathematical: well, we can consider these values as describing degrees of certainty, so what? But this descriptions immediately becomes a mathematical fact if we take into consideration the fact that

Transformations of degrees of uncertainty have already been described. Such a description was obtained in [KK91], and it is based on the following idea. The class F of reasonable transformations of degrees of uncertainty must satisfy the following properties:

(1) If a function $x \rightarrow f(x)$ is a reasonable transformation from a scale A to some scale B , and a function $y \rightarrow g(y)$ is a reasonable transformation from B into some other scale C , then it is reasonable to demand that the transformation $x \rightarrow g(f(x))$ is also a reasonable transformation. In other words, the class F of all reasonable transformations must be closed under composition.

(2) If $x \rightarrow f(x)$ is a reasonable transformation from a scale A to scale B , then the inverse function is a reasonable transformation from B to A .

Thus, the family F must contain the inverse of every function that belongs to it, and the composition of every two functions from F . In mathematical terms, it means that F must be a *transformation group*.

(3) The next reasonable property is that this class F must be not too big: its elements must be uniquely determined by fixing finitely many parameters. In mathematical terms this can be expressed by saying that the transformation group F is *finite-dimensional*.

Comment. Actually, in our case we only need $2n - 1$ parameters to describe all the functions of one variable that we have constructed: Namely, the $2n - 1$ parameters that we have fixed ($x_1, \dots, x_n, \mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n$ or $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, \mu_1, \dots, \mu_n$) are the parameters uniquely determining this function. Of course, it is not necessary to assume that we have exhausted all possible reasonable transformations, so $2n - 1$ parameters are not necessary sufficient. However, the opposite situation, when practically every function can be viewed on as a reasonable rescaling, also does not seem intuitively true.

(4) In [KK91], we have shown that some linear functions can be represented as transformations of degrees of uncertainty between some reasonable scales.

From (1)-(4) we concluded in [KK91] that *reasonable transformations are fractionally linear*, i.e., that every reasonable transformation has the form $f(x) = (ax + b)/(cx + d)$ for some a, b, c, d .

Comment. The problem of classifying all finite-dimensional transformation groups of an n -dimensional space $R^n, n = 1, 2, 3, \dots$, that include a sufficiently big family of linear transformations, was formulated by N. Wiener (see, e.g., [W62]). His hypothesis was confirmed in [GS64], [SS65]. It turned out that for $n = 1$, then only two groups are possible: the group of all linear transformations and the group of all fractionally-linear transformations (the simplified proof for $n = 1$ is given in [K87]; for other applications of this result see [KK90], [CK91], [KQ91]).

Using this description, we can formulate the first result.

4. NATURAL AXIOMS LEAD TO THE CHOICE OF CENTROID AS A DEFUZZIFICATION PROCEDURE FOR FUZZY CONTROL

Definition 1. Suppose that a class F of functions of one real variable is fixed, and all the elements of this class are fractionally linear functions. The elements of this class will be called *reasonable transformations*.

Comment. The justification for this definition was given in the previous section.

Definition 2. Assume that some positive integer n is fixed. A function $f(x_1, \dots, x_n, \mu_1, \dots, \mu_n)$ of $2n$ real variables is called a *defuzzification* if it is defined whenever not all μ_i are equal to 0. A defuzzification is called *reasonable* if it satisfies the following demands:

D1: The value of f always lies between the smallest $\min x_i$ and the biggest $\max x_i$ of all the values.

D2 (Symmetry): The value of f must not change after any permutation, i.e.:

$$\begin{aligned} f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n, \mu_1, \dots, \mu_{i-1}, \mu_i, \mu_{i+1}, \dots, \mu_{j-1}, \mu_j, \dots, \mu_n) = \\ f(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_n, \mu_1, \dots, \mu_{i-1}, \mu_j, \mu_{i+1}, \dots, \mu_{j-1}, \mu_i, \dots, \mu_n) \end{aligned}$$

D3: If $\mu_i = 0$ for some i , then the result of defuzzification must not depend on x_i , i.e.,

$$f(\dots, x_{i-1}, x_i, x_{i+1}, \dots, \mu_{i-1}, 0, \mu_{i+1}, \dots) = f(\dots, x_{i-1}, x'_i, x_{i+1}, \dots, \mu_{i-1}, 0, \mu_{i+1}, \dots) \text{ for all } x_i \text{ and } x'_i.$$

D4: If we fix the values of all its variables, except one of them, then the resulting functions $x_i \rightarrow f(x_1, \dots, x_{i-1}, x_i, \dots, x_n, \mu_1, \dots, \mu_n)$ and $\mu_i \rightarrow f(x_1, \dots, x_n, \mu_1, \dots, \mu_{i-1}, \mu_i, \mu_{i+1}, \dots, \mu_n)$ are reasonable transformations.

From these demands we can conclude the following:

LEMMA. Every reasonable defuzzification has the form $f(x_1, \dots, x_n, \mu_1, \dots, \mu_n) = (\alpha_1 x_1 + \dots + \alpha_n x_n) / (\alpha_1 + \dots + \alpha_n)$, where $\alpha_i = \mu_i g_i$, and g_i is a multilinear symmetric function of $\mu_1, \mu_2, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n$.

Idea of the proof. According to D4, the dependency of f on x_i (if we fix all other variables) is fractional linear: $f = (ax_i + b)/(cx_i + d)$. If $c \neq 0$, then this function is not defined for $x_i = -d_i/c_i$, and we assumed that it is everywhere defined. So $c_i = 0$, and f is a linear function of each of x_i . Therefore for fixed μ_i f is a multi-linear function of the variables x_1, \dots, x_n , i.e., $f = a_0 + a_1 x_1 + \dots + a_n x_n + a_{12} x_1 x_2 + \dots$. If the terms of higher than first order are different from 0, then for big x_i they are prevailing, so f increases at least as k^2 if we multiply all x_i by k . For sufficiently big k we'll then have $f > \max x_i$, that contradicts to D1. So f can contain only linear terms: $f = a_0 + a_1 x_1 + \dots + a_n x_n$. This same assumption D1 allows us to prove that $a_0 = 0$ (take $x_i = 0$) and $\sum a_i = 1$ (take $x_1 = x_2 = \dots = x_n$). As for the dependency on μ_i , we can use the algebraic result from [KQ91] that it is a fraction of multilinear functions, and the assumptions of symmetry and independence on x_i for $\mu_i = 0$.

Comment. In other words, $\alpha_i = \mu_i (c_0 + c_1 \times \sum_{j \neq i} \mu_j + \dots)$. So we more or less know what a defuzzification can be for a finite fuzzy set. But finite fuzzy sets are only the simplest case. In real life, if an expert is not sure about the value of a physical quantity, he names the whole interval of possible values, and not just finitely many of them. A natural idea is to *approximate* the infinite fuzzy set μ by a sequence of finite sets A_n , apply the defuzzification to these finite fuzzy sets, and then take the limit of the resulting values $f(A_n)$. A natural way to approximate a continuous function μ by a set of finite pairs is to take its values on a grid. For example, if a fuzzy function μ is located on an interval $[0,1]$, we can take for A_n the set of pairs $(0, \mu(0)), (1/n, \mu(1/n)), \dots, (i/n, \mu(i/n)), \dots, (1, \mu(1))$.

If we add such a *continuity demand* to the previous list of demands, an interesting thing happens: in the above formula for α_i as $n \rightarrow \infty$, the values $\sum_{j \neq i} \mu_j$ for different i become almost equal because differ only by terms μ_i that become negligible in comparison with the whole sum. The same is true for all other sums in this expression. So in the limit the coefficients at μ_i become equal, and by dividing both the numerator and the denominator of the fraction for f by this coefficient, we arrive at a formula $f(x_1, \dots, x_n, \mu_1, \dots, \mu_n) \approx \sum \mu_i x_i / \sum \mu_i$, that in the limit $n \rightarrow \infty$ leads to a centroid formula $f(\mu(x)) = (\int x \mu(x) dx) / (\int \mu(x) dx)$.

Conclusion. We explained the centroid formula that is really very efficient in fuzzy control.

5. THE SAME MATHEMATICAL FORMALISM HELPS TO CHOOSE MEMBERSHIP FUNCTION AND HOW TO DEAL WITH UNCERTAINTIES

How to choose membership functions. Suppose that we have a fuzzy notion like "small". Then for $x = 0$ and maybe for extremely small values of some physical quantity x we are sure that it is a small value; for some sufficiently big x we are absolutely sure that it is not small. However, for intermediate values x we are

uncertain whether x is small or not. The bigger the value x , the less we are certain that this value is small. In this case there are two ways to represent our uncertainty: first, we can use a general tool that translates our uncertainty into a number (value of a membership function $\mu(x)$), and, second, we can use this very value x : because the bigger x , the bigger is our uncertainty.

Of course, this is true not for all possible x , but only for those x that lie in the “gray zone”, between the values that are for sure small (and then $\mu(x) = 1$) and that are for sure not small (and then $\mu(x) = 0$). So on every such zone x and $\mu(x)$ are two different scales that express the same uncertainty, and therefore the transformation between them (i.e., a function μ) must be a transformation between two reasonable scales.

In Section 3 we already listed the arguments that such a transformation must be expressed by a fractionally linear function. So we come to a conclusion that *a reasonable membership function must be fractionally-linear between its 0 and 1 zones*. Since linear functions are a particular case of fractionally linear ones, *we get an explanation of the above-mentioned fact that linear and fractionally-linear functions are really very efficient in fuzzy control*.

Comments. 1. In this case, unlike defuzzification, we do not come out with a single membership function, but with a small family of them (3 parameters are sufficient to describe a fractional-linear function), for which it is easy to perform an exhaustive search.

2. One can ask the following natural question: OK, we found *reasonable*, or *natural* membership functions that may be the best one to represent the experts’ knowledge. But even the best experts are not necessarily ideal controllers. So why should we stick to what the experts say? Maybe in some cases a slight modification of those membership functions can lead to a better control? For example, one of the natural criteria for a control is the smoothness of the resulting trajectory (that’s what the Japanese fuzzy-controlled trains achieve). Fuzzy control is obtained by some transformations (integration, etc) from the initial membership functions. So it is reasonable in this case to look for the most smooth membership functions. Namely, by interviewing the experts we can determine the values of the membership function in several points, and then find the “smoothest” curve going through all these points. For usual numerical criteria of smoothness we conclude that the piecewise-polynomial (spline) extrapolation is the best. This fact explains why not only piecewise-linear, but also piecewise-quadratic membership functions are successful in fuzzy control.

How to deal with uncertainties? We can use the same kind of arguments for choosing the functions $f_{\&}$ and f_{\vee} from $[0, 1] \times [0, 1] \rightarrow [0, 1]$ that correspond to $\&$ and \vee .

Suppose that a function $f_{\&}$ is fixed. Let’s fix also some uncertain event A with a certainty degree a . Then for every other event B we can express our uncertainty in two different ways: either by giving the certainty value b of this B , or by giving a certainty value $b' = f_{\&}(a, b)$ of $A\&B$. Both scales for representing uncertainty sound reasonable, so the transformation from one scale to another must belong to the class of reasonable transformations.

From this we conclude that the function $a \rightarrow f_{\&}(a, b)$ is fractionally-linear for every b . In [KK90] we proved that this demand (and a similar demand for f_{\vee}) lead to a narrow choice of possible functions $f_{\&}$ and f_{\vee} : \min , \max of traditional fuzzy logic, ab and $a + b - ab$ that correspond to the so-called *probabilistic logic* and fractional-bilinear operations from [H75]. *We thus explained the success of the traditional choices of $\&$ and \vee -functions in fuzzy control*.

Comment. In [KK90] we actually proved a stronger statement: that if we assume that a pair of operations $f_{\&}$ and f_{\vee} is *optimal* with respect to some optimality criterion, and this criterion is *invariant* with respect to reasonable rescaling transformations (i.e., if one pair was better than another in one scale, it will still be better if we start expressing uncertainty in another scale), then the optimal family must be invariant (and thus coincide with one of the above-given functions). So whatever optimality criterion we choose, the optimal functions are among those enumerated above.

But what exactly operations correspond to what criterion? Of course, since we have narrowed down the set of possible choices to a finite-dimensional family of functions, we can apply the exhaustive search and find the best choice of $f_{\&}$ and f_{\vee} . Can we avoid this exhaustive choice? Sometimes yes. For example, in [KFL91] we proved that if we are interested in maximal stability (i.e., in the smallest possible relaxation time), then the optimal choices are $f_{\&} = \min$ and $f_{\vee}(a, b) = \min(a + b, 1)$. This formulation corresponds to the tracing problem: if we lost an object we must return to it as quickly as possible.

In *docking* problems this criterion makes no sense: if we unnecessarily speed up, we’ll crash into a space station instead of smoothly approaching it. So here a reasonable criterion is *smoothness*. The same techniques as we used in [KFL91] enables us to prove that in this case the optimal choice is $f_{\&}(a, b) = ab$ and

$f_{\vee}(a, b) = \max(a, b)$. Both results are in good accordance with the results of experiments with the Space Shuttle simulator.

CONCLUSIONS. In the present paper we analyzed the process of designing the fuzzy control. In order to design a specific control procedure one must make three choices: choose membership functions that correspond to fuzzy words, choose operations corresponding to $\&$ and \vee , and choose a defuzzification procedure. For each of these stages we formulate reasonable restrictions on the set of possible choices. In all three cases these restrictions are naturally formalized in a special mathematical formalism (group theory). This formalization allows us to apply the known deep results of group theory and conclude that the reasonable choice of a membership function is linear or fractionally-linear, to show that a reasonable defuzzification is a centroid and to enumerate all possible $\&$ and \vee operations. Thus:

- (1) we give theoretical explanations to the existing semi-empirical choices; these explanations are based on a single formalism and thus form a unifying theory for all 3 stages of control design;
- (2) we formulate the class of possible choices, so that for every specific situation the optimal choice can be found by analyzing only these choices; we also actually find the best choices for several typical situations (tracking and docking);
- (3) we show that fuzzy control is not a semi-empirical craft but it can be based on the same theoretical foundations as theoretical physics and some parts of Computer Science; we hope that this formalism will be helpful to solve other problems of fuzzy control theory.

ACKNOWLEDGEMENTS. This work was supported by a NSF Grant No. CDA-9015006, NASA Research Grant No. 9-482 and the Institute for Manufacturing and Materials Management grant. The authors are greatly thankful to A. Bernat, R. Bell, C.-C. Chang, M. Gelfond, J. Perliussi, H. Przymusinska and T. Przymusinski (El Paso, TX), Yu. Gurevich (Ann Arbor, Michigan), Y. Jani (NASA Johnson Space Center), J. McCarthy and P. Suppes (Stanford), C. McDonald (White Sands) and to all the participants of the ACM Rio Grande Chapter 1991 Winter Meeting and Sigma-Xi Chapters of the University of Texas at El Paso for valuable discussions.

REFERENCES

- [CK91] J. Corbin, V. Kreinovich. *Dynamic tuning of communication network parameters: why fractionally linear formulas work well*. University of Texas at El Paso, Computer Science Department, Technical Report UTEP-CS-91-4, June 1991.
- [GS64] V. M. Guillemin, S. Sternberg. *Bull. Amer. Math. Soc.*, 1964, Vol. 70, pp. 16–47.
- [H75] H. Hamacher, H. In: *Progress in Cybernetics and Systems Research*, 1975, Vol. 3 (R. Trappl, G. J. Klir and L. Ricciardi, Eds.) Hemisphere, N. Y., pp. 276–287.
- [K92] B. Kosko. *Neural networks and fuzzy systems*, Prentice-Hall, Englewood Cliffs, NJ, 1992.
- [K87] V. Kreinovich. *Measurement Techniques*, 1987, No. 10, pp. 8–10.
- [KFLL91] V. Kreinovich, O. Fuentes, R. Lea, A. Lokshin. *Expert systems become suspiciously smart*. University of Texas at El Paso, Computer Science Department, Technical Report UTEP- CS-91-12, 1991.
- [KK90] V. Kreinovich, S. Kumar. *Proceedings of the 3rd University of New Brunswick Artificial Intelligence Workshop*, Fredericton, N.B., Canada, 1990, pp. 169–178.
- [KK91] V. Kreinovich, S. Kumar. *Cybernetics and Systems*, 1991, Vol. 22, pp. 217–222.
- [KQ91] V. Kreinovich, C. Quintana. *How does new evidence change our estimates of probabilities: Carnap's formula revisited*. University of Texas at El Paso, Computer Science Department, Technical Report UTEP-CS-91-7, 1991.
- [L88] R. N. Lea, R. N. *Telemechanics and Informatics*, 1988, Vol. 5, pp. 179–185.
- [LJB90] R. N. Lea, Y. K. Jani, H. Berenji. *Proceedings of the 5th IEEE International Symposium on Intelligent Control*, 1990, Vol. 2, pp. 903–906.
- [LTTJ89] R. N. Lea, M. Togai, J. Teichrow, Y. Jani. *Proceedings of the 3rd International Fuzzy Systems Association Congress*, 1989, pp. 23–29.
- [MYI87] S. Miyamoto, S. Yasunobu, H. Ikara. In J. C. Bezdek (Ed.) *Analysis of Fuzzy Information*. Vol. 2. CRC Press, Boca Raton, FL, 1987, pp. 59–72.
- [S54] L. J. Savage. *The foundations of statistics*. Wiley, N. Y., 1954.
- [SS65] I. M. Singer, S. Sternberg. *Journal d'Analyse Mathematique*, 1965, Vol. XV, pp. 1–113.
- [W65] N. Wiener. *Cybernetics*, MIT Press, Cambridge MA, 1962.
- [Z65] L. Zadeh. *Information and control*, 1965, Vol. 8, pp. 338–353.