

INTERPOLATION THAT LEADS TO THE NARROWEST INTERVALS, AND ITS APPLICATION TO EXPERT SYSTEMS AND INTELLIGENT CONTROL

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Abstract. In many real-life situations, we want to reconstruct the dependency $y = f(x_1, \dots, x_n)$ from the known experimental results $x_i^{(k)}, y^{(k)}$. In other words, we want to interpolate the function f from its known values $y^{(k)} = f(x_1^{(k)}, \dots, x_n^{(k)})$ in finitely many points $\vec{x}^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$, $1 \leq k \leq N$. There are many functions that go through given points. How to choose one of them?

The main goal of finding f is to be able to predict y based on x_i . If we get x_i from measurements, then usually, we only get intervals that contain x_i . As a result of applying f , we get an interval \mathbf{y} of possible values of y . It is reasonable to choose f for which the resulting interval is the narrowest possible. In this paper, we formulate this choice problem in mathematical terms, solve the corresponding problem for several simple cases, and describe the application of these solutions to intelligent control.

1. INTERPOLATION: BRIEF FORMULATION OF THE PROBLEM; OUR IDEA

In many real-life applications, we know that a physical quantity y depends on the quantities x_1, \dots, x_n , but we do not know the exact dependency. To determine this dependency, we measure the values of x_i and y in several real-life situations. As a result, we get the values $y^{(k)} = f(x_1^{(k)}, \dots, x_n^{(k)})$ of the unknown function f at several points $\vec{x}^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$. From the results of these measurements, we want to reconstruct the function f , i.e., we want to know the values $f(\vec{x})$ for all other points \vec{x} . In mathematics, this problem is called *interpolation* (if \vec{x} is in between $\vec{x}^{(k)}$), and *extrapolation* otherwise. In real-life applications, this problem is also called *identification*.

There are many ways to extend a function defined in finitely many points to the entire area. So, we must somehow choose one of the resulting functions f . In some cases, we have some a priori information about f : e.g., that $f(\vec{x}) \geq 0$, or that f is monotonic. In the majority of real-life situations, however, this information is not sufficient to pick a function f uniquely, so, we need some additional choice criteria.

In interval computations (see, e.g., [M66], [M79], [AH83], [RR84]), several such criteria have been formulated; these criteria and corresponding algorithms are described, e.g., in [S73], [S73a], [KP81], [P84], [IKK85], [WP86], [WP86a], [KPK87], [ST88], [VBS89], [VS89], [VBS90], [WP90], [K91], [D92], [V92], [JW93].

In this paper, we propose a *new criterion*. Its idea is as follows: The goal of the interpolation (extrapolation) is to predict y based on our knowledge of x_i . Measurements are never absolutely accurate, so, after measuring x_i , we will only get *intervals* \mathbf{x}_i of possible values of x_i . After applying the interpolated function f to these intervals, we get the interval $\mathbf{y} = \mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of possible values of y . We would like to make the resulting predictions as precise as possible. For the same accuracies of measuring x_i (i.e., in mathematical terms, for the same widths of the intervals \mathbf{x}_i), different interpolation methods will lead to intervals \mathbf{y} of different width. For some methods, the resulting precision in y will be comparable with the accuracy with which we measured x_i , for some other extrapolation techniques, we will get y with much lower accuracy. For example, if $n = 1$, and we know that $f(0) = f(1) = 0$, then the constant interpolation $f(x) = 0$ will lead to $f([0.3, 0.5]) = 0$, while the sine interpolation $f(x) = \sin(100\pi x)$ (which is quite consistent with the initial data $f(0) = f(1) = 0$) will lead to $f([0.3, 0.5]) = [-1, 1]$.

It is natural to choose an interpolation method that does not add unnecessary additional uncertainty to the inaccuracy of measuring x_i , i.e., a method that *minimizes* the guaranteed width of \mathbf{y} for a given width δ of intervals \mathbf{x}_i .

We will see that it is sometimes impossible to minimize the width of \mathbf{y} for *all* δ . In this case, it is necessary to recall that the reason for this minimization is that we do not want to ruin the accuracy of measuring x_i . This is not such a big problem when we measure x_i with low accuracy, because in this case, the accuracy will be low anyway, but it is important for accurate measurements. So, if we cannot guarantee the narrowest intervals for all δ , we can at least try to guarantee the narrowest intervals for *all sufficiently small* δ .

This idea does not always lead us to a unique choice of interpolation. However, as we will see, in several simple case, it does. Several real-life applications for the resulting procedures (to expert systems and intelligent control) are presented.

2. DEFINITIONS AND THE MAIN RESULTS

2.1. Definitions

Definition 1. By an *interpolation problem*, we mean the tuple $\mathcal{P} = (n, U, \mathcal{F}, N, \vec{x}^{(1)}, \dots, \vec{x}^{(N)}, y^{(1)}, \dots, y^{(N)})$, where:

n is a positive integer;

U is a subset of R^n ;

\mathcal{F} is a set of functions from U to R ;

N is a positive integer;

$\vec{x}^{(k)}$ ($1 \leq k \leq N$) are elements of U ;

$y^{(k)}$ ($1 \leq k \leq N$) are real numbers.

We say that a function $f \in \mathcal{F}$ is a *possible solution* to the interpolation problem if $f(\vec{x}^{(k)}) = y^{(k)}$ for all k .

Definition 2. Let f be a possible solution to an an interpolation problem \mathcal{P} , and let $\delta > 0$ be a positive real number. We say that a δ -input uncertainty leads to a $\leq \alpha$ -output error, if for every $\vec{x} \in U$ and $\vec{x}' \in U$, for which $|x_i - x'_i| \leq \delta$ for all i , we have $|f(\vec{x}) - f(\vec{x}')| \leq \alpha$.

Remark. In other words, if for all i , $x'_i \in [x_i - \delta, x_i + \delta]$, then $f(\vec{x}') \in [f(\vec{x}) - \alpha, f(\vec{x}) + \alpha]$.

Definition 3. Let f be a possible solution to an interpolation problem \mathcal{P} , and let $\delta > 0$ be a positive real number. By a δ -sensitivity of a function $f(\vec{x})$ we mean the smallest of real numbers α , for which a δ -input uncertainty leads to a $\leq \alpha$ -output error. The δ -sensitivity of a function $f(\vec{x})$ will be denoted by $s_f(\delta)$.

Remark. It is easy to check that

$$s_f(\delta) = \sup\{|f(x_1, \dots, x_n) - f(x'_1, \dots, x'_n)| : |x_1 - x'_1| \leq \delta, \dots, |x_n - x'_n| \leq \delta\}.$$

When f is continuous, $s_f(\delta)$ is the well-known modulus of continuity of f [L66]. The above sup is in fact max: see Proposition 1 below; for reader's convenience, its proof, as well as all the proofs of the results are given in the last Section.

PROPOSITION 1. For every function $f(\vec{x})$, and for every $\delta > 0$, there exists a δ -sensitivity (i.e., the smallest of real numbers α , for which a δ -input uncertainty leads to a $\leq \alpha$ -output error).

Definition 4.

- We say that functions $f(\vec{x})$ and $g(\vec{x})$ are equally sensitive if for every δ , $s_f(\delta) = s_g(\delta)$.
- We say that a function $f(\vec{x})$ is less sensitive than a function $g(\vec{x})$, if for every δ , $s_f(\delta) \leq s_g(\delta)$, and at least for one $\delta > 0$, $s_f(\delta) < s_g(\delta)$.
- We say that a function $f(\vec{x})$ is asymptotically less sensitive than a function $g(\vec{x})$, if there exists a $\Delta > 0$ such that for every $\delta < \Delta$, $s_f(\delta) < s_g(\delta)$.
- We say that a function $f(\vec{x})$ is the least sensitive solution to an interpolation problem \mathcal{P} if f is a possible solution, and f is either less sensitive, or equally sensitive than any other possible solution.
- We say that a function $f(\vec{x})$ is the least asymptotically sensitive solution to an interpolation problem \mathcal{P} if f is a possible solution, and f is asymptotically less sensitive than any other possible solution.

2.2. Main result: 1-D case

Definition 5. By the simplest 1-D interpolation problem, we mean the interpolation problem, for which $n = 1$, $U = [a_1, a_2]$, \mathcal{F} = the set of all functions from U to R , $N = 2$, $x^{(1)} = a_1$, and $x^{(2)} = a_2$.

Comment. In other words, we are looking for a function $f : [a_1, a_2] \rightarrow R$ for which $f(a_1) = y^{(1)}$ and $f(a_2) = y^{(2)}$. It turns out that for this problem, there exists no least sensitive solution, but there does exist the least *asymptotically* sensitive one:

PROPOSITION 2. For every possible solution f of the simplest 1-D interpolation problem, there exists another possible solution g and a real number $\delta > 0$ such that $s_g(\delta) < s_f(\delta)$.

THEOREM 1. For every simplest 1-D interpolation problem:

- the linear function $f_0(x) = y^{(1)} + (x - a_1)(y^{(2)} - y^{(1)}) / (a_2 - a_1)$ is the least asymptotically sensitive solution;
- for every other possible solution f , there exists a positive real number $\Delta > 0$ and a positive real number $C < 1$ such that for all $\delta < \Delta$, $s_{f_0}(\delta) \leq C \cdot s_f(\delta)$.

2.3. Auxiliary results: 1-D case

Comment. The motivation for these results will be presented later.

Definition 6. Let $n = 1$ and $U = R$. Let us define the following 1-D interpolation problems:

- Let $x_- < x_+$ be two real numbers.
 - By \mathcal{P}_{pb} , we will denote the following problem: \mathcal{F} = the set of all functions $f : R \rightarrow R$ for which $f(x) = 0$ for $x \leq x_-$, and $f(x) = 1$ for $x > x_+$; $N = 2$, $f(x_-) = 0$, $f(x_+) = 1$.
 - By \mathcal{P}_{nb} , we will denote the following problem: \mathcal{F} = the set of all functions $f : R \rightarrow R$ for which $f(x) = 1$ for $x \leq x_-$, and $f(x) = 0$ for $x > x_+$; $N = 2$, $f(x_-) = 1$, $f(x_+) = 0$.
- Let $x_- < x_0 < x_+$ be real numbers such that $x_+ - x_0 = x_0 - x_-$. By \mathcal{P}_n , we mean the following problem: \mathcal{F} = the set of all functions $f : R \rightarrow R$ for which $f(x) = 0$ for $x \leq x_-$ and for $x \geq x_+$; $N = 3$, $f(x_-) = f(x_+) = 0$ and $f(x_0) = 1$.
- Let $x_- < a_- < a_+ < x_+$ be real numbers such that $x_+ - a_+ = a_- - x_-$. By $\mathcal{P}_{n'}$, we mean the following problem: \mathcal{F} = the set of all functions $f : R \rightarrow R$ for which $f(x) = 0$ for $x \leq x_-$ and for $x \geq x_+$, and $f(x) = 1$ for $a_- \leq x \leq a_+$; $N = 4$, $f(x_-) = f(x_+) = 0$, $f(a_-) = f(a_+) = 1$.

PROPOSITION 3. For the interpolation problems enumerated in Definition 6, the following piecewise linear interpolation functions are the least asymptotically sensitive solutions:

- For \mathcal{P}_{pb} , $f_0(x) = (x - x_-)/(x_+ - x_-)$ for $x_- \leq x \leq x_+$.
- For \mathcal{P}_{nb} , $f_0(x) = 1 - (x - x_-)/(x_+ - x_-)$ for $x_- \leq x \leq x_+$.
- For \mathcal{P}_n , $f_0(x) = |x - x_0|/(x_+ - x_0)$ for $x_- \leq x \leq x_+$.
- For $\mathcal{P}_{n'}$, $f_0(x) = (x - x_-)/(a_- - x_-)$ for $x \in [x_-, a_-]$ and $f_0(x) = (x_+ - x)/(x - a_+)$ for $x \in [a_+, x_+]$.

Definition 7. Let us define the following interpolation problem \mathcal{P}_- : $n = 1$, $U = [0, 1]$, \mathcal{F} consists of all decreasing functions $f : [0, 1] \rightarrow [0, 1]$ such that $f(f(x)) = x$ for all x ; $N = 2$, and the interpolation equations are $f(0) = 1$ and $f(1) = 0$ (i.e., $x^{(1)} = 0$, $x^{(2)} = 1$, $y^{(1)} = 1$, and $y^{(2)} = 0$).

PROPOSITION 4.

- $f_0(x) = 1 - x$ is the least sensitive solution to the interpolation problem \mathcal{P}_- .
- If f is a possible solution to \mathcal{P}_- , and $f \neq f_0$, then there exists a positive real number $\Delta > 0$ and a positive real number $C < 1$ such that for all $\delta < \Delta$, $s_{f_0}(\delta) \leq C s_f(\delta)$.

2.4. 2-D problems

Definition 8. Let $n = 2$, $U = [0, 1] \times [0, 1]$. Let us define the following 2-D interpolation problems:

- By $\mathcal{P}_{\&}$, we will denote the following problem:
 - \mathcal{F} = the set of all functions $f : U \rightarrow [0, 1]$ for which $f(x_1, x_2) \leq x_1$ and $f(x_1, x_2) = f(x_2, x_1)$ for all x_1 and x_2 .

- $N = 4, f(0, 0) = f(0, 1) = f(1, 0) = 0, f(1, 1) = 1.$
- By \mathcal{P}_\vee , we will denote the following problem:
 - \mathcal{F} = the set of all functions $f : U \rightarrow [0, 1]$ for which $f(x_1, x_2) \geq x_1$ and $f(x_1, x_2) = f(x_2, x_1)$ for all x_1 and $x_2.$
 - $N = 4, f(0, 0) = 0, f(0, 1) = f(1, 0) = 0 = (1, 1) = 1.$

THEOREM 2.

- $\min(x_1, x_2)$ is the least sensitive solution to the interpolation problem $\mathcal{P}_\&$.
- If f is a possible solution to the problem $\mathcal{P}_\&$, and $f \neq \min$, then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $s_{\min}(\delta) \leq C \cdot s_f(\delta).$
- $\max(x_1, x_2)$ is the least sensitive solution to the interpolation problem $\mathcal{P}_\vee.$
- If f is a possible solution to the problem \mathcal{P}_\vee , and $f \neq \max$, then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $s_{\max}(\delta) \leq C \cdot s_f(\delta).$

To compare these results, we describe the δ -sensitivity for \min , \max , and for several other possible solutions of the interpolation problems $\mathcal{P}_\&$ and \mathcal{P}_\vee :

PROPOSITION 5.

- For $f = \min$ and $f = \max$, $s_f(\delta) = \delta.$
- For $f(a, b) = ab$ and $f(a, b) = a + b - ab$, $s_f(\delta) = 2\delta - \delta^2;$
- For $f(a, b) = \min(a + b, 1)$, $s_f(\delta) = \min(2\delta, 1).$

Comment. The fact that \min and \max , and ab and $a + b - ab$ are equally sensitive stems from the fact that in general *dual* functions have the same modulus of continuity, where $g(a, b)$ is *dual* to an $f(a, b)$ if $g(a, b) = 1 - f(1 - a, 1 - b).$

PROPOSITION 6. *Dual operations are equally sensitive.*

3. APPLICATIONS OF THESE RESULTS TO EXPERT SYSTEMS AND INTELLIGENT CONTROL

3.1. What is intelligent control

In case we do not have the precise knowledge of a controlled system, we are unable to apply traditional control theory. In such cases, we can find an expert who is good at control, extract as many rules as possible from him, and try to transform these rules into the precise control strategy. Zadeh and Mamdani initiated a methodology for such a translation ([CZ72, M74]) that is based on *fuzzy set theory* [Z65] and is therefore called *fuzzy control* (see, e.g., the surveys [S85, L90, B91]). In order to apply this methodology, we must:

- 1) describe the expert's uncertainty about every natural-language term A (such as "small") that he uses while describing the control rules; this is done by ascribing to every possible value x of the related physical quantity a value $\mu_A(x)$ from the interval $[0,1]$ that describes to what extent this expert believes that x satisfies the property A (e.g., $\mu_{small}(0.3)$ is his degree of belief that 0.3 is small). The resulting function μ_A is called a *membership function*;

- 2) experts' rules contain natural-language words combined by logical connectives (e.g., “if x is small, and \dot{x} is medium, then u must be small”). Therefore, we must be able to estimate the experts' degree of belief in $A \& B$, $A \vee B$, $\neg A$ (where \neg stands for “not”) from the known values of degrees of belief of A and B . In other words, we must describe the fuzzy analogues of $\&$, \vee , and \neg to combine the original membership functions into a membership function $\mu_C(u)$ for control;
- 3) finally, we must transform this membership functions into an actual control value by a proper *defuzzification* procedure.

As concerns the first stage, there exist several methods that allow us to ask several questions to an expert or experts and come out with the desired values of membership functions (see, e.g., [DP80, KF88]). This makes perfect sense if the experts (whom we ask) give “yes” or “no” answers to all these questions, i.e., when they are absolutely sure of what they are doing. They may be unable to describe their control strategy in precise mathematical terms, but they are absolutely confident in what they are doing (a good example is a person driving his car: he has no doubts about his ability to drive, but he usually cannot formulate his strategy in precise terms).

However, if we are planning a trip to the unknown (e.g., a mission to Mars), then operators are often not that confident in their control abilities. For example, they can formulate a rule in terms of a certain angle being small, but they are uncertain of whether, say, 10° is a small angle or not. As a result, the values of membership functions that we extract from the same expert can differ drastically. Different membership functions, in their turn, can lead to drastically different control strategies, with different quality of the resulting control.

This situation can be viewed as one step further away from the precision of traditional control:

precise knowledge \rightarrow *uncertain knowledge with known degrees of certainty* \rightarrow
uncertain knowledge with uncertain degrees of certainty

What to do in these maximally uncertain situations? Since fuzzy control proved to be a very efficient methodology [S85, L90, B91], we still want to use it, but we must now be very cautious in choosing $\&$ -, \vee -, and \neg - operations, and in choosing a defuzzification procedure. In all these choices, we want to result to be as *least sensitive* to the possible changes in the values of membership functions as possible. In other words, we want to develop *the least sensitive control*.

3.2. Sensitivity of $\&$ - and \vee -operations

Let us first analyze the case of $\&$ - and \vee -operations (this section subsumes [NK92]).

The first paper by L. Zadeh [Z65] that introduced this approach to knowledge representation proposed $\min(a, b)$ and ab as $\&$ -operations, and $\max(a, b)$ and $a + b - ab$ as \vee -operations. Zadeh himself stressed that these operations “are not the only operations in terms of which the union and intersection can be defined”, and “which of these ... definitions are more appropriate depends on the context” [Z75, pp. 225–226]. Since then several

dozens different $\&$ - and \vee -operations have been proposed and successfully used. Some operations have been discovered empirically while working on real expert systems (e.g., the famous MYCIN [BS84]) or while analyzing commonsense reasoning [O77, Z78]); some of them were proposed on a more theoretical basis (see, e.g., [DP80, KF88]). A survey of such operations is given in [KQLFLKBR92].

The natural properties of an $\&$ -operation f are as follows:

- First, since $A\&B$ and $B\&A$ mean the same, we must demand that $f(a, b) = f(b, a)$ for all a and b .
- Second, when each of the statements A and B is either definitely true, or definitely false, we must get the same truth values as the normal $\&$ operation of binary logic, i.e., we must have $f(0, 0) = f(0, 1) = f(1, 0) = 0$ and $f(1, 1) = 1$.
- Third, the degree of belief in $A\&B$ cannot exceed the degree of belief in A . So, we demand that $f(a, b) \leq a$ for all a and b .

Summarizing, we conclude that an $\&$ -operation f must be a possible solution to the interpolation problem $\mathcal{P}_{\&}$ (this is the explanation of the term that we used while proving Theorem 2). Similarly, we can conclude that an \vee -operation f must be a possible solution to the interpolation problem \mathcal{P}_{\vee} . So, from Theorem 2, we can now conclude that $f(a, b) = \min(a, b)$ is the least sensitive $\&$ -operation, and $f(a, b) = \max(a, b)$ is the least sensitive \vee -operation.

Comments.

1. In [KK90], [KQL91], [KQLFLKBR92] general optimization problem are analyzed on the set of all possible $\&$ - and \vee - operations. As a result of this mathematical analysis, lists are given that include all $\&$ - and \vee - operations that can be optimal under reasonable optimality criteria. Our Theorem 2 are in good accordance with that general result, because both min and max are elements of those lists.

2. Similar questions of sensitivity in the context of neural networks are analyzed in [DF92].

3.3. Sensitivity of negation operations

We can define a *negation operation* as a function $f : [0, 1] \rightarrow [0, 1]$ that interpolates the values coming from the definition of the classical negation: $f(0) = 1$ and $f(1) = 0$. From Theorem 1, we can now conclude that $f(x) = 1 - x$ is asymptotically the least sensitive negation operation.

In addition to that, since $\neg(\neg A)$ means the same as A , we can demand that $f(f(x)) = x$ for all x . Also, if we increase our degree of belief in A , then the degree of belief in $\neg A$ should decrease. So, the function f must be *decreasing*. We arrive at the conclusion that f must be a possible solution to the problem \mathcal{P}_{\neg} , and therefore, due to Proposition 4, that $f(x) = 1 - x$ is the least sensitive negation operation (and not only asymptotically the least sensitive one).

3.4. The least sensitive normalization

In some cases, before making a decision an auxiliary operation is performed with a membership function $\mu_C(u)$ that is called a *normalization*. The reason for this operation is that

for many notion from natural language, there is a value about which all the experts (or at least the vast majority of them) agree that this value satisfies the desired property: for example, for “negligible” it is 0, for “big” it is 1000 (or 10^6 if 1000 is not enough). So, for the corresponding membership functions $\mu(x)$, there exists a value x_0 for which $\mu(x_0) = 1$, hence $\sup_x \mu(x) = 1$.

However, after applying the $\&-$, $\vee-$ and $\neg-$ operations, we sometimes obtain a membership function $\mu(x)$, for which $v = \sup_x \mu(x) < 1$, and which is thus difficult to interpret. So, before we apply a defuzzification procedure to it, we first want to *normalize* this membership function, i.e., apply some transformation $t : [0, v] \rightarrow [0, 1]$ and get a new function $\mu'(x) = t(\mu(x))$ whose biggest value is already equal to 1. Usually, the function $t(x) = x/v$ is taken. The question is: which of the possible normalization procedures is the least sensitive? From Theorem 1, we can conclude that $f(x) = x/v$ is the *least asymptotically sensitive normalization*.

3.5. The least sensitive choice of membership functions

All the above applications are about the case when the experts can be uncertain, but the inputs for the control decision (i.e., the values of x , \dot{x} , etc) are considered to be precise. In real-life situations, especially in the case of the future space missions, it is important to take into considerations that the input data can also be imprecise. In this case, we want to choose membership functions in such a way that the change in an input value x will lead to the smallest possible change in the value of $\mu(x)$ (and thus in the resulting control). In other words, we want to guarantee that the interval of possible values of $\mu(x)$ is the least possible.

We want to use this idea to choose the most sensitive extrapolation procedure for membership functions. In other words, when we have a fuzzy notion for which we want to describe a membership function, we describe when this notion is absolutely true, and when it is absolutely false (i.e., when the membership function is equal to 1 and 0) and get all other values of membership function by extrapolation.

In fuzzy control, four types of natural-languages terms (*fuzzy variables*) variables are mainly used:

- 1) Variables like “negligible”, where one can name a value x_0 for which the corresponding property is absolutely true ($\mu(x_0) = 1$), (for negligible it is $x_0 = 0$), and the values x_- and x_+ such that for $x < x_-$ and $x > x_+$ the corresponding property is absolutely false (e.g., values with $x < x_-$ or $x > x_+$ are absolutely not negligible).
- 2) (similar case) Variables, for which we can name an interval $[a_-, a_+]$, inside which the corresponding property is absolutely true, and a bigger interval $[x_-, x_+]$, outside which this property is absolutely false (the first case can be considered as a particular case of this one, when $a_- = a_+ = x_0$).
- 3) Variable like “positive big”, for which we can name values $x_- < x_+$ such that for $x < x_-$ the corresponding property is absolutely false, and for $x > x_+$ this property is absolutely true.
- 4) Variable like “negative big”, for which we can name values $x_- < x_+$ such that for $x < x_-$ the corresponding property is absolutely true, and for $x > x_+$ this property is absolutely false.

In the first and second cases, usually the intervals are symmetric, i.e., in the first case, $x_+ - x_0 = x_0 - x_-$, and, in the second case, $x_+ - a_+ = a_- - x_-$. Applying Proposition 3, we can now conclude that *the least asymptotically sensitive membership functions can be obtained by linear interpolation* (i.e., triangular, trapezoidal, etc).

3.6. Conclusions

As far as combining degrees of belief of experts is concerned, in situations where estimates can vary drastically, it is reasonable to use fuzzy logic connectives, which are the least sensitive to these variations, i.e., for which the resulting intervals of uncertainty are the smallest possible. We have proved that in this situation, the dual pair $\min(a, b)$, $\max(a, b)$ are the least sensitive operations. Results are also given for choosing the least sensitive negation operations and membership functions.

4. PROOFS

Proof of Proposition 1.

The set S of all real numbers α , for which a δ -input uncertainty leads to a $\leq \alpha$ -output error, is bounded from below (by 0), and therefore, has an infimum (the greatest lower bound) r . r is the value of δ -sensitivity. Indeed, since r is the greatest lower bound of the set S , for every positive integer k there exists a number $r_k \in S$ such that $r_k < r + 1/k$. According to the definition of S , from $r_k \in S$ we conclude that if $|x_i - x'_i| \leq \delta$ for all i , then $|f(x_1, \dots, x_n) - f(x'_1, \dots, x'_n)| \leq r_k$. Letting $k \rightarrow \infty$, we conclude that $|f(x_1, \dots, x_n) - f(x'_1, \dots, x'_n)| \leq \lim_k r_k = r$. Q.E.D.

Before proving Proposition 2, let us first prove Theorem 1.

Proof of Theorem 1.

1. Let us first prove that if f is non-linear, then $s_{f_0}(\delta) > s_f(\delta)$ for sufficiently small δ . Without losing generality, let us assume that $y^{(2)} \geq y^{(1)}$ (the proof for the case when $y^{(2)} < y^{(1)}$ is similar). For $f_0(x)$, one can easily compute that $s_{f_0}(\delta) = K\delta$, where $K = |y^{(2)} - y^{(1)}|/(a_2 - a_1)$. Since f is different from f_0 , we have $f(x) \neq f_0(x)$ for some x . For this x , either $f(x) < f_0(x)$, or $f(x) > f_0(x)$. Let us analyze these two cases.

- In the first case, for $x_1 = x$ and $x'_1 = a_2$, we have $|x_1 - x'_1| = a_2 - x$ and $|f(x_1) - f(x'_1)| = y^{(2)} - f(x) > y^{(2)} - f_0(x) = f_0(a_2) - f_0(x) = s_{f_0}(a_2 - x)$. Hence, for $\delta_0 = a_2 - x$, we have $s_f(\delta_0) > s_{f_0}(\delta_0) = K\delta_0$.
- In the second case, for $x_1 = a_1$ and $x'_1 = x$, we have $|x_1 - x'_1| = x - a_1$ and

$$|f(x_1) - f(x'_1)| = f(x) - y^{(1)} > f_0(x) - y^{(1)} = f_0(x) - f_0(a_1) = s_f(x - a_1),$$

hence, for $\delta_0 = x - a_1$, we have $s_f(\delta_0) > s_{f_0}(\delta_0) = K\delta_0$.

2. To prove the second part of Theorem 1, we need the following Lemma (we will use it for $K = |y^{(2)} - y^{(1)}|/(a_2 - a_1)$):

LEMMA. *If $s_f(\delta_0) > K\delta_0$ for some $K > 0$ and $\delta > 0$, then there exists a positive real number $\Delta > 0$ and positive real number $C < 1$ such that for all $\delta < \Delta$, $s_f(\delta) \geq (K\delta)/C$.*

Proof of the Lemma. As we have already noticed, for continuous functions f , δ -sensitivity coincides with the modulus of continuity of f . The modulus of continuity is a subadditive function [L66], so, if f is continuous, then $s_f(\delta_1 + \delta_2 + \dots + \delta_n) \leq s_f(\delta_1) + s_f(\delta_2) + \dots + s_f(\delta_n)$ for all $\delta_1, \dots, \delta_n > 0$. The proof of this inequality does not use continuity of f and therefore, it can be applied to arbitrary functions f . In particular, for $\delta_1 = \delta_2 = \dots = \delta_n = \delta_0/n$, we conclude that $s_f(\delta_0) \leq n s_f(\delta_0/n)$. Therefore, $s_f(\delta_0/n) \geq s_f(\delta_0)/n$. If we denote $s_f(\delta_0)$ by D , then this inequality takes the form $s_f(\delta_0/n) \geq D/n$.

In order to continue the proof, we need to use one more property of the modulus of continuity [L66]: if $\delta < \delta'$, then $s_f(\delta) \leq s_f(\delta')$.

Let us now take any real number C between $c = K\delta_0/D$ and 1 ($c < C < 1$), and prove that there exists a $\Delta > 0$ such that for all $\delta < \Delta$, we have $K\delta \leq C s_f(\delta)$ (or, equivalently, $s_f(\delta) \geq K\delta/C$).

We already know how to estimate the values of $s_f(\delta)$ for $\delta = \delta_0/n$, where $n = 1, 2, 3, \dots$. So, to get the estimates for arbitrary δ , we can use these known estimates. For every $\delta < \delta_0$, we want to find an n such that $\delta_0/(n+1) \leq \delta \leq \delta_0/n$. This inequality is equivalent to $(n+1)/\delta_0 \geq 1/\delta \geq n/\delta_0$, which, after multiplying both sides by δ_0 , turns out to be equivalent to the inequality $n \leq \delta_0/\delta \leq n+1$. Therefore, we can take as n the integer part $\lfloor \delta_0/\delta \rfloor$ of the ratio δ_0/δ . From monotonicity, we can conclude that $s_f(\delta) \geq s_f(\delta_0/(n+1))$. We have already proved that $s_f(\delta_0/(n+1)) \geq D/(n+1)$. Therefore, $s_f(\delta) \geq D/(n+1)$. We defined c as $c = K\delta_0/D$; so, $D = K\delta_0/c$. Hence, $s_f(\delta) \geq K\delta_0/(c(n+1))$.

We want to get an inequality $s_f(\delta) \geq K\delta/C$. We will be able to deduce this inequality from the one that we have just proved if $\delta_0/(c(n+1)) \geq \delta/C$. Since $\delta \leq \delta_0/n$, this inequality is valid if $\delta_0/Cn \leq \delta_0/(c(n+1))$. Dividing both sides by δ_0 and then inverting both sides, we get an equivalent inequality $Cn \geq c(n+1)$, which, in its turn, is equivalent to $(C-c)n \geq c$ and $n \geq c/(C-c)$. Therefore, if $n \geq c/(C-c)$, then for $\delta \leq \delta_0/n$ we get the desired inequality $s_f(\delta) \geq \delta/c$.

The inequality $n \geq c/(C-c)$ is valid for all n starting from $N = \lfloor c/(C-c) \rfloor + 1$. Therefore, the desired inequality $s_f(\delta) \geq K\delta/c$ is true for all $\delta < \Delta$, where $\Delta = \delta_0/N$. The Lemma is proven.

3. Now, the second statement of Theorem 1 is proven. The first statement of Theorem 1 directly follows from the second one. Q.E.D.

Now we are ready to prove Proposition 2.

Proof of Proposition 2.

If $f(x) \neq f_0(x)$, then the existence of the desired $g(x)$ follows from Theorem 1: we can take $g(x) = f_0(x)$. So in order to prove this Proposition, it is sufficient to prove it for $f(x) = f_0(x)$, i.e., it is sufficient to find a function $g(x)$ such that $s_g(\delta) < s_{f_0}(\delta)$ for some $\delta > 0$.

Let us define the following function $F(x)$:

- $F(x) = 100/3x$ for $x \leq 0.01$,
- $F(x) = 1/3$ for $0.01 \leq x \leq 0.495$,
- $F(x) = 1/3 + 100/3 * (x - 0.495)$ for $0.495 \leq x \leq 0.505$,
- $F(x) = 2/3$ for $0.505 \leq x \leq 0.99$, and
- $F(x) = 2/3 + 100/3 * (x - 0.99)$ for $x > 0.99$.

This is a continuous function from $[0,1]$ to $[0,1]$. Let us prove that for $g(x) = F(x)$, $s_g(0.4) \leq 1/3 < 0.4 = s_{f_0}(0.4)$. In other words, we want to prove that if $|a - a'| \leq 0.4$, then $|g(a) - g(a')| \leq 1/3 < 0.4$.

Without losing generality, we can assume that $a < a'$; then $a' \leq 0.4 + a$, and the desired inequality takes the form $g(a') - g(a) \leq 1/3 < 0.4$. Let us consider all possible locations of a .

- If $0 \leq a \leq 0.01$, then $a' \leq 0.4 + a \leq 0.4 + 0.01$, and, therefore $g(a') \leq g(0.45) = 1/3$. Hence, $g(a') - g(a) \leq g(a') \leq 1/3$.
- If $0.01 \leq a \leq 0.495$, then $g(a) = 1/3$, and $a' \leq 0.4 + 0.495 = 0.895$, hence $g(a') \leq g(0.895) = 2/3$. Therefore, $g(a') - g(a) \leq g(0.895) - g(a) = 1/3$.
- If $0.495 \leq a \leq 0.505$, then $g(a) \geq g(0.495) = 1/3$; here $a' \leq 0.505 + 0.4$, hence $g(a') \leq g(0.905) = 2/3$. Therefore, $g(a') - g(a) \leq g(a') - g(0.495) = 1/3$.
- For the cases $0.505 \leq a \leq 0.99$ and $a \geq 0.99$, the proofs are similar.

So, in all the cases, $|g(a) - g(a')| \leq 1/3 < 0.4$.

This proves the Proposition for the case when $a_1 = 0$, $a_2 = 1$, $y^{(1)} = 0$, and $y^{(2)} = 1$. In the general case, we can take $g(x) = y^{(1)} + (y^{(2)} - y^{(1)}) \cdot F((x - a_1)/(a_2 - a_1))$. Q.E.D.

Proof of Proposition 3. For the piecewise-linear functions, $s_{f_0}(\delta) = k\delta$, where $k = 1/(x_+ - x_-)$ for functions of types 1 and 2, $k = 1/(x_+ - x_0)$ for functions of type 3, and $k = 1/(x_+ - a_+)$ for functions of type 4. The fact that these functions are less asymptotically sensitive than the others can be proven just like in the proof of Theorem 1.

Proof of Theorem 2. Let us call a possible solution of a problem $\mathcal{P}_{\&}$ an $\&$ -operation, and a possible solution of a problem \mathcal{P}_{\vee} an \vee -operation.

1°. Let us first prove that we have $f(a, b) \leq \min(a, b)$ for any $\&$ -operation f . Since $f(a, b) \leq a$ and $f(a, b) = f(b, a) \leq b$, it follows that $f(a, b) \leq \min(a, b)$.

2°. Next, let us show that $s_{\min}(\delta) = \delta$.

Indeed, for $|a - a'| \leq \delta$, we have $a \leq a' + \delta$ and likewise $b \leq b' + \delta$. Hence, $\min(a, b) \leq \min(a' + \delta, b' + \delta) = \min(a', b') + \delta$, therefore, $\min(a, b) \leq \min(a', b') + \delta$. Likewise, $\min(a', b') \leq \min(a, b) + \delta$, so $-\delta \leq \min(a, b) - \min(a', b') \leq \delta$, and

$$|\min(a, b) - \min(a', b')| \leq \delta.$$

Take $a = b = \delta$, $a' = b' = 0$. Then

$$|\min(a, b) - \min(a', b')| = \delta,$$

and therefore, the output error is precisely δ . So, we cannot take $\alpha < \delta$, and so the δ -sensitivity of \min is really equal to δ .

3°. Let us now prove that for every $\&$ -operation $f(a, b)$: $s_f(\delta) \geq s_{\min}(\delta) = \delta$.

Indeed, suppose that for some $\delta \in (0, 1)$, $s_f(\delta) < \delta$. This means that if $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, then $|f(a, b) - f(a', b')| \leq s_f(\delta) < \delta$. In particular, if we take $a = b = 1$ and $a' = b' = 1 - \delta$, we conclude that $|f(1, 1) - f(1 - \delta, 1 - \delta)| < \delta$. But according to the definition of a $\&$ -operation, $f(1, 1) = 1$, therefore, this inequality turns into $|1 - f(1 - \delta, 1 - \delta)| < \delta$. Hence, $1 - f(1 - \delta, 1 - \delta) \leq |1 - f(1 - \delta, 1 - \delta)| < \delta$, therefore, $f(1 - \delta, 1 - \delta) > 1 - \delta$. But we have already proved in 1° that $f(a, b) \leq \min(a, b)$, therefore, $f(1 - \delta, 1 - \delta) \leq 1 - \delta$. These two inequalities contradict to each other. Therefore, our assumption that $s_f(\delta) < \delta$ is incorrect. Hence, $s_f(\delta) \geq \delta$.

4°. Finally, let us show that $\min(a, b)$ is the only $\&$ -operation, for which $s_f(\delta) = \delta$ for all δ .

Indeed, suppose that f is different from \min . Then for some a and b , $f(a, b) \neq \min(a, b)$, hence $f(a, b) < \min(a, b)$. Without loss of generality, assume $a \leq b$, resulting in $f(a, b) < \min(a, b) = a$. For $a' = b' = 1$, we have $|a - a'| = 1 - a$, $|b - b'| = 1 - b \leq 1 - a$, but $|f(a, b) - f(a', b')| = 1 - f(a, b) > 1 - a$. So, for $\delta_0 = 1 - a$, $s_f(\delta_0) > \delta_0$. The first statement of Theorem 2 (for $\&$ -operations) is proven.

The second statement follows from the above Lemma.

The third and the fourth statements (about \vee -operations) can be proven in a similar manner. Q.E.D.

Proof of Proposition 4.

Let us call a possible solution to a problem \mathcal{P}_- , a *negation operation*. Let us first prove that for every negation operation, $s_f(\delta) \geq \delta$ for all δ . For a standard negation operation $f_0(x) = 1 - x$, $f_0(\delta) = 1 - \delta$. So, let us consider three possible cases:

- $f(\delta) < 1 - \delta$,
- $f(\delta) = 1 - \delta$, and
- $f(\delta) > 1 - \delta$.

Let us prove the inequality $s_f(\delta) \geq \delta$ for all these three cases.

- In the first case, for $a = 0$, $a' = \delta$, we have $|a - a'| \leq \delta$ and $|f(a) - f(a')| = |1 - f(\delta)| > \delta$. Therefore, $s_f(\delta) \geq |f(a) - f(a')| > \delta$, and $s_f(\delta) > \delta$.
- In the second case, likewise $s_f(\delta) \geq \delta$.
- In the third case, for $a = 1$ and $a' = f(\delta)$, we have $|a - a'| \leq \delta$, but $|f(a) - f(a')| = |0 - \delta| = \delta$. Therefore, $s_f(\delta) \geq |f(a) - f(a')| \geq \delta$, and $s_f(\delta) \geq \delta$.

So, in all three cases, we have $s_f(\delta) \geq \delta$ for all δ .

We can also easily show that for $f_0(x) = 1 - x$, we have $s_{f_0}(\delta) = \delta$. So, to complete the proof of the theorem, it is sufficient to prove that if a negation operation is not standard, i.e., $f(x) \neq 1 - x$ for some x , then $s_f(\delta) > \delta$ for some δ . If $f(x) \neq 1 - x$, this means that either $f(x) < 1 - x$, or $f(x) > 1 - x$. The case when $f(x) < 1 - x$ has already been considered above, and in this case, as we have already proved, $s_f(\delta) > \delta$ for $\delta = x$.

Suppose now that $f(x) > 1 - x$. Then $x > 1 - f(x)$. So, for $a = 1$ and $a' = f(x)$, we have $|a - a'| = 1 - f(x)$, and $|f(a) - f(a')| = |0 - x| = x$. Therefore, for $y = 1 - f(x)$, $s_f(y) \geq |f(a) - f(a')| = x > y$, and $s_f(y) > y$.

The second statement of the Proposition follows from the above Lemma. Q.E.D.

Before proving Proposition 5, let us prove Proposition 6.

Proof of Proposition 6.

1°. Let us first prove that if f and g are dual, i.e., $g(a, b) = 1 - f(1 - a, 1 - b)$, then for every δ , $s_g(\delta) \geq s_f(\delta)$.

Indeed, suppose that $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, and let us prove that

$$|g(a, b) - g(a', b')| \leq s_f(\delta).$$

Since $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, we have $|A - A'| = |a - a'| \leq \delta$ and $|B - B'| = |b - b'| \leq \delta$, where we denoted $A = 1 - a$, $A' = 1 - a'$, $B = 1 - b$, and $B' = 1 - b'$. Due to the definition of $s_f(\delta)$, we can conclude that $|f(A, B) - f(A', B')| \leq s_f(\delta)$. But $g(a, b) = 1 - f(A, B)$ and $g(a', b') = 1 - f(A', B')$, therefore $|g(a, b) - g(a', b')| = |f(A, B) - f(A', B')| \leq s_f(\delta)$. So, for $\alpha = s_f(\delta)$, if $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, then $|g(a, b) - g(a', b')| \leq \alpha$. Since $s_g(\delta)$ is defined as the smallest of all α with this property, we conclude that $s_g(\delta) \leq s_f(\delta)$.

2°. One can easily check that if g is dual to f , then f is dual to g . Therefore, we have both $s_g(\delta) \leq s_f(\delta)$ and $s_f(\delta) \leq s_g(\delta)$, hence $s_g(\delta) = s_f(\delta)$. Q.E.D.

Proof of Proposition 5.

In the proof of Theorem 2, we have already shown that $s_{\min}(\delta) = s_{\max}(\delta) = \delta$.

1) Let us prove the result about s_f for $f(a, b) = ab$. We must prove, first, that if $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, then $|ab - a'b'| \leq 2\delta - \delta^2$, and, second, that there exist such a, b, a', b' for which $|a - a'| \leq \delta$, $|b - b'| \leq \delta$, and $|ab - a'b'| = 2\delta - \delta^2$.

The second statement is easy to prove: take $a = b = 1, a' = b' = 1 - \delta$, then $|ab - a'b'| = 1 - (1 - \delta)^2 = 2\delta - \delta^2$. Let us now prove the first one.

Let us denote $|a - a'|$ by Δ_a , and $|b - b'|$ by Δ_b . Then $\Delta_a \leq \delta$ and $\Delta_b \leq \delta$. Without losing any generality we can assume that $a \geq a'$. Then $a' = a - \Delta_a$. With respect to b and b' , there are two possible cases: $b \geq b'$ and $b < b'$. Let us consider both of them.

If $b \geq b'$, then $b' = b - \Delta_b$, and $ab \geq a'b'$, so the desired absolute value $d = |ab - a'b'|$ can be computed as follows: $d = |ab - a'b'| = ab - a'b' = ab - (a - \Delta_a)(b - \Delta_b) = a\Delta_b + b\Delta_a - \Delta_a\Delta_b$. Since $a \leq 1$ and $b \leq 1$, we have $d \leq \Delta_a + \Delta_b - \Delta_a\Delta_b$. The right-hand side of this inequality can be expressed as $1 - (1 - \Delta_a)(1 - \Delta_b)$. Therefore, it is a monotonely increasing function of both Δ_a and Δ_b . So, its maximal value is attained when both of these variables take their biggest possible values. Since $\Delta_a \leq \delta$ and $\Delta_b \leq \delta$, the maximal possible value is attained when $\Delta_a = \Delta_b = \delta$, and is equal to $2\delta - \delta^2$. Therefore, $d \leq \Delta_a + \Delta_b - \Delta_a\Delta_b \leq 2\delta - \delta^2$. So for this case the desired inequality is proved.

Let us now consider the case when $b < b'$. Then $b = b' - \Delta_b$, and $d = |ab - a'b'| = |a(b' - \Delta_b) - (a - \Delta_a)b'| = |a\Delta_b - b'\Delta_a|$. Let us consider two subcases: when the expression under the absolute value is positive or negative, i.e., when $a\Delta_b \geq b'\Delta_a$ and $a\Delta_b < b'\Delta_a$. In the first subcase, $d = a\Delta_b - b'\Delta_a$, therefore, $d \leq b'\Delta_a$. Since $\Delta_a \leq \delta$ and $b' \leq 1$, we get $d \leq \delta$.

In the second subcase similarly $d = b'\Delta_a - a\Delta_b \leq b'\Delta_a \leq \delta$. So, in both cases $d \leq \delta$.

So, to complete the proof, it is sufficient to show that $\delta \leq 2\delta - \delta^2$ for all δ from 0 to 1. Indeed, by dividing both sides by δ and moving all terms to the right-hand side, we conclude that this inequality is equivalent to $0 \leq 1 - \delta$, which is certainly true for $\delta \leq 1$.

For $f(a, b) = a + b - ab$, the expression for $s_f(\delta)$ follows from Proposition 6.

3) Let us now prove that $s_f(\delta)$ for the last function f . Let us first consider the case, when $\delta < 1/2$. Then $2\delta < 1$, and $\min(2\delta, 1) = 2\delta$. Let us prove that in this case, if $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, then $|f(a, b) - f(a', b')| \leq 2\delta$.

Indeed, if $|a - a'| \leq \delta$ and $|b - b'| \leq \delta$, then $|(a + b) - (a' + b')| = |(a - a') + (b - b')| \leq 2\delta$. In particular, this means that $a' + b' \leq a + b + 2\delta$. Evidently, $a' + b' \leq 1$, therefore, $a' + b' \leq 1 < 1 + 2\delta$. So, $a' + b'$ is not bigger than the smallest of these two numbers: $a' + b' \leq \min(a + b + 2\delta, 1 + 2\delta)$. But $\min(a + b + 2\delta, 1 + 2\delta) = \min(a + b, 1) + 2\delta = f(a, b) + 2\delta$. So, $a' + b' \leq f(a, b) + 2\delta$. Since $f(a', b') = \min(a' + b', 1)$ and therefore, $f(a', b') \leq a' + b'$, we conclude that $f(a', b') \leq f(a, b) + 2\delta$. In a similar manner we can prove that $f(a, b) \leq f(a', b') + 2\delta$. Combining these two inequalities, we conclude that $|f(a, b) - f(a', b')| \leq 2\delta$. So, for $\delta < 1/2$, $s_f(\delta) \leq 2\delta$.

Let us now show that $\alpha = 2\delta$ is the smallest value, for which δ -input uncertainty leads to a $\leq \alpha$ -output error, and thus, $s_f(\delta) = 2\delta$. Indeed, if we take $a = b = 0, a' = b' = \delta$, then $|a - a'| \leq \delta, |b - b'| \leq 2\delta$, and $|f(a, b) - f(a', b')| = |0 - 2\delta| = 2\delta$, so the values $\alpha < 2\delta$ do not work in this case. So, for $\delta < 1/2$, we proved that $s_f(\delta) = 2\delta$.

Now let us consider the case when $\delta \geq 1/2$. In this case, $\min(2\delta, 1) = 1$. If we take $a = b = 0, a' = b' = \delta$, then $f(a, b) = 0, f(a', b') = 1, |a - a'| \leq \delta, |b - b'| \leq 2\delta$, and $|f(a, b) - f(a', b')| = |0 - 1| = 1$. Therefore, nothing smaller than 1 can serve as α , hence $s_f(\delta) = 1$. Q.E.D.

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