

# ON ROBUSTNESS OF FUZZY LOGICS

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**Abstract.** In this paper we propose to investigate the concept of robustness of combining operations in fuzzy systems. We use a concept similar to modulus of continuity in approximation theory to characterize robustness of fuzzy logic connectives. Specifically, it is shown that min and max are the most robust  $\&$ - and  $\vee$ -operations.

## 1. INTRODUCTION

In the majority of expert systems and intelligent control systems uncertainty of expert's statements is represented by a number from the interval  $[0,1]$ :  $t(A) = 1$  means that an expert is absolutely sure in  $A$ ,  $t(A) = 0$  means that the expert is absolutely sure that  $A$  is false, and values from 0 to 1 represent different degrees of the expert's uncertainty. These numbers are called certainty values, degrees of belief, truth values, etc. If we know the degrees of belief  $t(A)$  and  $t(B)$  for two statements  $A$  and  $B$ , and nothing else is known about  $A$  and  $B$ , then what is the reasonable degree of belief in  $A\&B$ ? in  $A\vee B$ ? We must compute these estimates for  $t(A\&B)$  and  $t(A\vee B)$  from the only information that we have: from the values  $t(A)$  and  $t(B)$ . The operations  $f(a,b)$  that transform  $t(A)$  and  $t(B)$  into the estimates  $f(t(A),t(B))$  for  $t(A\&B)$  and  $t(A\vee B)$  are called correspondingly an  $\&$ -operation and an  $\vee$ -operation.

The first paper by L. Zadeh [Z65] that introduced this approach to knowledge representation proposed  $\min(a,b)$  and  $ab$  as  $\&$ -operations, and  $\max(a,b)$  and  $a + b - ab$  as  $\vee$ -operations. Zadeh himself stressed that these operations "are not the only operations in terms of which the union and intersection can be defined", and "which of these ... definitions are more appropriate depends on the context" [Z75, pp. 225–226]. Since then several dozens different  $\&$ - and  $\vee$ -operations have been proposed and successfully used. Some operations have been discovered empirically while working on real expert systems (e.g., the famous MYCIN [BS84]) or while analyzing commonsense reasoning [O77, Z78]); some of them were proposed on a more theoretical basis (see, e.g., [DP80, KF88]). A survey of such operations is given in [KQLFLKBR92].

One of the most successful applications of this type of uncertainty representation is in the area of intelligent control. This area was first outlined by L. Zadeh [CZ72] and experimentally tested by E. Mamdani [M74] in the framework of fuzzy set theory [Z65], therefore this area of research is also called *fuzzy control*. For the current state of fuzzy control the reader is referred to the surveys [S85, L90, B91].

There is a necessity for robust operations. Indeed, there are many procedures to estimate degrees of belief ([DP80, KF88]). Sometimes, an expert has a strong feeling about his degree of certainty; in such cases, this degree can be estimated more or less precisely. Sometimes, however, an expert is not sure about the strength of his own beliefs; in these

cases the known procedures can lead to drastically different estimates  $a, a'$  for the same degree of belief  $t(A)$ .

Some  $\&$ - and  $\vee$ -operations are very sensitive to such uncertainty in the sense that small changes in  $t(A)$  and  $t(B)$  can lead to absolutely different estimates for  $t(A\&B)$  and  $t(A\vee B)$ . We would like to restrict ourselves to the operations that are *robust* in the sense that they are the least sensitive to such changes.

## 2. DEFINITIONS AND THE MAIN RESULTS

Let us recall some tutorial facts.

### Definition 1.

(i). By a *binary operation* (or *operation* for short) we mean a function  $f(a, b)$  from  $[0, 1] \times [0, 1]$  into  $[0, 1]$ .

(ii). A binary operation is called a  $\&$ -operation if the following conditions are true:

- $f(0, 0) = f(0, 1) = f(1, 0) = 0, f(1, 1) = 1$ ;
- $f(a, b) = f(b, a)$  for all  $a, b$ ;
- $f(a, b) \leq a$  for all  $a$  and  $b$ .

(iii). A binary operation is called a  $\vee$ -operation if the following conditions are true:

- $f(0, 0) = 0, f(0, 1) = f(1, 0) = f(1, 1) = 1$ ;
- $f(a, b) = f(b, a)$  for all  $a, b$ ;
- $f(a, b) \geq a$  for all  $a$  and  $b$ .

*Remark.* The above binary operations are slightly more general than the usual  $t$ -norms and  $t$ -conorms in the literature [GN85].

**Definition 2.** Suppose that a binary operation  $f(a, b)$  is given. We say that a  $\delta$ -input uncertainty leads to a  $\leq \alpha$ -output error, if for every  $a, a', b, b'$ , for which  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , we have  $|f(a, b) - f(a', b')| \leq \alpha$ .

**Definition 3.** Suppose that  $f(a, b)$  is a binary operation, and  $\delta > 0$  is a positive real number. By a  $\delta$ -robustness of an operation  $f(a, b)$  we mean the smallest of real numbers  $\alpha$ , for which a  $\delta$ -input uncertainty leads to a  $\leq \alpha$ -output error. The  $\delta$ -robustness of an operation  $f(a, b)$  will be denoted by  $r_f(\delta)$ .

*Remark.* It is easy to check that  $r_f(\delta) = \sup\{|f(a, b) - f(a', b')| : |a - a'| \leq \delta, |b - b'| \leq \delta\}$ . When  $f$  is continuous,  $r_f(\delta)$  is the well-known modulus of continuity of  $f$  [L66]. The above sup is in fact max: see Proposition 1 below; its proof, as well as all the proofs of the results in this section are in Section 3 for easy reading.

**PROPOSITION 1.** For every operation  $f(a, b)$ , and for every  $\delta > 0$ , there exists a  $\delta$ -robustness (i.e., the smallest of real numbers  $\alpha$ , for which a  $\delta$ -input uncertainty leads to a  $\leq \alpha$ -output error).

In order to compare different operations in terms of robustness, we will proceed as in standard decision theory, where  $r_f(\delta)$  plays the role of the “risk” (see, e.g., [BG79]).

**Definition 4.** We say that operations  $f(a, b)$  and  $g(a, b)$  are *equally robust* if for every  $\delta$ ,  $r_f(\delta) = r_g(\delta)$ . We say that an operation  $f(a, b)$  is *more robust* than an operation  $g(a, b)$ , if for every  $\delta$ ,  $r_f(\delta) \leq r_g(\delta)$ , and at least for one  $\delta > 0$ ,  $r_f(\delta) < r_g(\delta)$ .

**Definition 5.** We say that an  $\&$ -operation  $f(a, b)$  is *the most robust  $\&$ -operation*, if it is either more robust, or equally robust than any other  $\&$ -operation. We say that an  $\vee$ -operation  $g(a, b)$  is *the most robust  $\vee$ -operation*, if it is either more robust, or equally robust than any other  $\vee$ -operation.

**THEOREM 1.**  $f(a, b) = \min(a, b)$  is the most robust  $\&$ -operation.

**THEOREM 2.**  $f(a, b) = \max(a, b)$  is the most robust  $\vee$ -operation.

*Remarks.*

1. In [KK90], [KQL91], [KQLFLKBR92] general optimization problems are analyzed on the set of all possible  $\&$ - and  $\vee$ - operations. As a result of this mathematical analysis, lists are given that include all  $\&$ - and  $\vee$ - operations that can be optimal under reasonable optimality criteria. Our Theorems 1 and 2 are in good accordance with that general result, because both  $\min$  and  $\max$  are elements of those lists.

2. Similar questions of robustness in the context of neural networks are analyzed in [DF92].

3. It is interesting to know to what extent the robustness functions that correspond to  $\min$  and  $\max$  are smaller than those of the other binary operations: are they smaller in a few points only, or essentially smaller for all  $\delta$ ? The answer is given by the following theorems:

**THEOREM 3.** Suppose that  $f(a, b)$  is an  $\&$ -operation, and  $f(a, b)$  is different from  $\min$ . Then there exists a positive real number  $\Delta > 0$  and positive real number  $C < 1$  such that for all  $\delta < \Delta$ ,  $r_{\min}(\delta) \leq Cr_f(\delta)$ .

**THEOREM 4.** Suppose that  $f(a, b)$  is a  $\vee$ -operation, and  $f(a, b)$  is different from  $\max$ . Then there exists a positive real number  $\Delta > 0$  and positive real number  $C < 1$  such that for all  $\delta < \Delta$ ,  $r_{\max}(\delta) \leq Cr_f(\delta)$ .

The following Theorem describes  $\delta$ -robustness for several other operations:

**THEOREM 5.**

- 1) if  $f(a, b) = ab$ , then  $r_f(\delta) = 2\delta - \delta^2$ ;
- 2) if  $f(a, b) = a + b - ab$ , then  $r_f(\delta) = 2\delta - \delta^2$ ;
- 3) if  $f(a, b) = \min(a + b, 1)$ , then  $r_f(\delta) = \min(2\delta, 1)$ .

*Remarks.*

1. In these three cases,  $\lim_{\delta \rightarrow 0} r_f(\delta)/r_{\min}(\delta) = 2$ , so in Theorems 3 and 4 we can take  $C = 1/2 + \alpha$  for arbitrary small  $\alpha > 0$ .

2. From Theorem 5 we can conclude that the operations  $ab$  and  $a + b - ab$  are equally robust, and that  $f(a, b) = \min(a + b, 1)$  is more robust than both of them.

3. The fact that  $ab$  and  $a + b - ab$  are equally robust stems from the fact that in general *dual* operations have the same modulus of continuity, where  $g(a, b)$  is *dual* to an  $f(a, b)$  if  $g(a, b) = 1 - f(1 - a, 1 - b)$ .

**PROPOSITION 2.** *Dual operations are equally robust.*

### 3. PROOFS

**Proof of Proposition 1.** The set  $S$  of all real numbers  $\alpha$ , for which a  $\delta$ -input uncertainty leads to a  $\leq \alpha$ -output error, is bounded from below (by 0), and therefore, has an infimum (the greatest lower bound)  $r$ .  $r$  is the value of  $\delta$ -robustness. Indeed, since  $r$  is the greatest lower bound of the set  $S$ , for every positive integer  $k$  there exists a number  $r_k \in S$  such that  $r_k < r + 1/k$ . According to the definition of  $S$ , from  $r_k \in S$  we conclude that if  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , then  $|f(a, b) - f(a', b')| \leq r_k$ . Letting  $k \rightarrow \infty$ , we conclude that  $|f(a, b) - f(a', b')| \leq \lim_k r_k = r$ . Q.E.D.

**Proof of Theorem 1.**

1°. Let us first prove that, as in the case of  $t$ -norm, we have  $f(a, b) \leq \min(a, b)$  for any  $\&$ -operation  $f$ . Since  $f(a, b) \leq a$  and  $f(a, b) = f(b, a) \leq b$ , it follows that  $f(a, b) \leq \min(a, b)$ .

2°. Next,  $r_{\min}(\delta) = \delta$ .

Indeed, for  $|a - a'| \leq \delta$ , we have  $a \leq a' + \delta$  and likewise  $b \leq b' + \delta$ . Hence,  $\min(a, b) \leq \min(a' + \delta, b' + \delta) = \min(a', b') + \delta$ , therefore,  $\min(a, b) \leq \min(a', b') + \delta$ . Likewise,  $\min(a', b') \leq \min(a, b) + \delta$ , so  $-\delta \leq \min(a, b) - \min(a', b') \leq \delta$ , and  $|\min(a, b) - \min(a', b')| \leq \delta$ . Take  $a = b = \delta$ ,  $a' = b' = 0$ . Then  $|\min(a, b) - \min(a', b')| = \delta$ , and therefore, the output error is precisely  $\delta$ . So, we cannot take  $\alpha < \delta$ , and so the  $\delta$ -robustness of  $\min$  is really equal to  $\delta$ .

3°. For every  $\&$ -operation  $f(a, b)$ :  $r_f(\delta) \geq r_{\min}(\delta) = \delta$ .

Suppose that for some  $\delta \in (0, 1)$ ,  $r_f(\delta) < \delta$ . This means that if  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , then  $|f(a, b) - f(a', b')| \leq r_f(\delta) < \delta$ . In particular, if we take  $a = b = 1$  and  $a' = b' = 1 - \delta$ , we conclude that  $|f(1, 1) - f(1 - \delta, 1 - \delta)| < \delta$ . But according to the definition of a  $\&$ -operation,  $f(1, 1) = 1$ , therefore, this inequality turns into  $|1 - f(1 - \delta, 1 - \delta)| < \delta$ . Hence,  $1 - f(1 - \delta, 1 - \delta) \leq |1 - f(1 - \delta, 1 - \delta)| < \delta$ , therefore,  $f(1 - \delta, 1 - \delta) > 1 - \delta$ . But we have already proved in 1° that  $f(a, b) \leq \min(a, b)$ , therefore,  $f(1 - \delta, 1 - \delta) \leq 1 - \delta$ . These two inequalities contradict to each other. Therefore, our assumption that  $r_f(\delta) < \delta$  is incorrect. Hence,  $r_f(\delta) \geq \delta$ .

4°.  $\min(a, b)$  is the only  $\&$ -operation, for which  $r_f(\delta) = \delta$  for all  $\delta$ .

Indeed, suppose that  $f$  is different from  $\min$ . Then for some  $a$  and  $b$ ,  $f(a, b) \neq \min(a, b)$ , hence  $f(a, b) < \min(a, b)$ . Without loss of generality, assume  $a \leq b$ , resulting in  $f(a, b) < \min(a, b) = a$ . For  $a' = b' = 1$ , we have  $|a - a'| = 1 - a$ ,  $|b - b'| = 1 - b \leq 1 - a$ , but  $|f(a, b) - f(a', b')| = 1 - f(a, b) > 1 - a$ . So, for  $\delta_0 = 1 - a$ ,  $r_f(\delta_0) > \delta_0$ . Q.E.D.

**Proof of Theorem 2** is similar.

**Proof of Theorem 3.**

1°. In part 4° of the proof of Theorem 1, we showed that if an  $\&$ -operation  $f(a, b)$  is different from  $\min(a, b)$ , then we can find the values  $a, b, a' = 1$  and  $b' = 1$ , for which  $a \leq b, |a - a'| = \delta_0, |b - b'| \leq \delta_0$ , and  $D > \delta_0$ , where  $D = f(a', b') - f(a, b)$  and  $\delta_0 = 1 - a$ .

From this we concluded that  $r_f(\delta_0) \geq D > \delta_0$ . Let us denote the ratio  $\delta_0/D$  by  $c$ . From  $D > \delta_0$  we conclude that  $c < 1$ .

2°. Let us now prove that for every positive integer  $n$ ,  $r_f(\delta_0/n) \geq D/n$ .

Indeed, let us divide the intervals  $(a, a')$  and  $(b, b')$  into  $n$  equal parts, i.e., let us consider the values  $a_i = a + i(1-a)/n$  and  $b_i = b + i(1-b)/n$ , where  $i = 0, 1, 2, \dots, n$ . Then, for every  $i$  from 0 to  $n-1$ ,  $|a_i - a_{i+1}| = (1-a)/n = \delta_0/n$ , and  $|b_i - b_{i+1}| = (1-b)/n \leq \delta_0/n$ . Therefore, according to the definition of  $\delta_0$ -robustness,  $|f(a_i, b_i) - f(a_{i+1}, b_{i+1})| \leq r_f(\delta_0/n)$ .

Let us now use the fact that  $a_0 = a, b_0 = b, a_n = a', b_n = b'$ , and  $f(a_0, b_0) - f(a_n, b_n) = -D$ . We have

$$f(a_0, b_0) - f(a_n, b_n) = \\ (f(a_0, b_0) - f(a_1, b_1)) + (f(a_1, b_1) - f(a_2, b_2)) + \dots + (f(a_{n-1}, b_{n-1}) - f(a_n, b_n)).$$

The absolute value of each of  $n$  terms in the sum is bounded by  $r_f(\delta_0/n)$ , therefore, the absolute value of the entire sum is bounded by  $nr_f(\delta_0/n)$ . But this sum equals to  $f(a_0, b_0) - f(a_n, b_n) = -D$ , so its absolute value is  $D$ , and thus  $D \leq nr_f(\delta_0/n)$ . Hence,  $r_f(\delta_0/n) \geq D/n$ .

3°. In order to continue the proof, we need to prove the following simple fact: if  $\delta < \delta'$ , then  $r_f(\delta) \leq r_f(\delta')$  (it is well known for moduli of continuity, but we will reproduce a proof for completeness).

Indeed, from the definition of the function  $r_f(\delta)$  we conclude that if  $|a - a'| \leq \delta'$  and  $|b - b'| \leq \delta'$ , then  $|f(a, b) - f(a', b')| \leq r_f(\delta')$ . But if  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , then (since  $\delta < \delta'$ )  $|a - a'| \leq \delta'$  and  $|b - b'| \leq \delta'$ , and therefore,  $|f(a, b) - f(a', b')| \leq r_f(\delta')$ . So, a  $\delta$ -input uncertainty leads to a  $\leq \alpha$ -output error, where  $\alpha = r_f(\delta')$ . Since by definition  $r_f(\delta)$  is the smallest of all such values  $\alpha$ , we conclude that  $r_f(\delta) \leq r_f(\delta')$ .

4°. Let us now take any real number  $C$  between  $c = \delta_0/D$  and 1 ( $c < C < 1$ ), and prove that there exists a  $\Delta > 0$  such that for all  $\delta < \Delta$ , we have  $r_{\min}(\delta) \leq Cr_f(\delta)$ .

Since  $r_{\min}(\delta) = \delta$  for all  $\delta$ , the inequality can be rewritten as  $\delta \leq Cr_f(\delta)$ , or, equivalently,  $r_f(\delta) \geq \delta/C$ .

We already know how to estimate the values of  $r_f(\delta)$  for  $\delta = \delta_0/n$ , where  $n = 1, 2, 3, \dots$ . So, to get the estimates for arbitrary  $\delta$ , we can use these known estimates. For every  $\delta < \delta_0$ , we want to find an  $n$  such that  $\delta_0/(n+1) \leq \delta \leq \delta_0/n$ . This inequality is equivalent to  $(n+1)/\delta_0 \geq 1/\delta \geq n/\delta_0$ , which, after multiplying both sides by  $\delta_0$ , turns out to be equivalent to the inequality  $n \leq \delta_0/\delta \leq n+1$ . Therefore, we can take for  $n$  the integer part  $\lfloor \delta_0/\delta \rfloor$  of the ratio  $\delta_0/\delta$ . From 3° we can conclude that  $r_f(\delta) \geq r_f(\delta_0/(n+1))$ . In 2° we proved that  $r_f(\delta_0/(n+1)) \geq D/(n+1)$ . Therefore,  $r_f(\delta) \geq D/(n+1)$ . We defined  $c$  as  $c = \delta_0/D$ ; so,  $D = \delta_0/c$ , so  $r_f(\delta) \geq \delta_0/(c(n+1))$ .

We want to get an inequality  $r_f(\delta) \geq \delta/C$ . We will be able to deduce this inequality from the one that we have just proved if  $\delta_0/(c(n+1)) \geq \delta/C$ . Since  $\delta \leq \delta_0/n$ , this inequality is valid if  $\delta_0/Cn \leq \delta_0/(c(n+1))$ . Dividing both sides by  $\delta_0$  and then inverting both sides, we get an equivalent inequality  $Cn \geq c(n+1)$ , which, in its turn, is equivalent to  $(C-c)n \geq c$

and  $n \geq c/(C - c)$ . Therefore, if  $n \geq c/(C - c)$ , then for  $\delta \leq \delta_0/n$  we get the desired inequality  $r_f(\delta) \geq \delta/c$ .

The inequality  $n \geq c/(C - c)$  is valid for all  $n$  starting from  $N = \lfloor c/(C - c) \rfloor + 1$ . Therefore, the desired inequality  $r_f(\delta) \geq \delta/c$  is true for all  $\delta < \Delta$ , where  $\Delta = \delta_0/N$ . Q.E.D.

**Proof of Theorem 4** is similar.

Before proving Theorem 5 let us prove Proposition 2.

**Proof of Proposition 2.**

1°. Let us first prove that if  $f$  and  $g$  are dual, i.e.,  $g(a, b) = 1 - f(1 - a, 1 - b)$ , then for every  $\delta$ ,  $r_g(\delta) \geq r_f(\delta)$ .

Indeed, suppose that  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , and let us prove that  $|g(a, b) - g(a', b')| \leq r_f(\delta)$ . Since  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , we have  $|A - A'| = |a - a'| \leq \delta$  and  $|B - B'| = |b - b'| \leq \delta$ , where we denoted  $A = 1 - a$ ,  $A' = 1 - a'$ ,  $B = 1 - b$ , and  $B' = 1 - b'$ . Due to the definition of  $r_f(\delta)$ , we can conclude that  $|f(A, B) - f(A', B')| \leq r_f(\delta)$ . But  $g(a, b) = 1 - f(A, B)$  and  $g(a', b') = 1 - f(A', B')$ , therefore  $|g(a, b) - g(a', b')| = |f(A, B) - f(A', B')| \leq r_f(\delta)$ . So, for  $\alpha = r_f(\delta)$ , if  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , then  $|g(a, b) - g(a', b')| \leq \alpha$ . Since  $r_g(\delta)$  is defined as the smallest of all  $\alpha$  with this property, we conclude that  $r_g(\delta) \leq r_f(\delta)$ .

2°. One can easily check that if  $g$  is dual to  $f$ , then  $f$  is dual to  $g$ . Therefore, we have both  $r_g(\delta) \leq r_f(\delta)$  and  $r_f(\delta) \leq r_g(\delta)$ , hence  $r_g(\delta) = r_f(\delta)$ . Q.E.D.

**Proof of Theorem 5.**

1) We must prove, first, that if  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , then  $|ab - a'b'| \leq 2\delta - \delta^2$ , and, second, that there exist such  $a, b, a', b'$  for which  $|a - a'| \leq \delta$ ,  $|b - b'| \leq \delta$ , and  $|ab - a'b'| = 2\delta - \delta^2$ .

The second statement is easy to prove: take  $a = b = 1$ ,  $a' = b' = 1 - \delta$ , then  $|ab - a'b'| = 1 - (1 - \delta)^2 = 2\delta - \delta^2$ . Let us now prove the first one.

Let us denote  $|a - a'|$  by  $\Delta_a$ , and  $|b - b'|$  by  $\Delta_b$ . Then  $\Delta_a \leq \delta$  and  $\Delta_b \leq \delta$ . Without losing any generality we can assume that  $a \geq a'$ . Then  $a' = a - \Delta_a$ . With respect to  $b$  and  $b'$ , there are two possible cases:  $b \geq b'$  and  $b < b'$ . Let us consider both of them.

If  $b \geq b'$ , then  $b' = b - \Delta_b$ , and  $ab \geq a'b'$ , so the desired absolute value  $d = |ab - a'b'|$  can be computed as follows:  $d = |ab - a'b'| = ab - a'b' = ab - (a - \Delta_a)(b - \Delta_b) = a\Delta_b + b\Delta_a - \Delta_a\Delta_b$ . Since  $a \leq 1$  and  $b \leq 1$ , we have  $d \leq \Delta_a + \Delta_b - \Delta_a\Delta_b$ . The right-hand side of this inequality can be expressed as  $1 - (1 - \Delta_a)(1 - \Delta_b)$ . Therefore, it is a monotonely increasing function of both  $\Delta_a$  and  $\Delta_b$ . So, its maximal value is attained when both of these variables take their biggest possible values. Since  $\Delta_a \leq \delta$  and  $\Delta_b \leq \delta$ , the maximal possible value is attained when  $\Delta_a = \Delta_b = \delta$ , and is equal to  $2\delta - \delta^2$ . Therefore,  $d \leq \Delta_a + \Delta_b - \Delta_a\Delta_b \leq 2\delta - \delta^2$ . So for this case the desired inequality is proved.

Let us now consider the case when  $b < b'$ . Then  $b = b' - \Delta_b$ , and  $d = |ab - a'b'| = |a(b' - \Delta_b) - (a - \Delta_a)b'| = |a\Delta_b - b'\Delta_a|$ . Let us consider two subcases: when the expression under the absolute value is positive or negative, i.e., when  $a\Delta_b \geq b'\Delta_a$  and  $a\Delta_b < b'\Delta_a$ .

In the first subcase,  $d = a\Delta_b - b'\Delta_a$ , therefore,  $d \leq b'\Delta_a$ . Since  $\Delta_a \leq \delta$  and  $b' \leq 1$ , we get  $d \leq \delta$ .

In the second subcase similarly  $d = b'\Delta_a - a\Delta_b \leq b'\Delta_a \leq \delta$ . So, in both cases  $d \leq \delta$ .

So, to complete the proof, it is sufficient to show that  $\delta \leq 2\delta - \delta^2$  for all  $\delta$  from 0 to 1. Indeed, by dividing both sides by  $\delta$  and moving all terms to the right-hand side, we conclude that this inequality is equivalent to  $0 \leq 1 - \delta$ , which is certainly true for  $\delta \leq 1$ . 2) follows from 1) and Proposition 2.

3) Let us first consider the case, when  $\delta < 1/2$ . Then  $2\delta < 1$ , and  $\min(2\delta, 1) = 2\delta$ . Let us prove that in this case, if  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , then  $|f(a, b) - f(a', b')| \leq 2\delta$ .

Indeed, if  $|a - a'| \leq \delta$  and  $|b - b'| \leq \delta$ , then  $|(a + b) - (a' + b')| = |(a - a') + (b - b')| \leq 2\delta$ . In particular, this means that  $a' + b' \leq a + b + 2\delta$ . Evidently,  $a' + b' \leq 1$ , therefore,  $a' + b' \leq 1 < 1 + 2\delta$ . So,  $a' + b'$  is not bigger than the smallest of these two numbers:  $a' + b' \leq \min(a + b + 2\delta, 1 + 2\delta)$ . But  $\min(a + b + 2\delta, 1 + 2\delta) = \min(a + b, 1) + 2\delta = f(a, b) + 2\delta$ . So,  $a' + b' \leq f(a, b) + 2\delta$ . Since  $f(a', b') = \min(a' + b', 1)$  and therefore,  $f(a', b') \leq a' + b'$ , we conclude that  $f(a', b') \leq f(a, b) + 2\delta$ . In a similar manner we can prove that  $f(a, b) \leq f(a', b') + 2\delta$ . Combining these two inequalities, we conclude that  $|f(a, b) - f(a', b')| \leq 2\delta$ . So, for  $\delta < 1/2$ ,  $r_f(\delta) \leq 2\delta$ .

Let us now show that  $\alpha = 2\delta$  is the smallest value, for which  $\delta$ -input uncertainty leads to a  $\leq \alpha$ -output error, and thus,  $r_f(\delta) = 2\delta$ . Indeed, if we take  $a = b = 0$ ,  $a' = b' = \delta$ , then  $|a - a'| \leq \delta$ ,  $|b - b'| \leq 2\delta$ , and  $|f(a, b) - f(a', b')| = |0 - 2\delta| = 2\delta$ , so the values  $\alpha < 2\delta$  do not work in this case. So, for  $\delta < 1/2$ , we proved that  $r_f(\delta) = 2\delta$ .

Now let us consider the case when  $\delta \geq 1/2$ . In this case,  $\min(2\delta, 1) = 1$ . If we take  $a = b = 0$ ,  $a' = b' = \delta$ , then  $f(a, b) = 0$ ,  $f(a', b') = 1$ ,  $|a - a'| \leq \delta$ ,  $|b - b'| \leq 2\delta$ , and  $|f(a, b) - f(a', b')| = |0 - 1| = 1$ . Therefore, nothing smaller than 1 can serve as  $\alpha$ , hence  $r_f(\delta) = 1$ . Q.E.D.

#### 4. CONCLUSIONS

As far as combining degrees of belief of experts is concerned, in situations where estimates can vary drastically, it is reasonable to use *robust* fuzzy logic connectives, which are the least sensitive to these variations. We have proved that in this situation, the dual pair  $\min(a, b)$ ,  $\max(a, b)$  are the most robust operations.

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