

IS 3K BACKGROUND OF COSMOLOGICAL ORIGIN OR IT IS A MIXTURE OF RADIATIONS OF SMALL SOURCES?

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ABSTRACT. It has been recently suggested that a background radiation can be explained as a combination of weak background sources. We provide a mathematical argument in favor of a cosmological origin of the background. Namely, the background radiation is well described by Planck's law. As for possible sources (stars, etc), for their spectra the black-body formula is a very crude approximation, because the individual differences between the stars with the same temperature T are very big. However, these differences are truly individual, they go in different directions for different stars. So, it makes sense to assume that for the total spectra of many sources, these differences more or less cancel each other, and the resulting total spectrum is close to the sum of the approximate black-body expressions. We prove that Planck's spectrum cannot be represented as a sum of black-body spectra.

Of course, we can always explain the Planck spectrum as a result of some process that "thermalizes" the initially non-Planck one. But if we exclude this rather ad hoc explanation, then, in the above- described approximation, cosmological interpretation is the only one possible.

Since we are dealing with approximations to real spectra, our result is not an ultimate proof of cosmological nature of background radiation, but we believe that it is a strong argument in favor of this hypothesis.

Key Words: *microwave background, cosmological origin*

A physical problem. Is 3K background really of cosmological origin, as it is traditionally assumed in astrophysics, or it is a mixture of radiations of small celestial bodies? This question was recently raised by R. L. Kurucz, who claims (in Section 5 of ¹) that it is impossible to tell whether the observed microwave background is a cosmological 3K radiation, or a combination of radiations of billions of background sources, that are too weak to be individually detectable by the current equipment.

Of course, if we consider all mathematical functions $f(\nu)$ as possible spectra of these background sources, then we can always find the set of functions $f_1(\nu), \dots, f_N(\nu)$, for which

the sum $\sum_{i=1}^N f_i(\nu)$ (i.e., the total spectrum) is precisely equal to the Planck spectrum $\rho_T(\nu)$ (or to any other background spectrum that we want to explain). Our point is that if we restrict ourselves to physically meaningful spectra $f_i(\nu)$, then it becomes extremely difficult (maybe even impossible?) to find $f_i(\nu)$ such that $\sum_i f_i(\nu) = \rho(\nu)$, and this to explain Planck's radiation as a mixture of radiations of small sources.

One more possibility to explain the observed Planck spectrum is to assume that it is a result of some process that "thermalizes" the initially non-Planck one. To our viewpoint, this is a rather ad hoc explanation, so we will not consider it.

Since there are very many different models that describe the spectra of possible background sources (stars and galaxies), we could not check this possibility for all these models. So, let us check this possibility on the following simplest possible model of a background source. The smallest information that we can get about a star's spectrum is when we know only its temperature T . If the temperature T is the only information that we have, then, as a crude approximation to the real spectrum, we can use a black-body (Planck) spectrum that corresponds to this value T . Spectra of real stars are different from this simplified model. However, these differences are individual: for the same frequency ν , some stars have the spectral density $\rho(\nu)$ that is bigger than the Planck's value, for some other stars the value $\rho(\nu)$ can be smaller than the one predicted by Planck's formula. Therefore, it makes sense to assume that if we average over billions of weak sources, then these individual differences more or less cancel each other, and the resulting spectrum is close to the sum of black-body spectra.

In this approximation (and in the absence of a subsequent thermalizing process), the question whether a background radiation can be explained as a combination of weak sources, can be reformulated as follows: is it possible to represent Planck's distribution with a temperature T_0 as a sum of several Planck's distributions with different temperatures T ? The answer to this question is negative: *we show that if the background spectrum actually follows Planck's law, then in this approximation, it cannot be explained as a combination of different sources.*

Of course, since we are dealing with approximations to real spectra, this mathematical result does not provide us with an ultimate proof that a cosmological interpretation of a background radiation is the only mathematically possible one. However, from our viewpoint, this result is a strong argument in favor of a cosmological interpretation.

Mathematical formulation of the problem. Let us reformulate this problem in mathematical terms. If the body is in the state of equilibrium with its radiation, then it can be

characterized by a temperature T . By Planck's law, the radiation density (energy per unit volume) of such a body on frequencies between ν and $\nu + d\nu$, is given by the expression $\rho_T(\nu)d\nu$, where

$$\rho_T(\nu) = \frac{8\pi h\nu^3}{c^3(\exp(h\nu/(kT)) - 1)}, \quad (1)$$

h is Planck's constant, c is the velocity of light, and k is the "gas constant". So, if we assume that the background microwave radiation is of cosmological origin, the observed density must be described by the above-given formula.

We consider the case when the radiation of each background source is described by Planck's formula for some temperature T . So, the combined spectrum of several sources can be therefore represented as

$$\rho(\nu) = \int \phi(T)\rho_T(\nu) dT = \int \frac{8\pi h\phi(T)\nu^3}{c^3(\exp(h\nu/(kT)) - 1)} dT, \quad (2)$$

where $\phi(T) \geq 0$ denotes the total intensity of radiation that is coming from the sources of temperature T .

If there is only one source, with temperature T_0 , then $\phi(T) = \delta(T - T_0)$ and $\rho(\nu) = \rho_{T_0}(\nu)$. In case we assume that many sources contribute to the observed radiation, it is impossible that they all have precisely the same temperature. Therefore, in this case, $\phi(T) \neq \delta(T - T_0)$.

So, the question is as follows: Suppose that $\rho(\nu) = \rho_{T_0}(\nu)$, i.e., the observed radiation $\rho(\nu)$ is described by the Planck's expression (1). Does there exist a function $\phi(T) \neq \delta(T - T_0)$, that leads to the same density $\rho(\nu)$? In other words, if $\phi(T) \geq 0$ and

$$\int \frac{8\pi h\phi(T)\nu^3}{c^3(\exp(h\nu/(kT)) - 1)} dT = \frac{8\pi h\nu^3}{c^3(\exp(h\nu/(kT_0)) - 1)}, \quad (3)$$

does it follow that $\phi(T) = \delta(T - T_0)$?

Main result. The author asked this mathematical question in ². As a result, several mathematicians pointed to him that the problem of whether the temperature distribution $\phi(T)$ can be uniquely reconstructed from the observed spectral density $\rho(\nu)$, had already been discussed by R. E. A. C. Paley and N. Wiener in ³. They proved that under certain mathematical assumptions $\phi(T)$ is uniquely determined by $\rho(\nu)$. We are going to prove that these mathematical assumptions are valid in our case, and therefore, $\phi(T)$ is uniquely determined by $\rho(\nu)$: i.e., $\phi(T) = \delta(T - T_0)$.

THEOREM. *If $\phi(T) \geq 0$, and*

$$\int \frac{8\pi h\phi(T)\nu^3}{c^3(\exp(h\nu/(kT)) - 1)} dT = \frac{8\pi h\nu^3}{c^3(\exp(h\nu/(kT_0)) - 1)},$$

then $\phi(T) = \delta(T - T_0)$.

Proof. For the readers' convenience, we will describe the methods from ³ in sufficient detail (for the same reason, we slightly changed the denotations).

From (3) it follows that

$$\int \frac{8\pi h\phi(T)\nu^3}{c^3(\exp(h\nu/(kT)) - 1)} dT = \int \frac{8\pi h\delta(T - T_0)\nu^3}{c^3(\exp(h\nu/(kT)) - 1)} dT.$$

Let us move all the terms from this equation into the left-hand side, and divide both sides by $8\pi h\nu^3/c^3$. Thus, we conclude that

$$\int \frac{\tilde{\phi}(T)}{\exp(h\nu/(kT)) - 1} dT = 0, \quad (4)$$

where $\tilde{\phi}(T)$ denotes $\phi(T) - \delta(T - T_0)$. In these terms, our question can be reformulated as follows: is $\tilde{\phi}(T) = 0$ the only solution of (4)?

Following ³, let us introduce a new variable $\mu = h/(kT)$ (so that $T = h/(k\mu)$ and $dT = -h/(k\mu^2)d\mu$), and a new unknown function $\tilde{\Phi}(\mu) = \tilde{\phi}(h/(k\mu))$. Then, $\tilde{\phi}(T) = \tilde{\Phi}(h/(kT))$, and equation (4) takes the following form:

$$\int -\frac{hk^{-1}\mu^{-2}\tilde{\Phi}(\mu)}{\exp(\mu\nu) - 1} d\mu = 0.$$

Let us now divide both sides by $-hk^{-1}$ and denote $A(\mu) = \mu^{-2}\tilde{\Phi}(\mu)$. Then, we get the following equation:

$$\int \frac{A(\mu)}{\exp(\mu\nu) - 1} d\mu = 0. \quad (5)$$

To simplify this expression, we introduce new variables $\xi = \ln \mu$ and $\eta = \ln \nu$. Then, $\mu = \exp(\xi)$, $d\mu = \exp(\xi)d\xi$, $\nu = \exp(\eta)$, $\mu\nu = \exp(\xi + \eta)$, and equation (5) takes the following form:

$$\int \frac{a(\xi)\exp(\xi)}{\exp(\exp(\xi + \eta)) - 1} d\xi = 0, \quad (6)$$

where by $a(\xi)$ we denoted $a(\xi) = A(\exp(\xi))$. For any λ , we can multiply both sides of (6) by $\exp(\lambda\eta)$, thus transforming (6) into an equation

$$\int \frac{a(\xi)\exp(\xi)\exp(\lambda\eta)}{\exp(\exp(\xi + \eta)) - 1} d\xi = 0.$$

This equation can be simplified if we represent $\exp(\xi)$ as $\exp(\lambda\xi)\exp((1-\lambda)\xi)$:

$$\int \frac{a(\xi)\exp((1-\lambda)\xi)\exp(\lambda(\xi+\eta))}{\exp(\exp(\xi+\eta))-1}d\xi=0. \quad (7)$$

The left-hand side of (7) is a convolution of the functions $F(\xi)=a(\xi)\exp((1-\lambda)\xi)$ and

$$G(\xi)=\frac{\exp(\lambda\xi)}{\exp(\exp(\xi))-1}.$$

Convolution is easily expressible in terms of Fourier transforms of F and G .

To apply this simplifying techniques, let us find λ for which $\int|F(\xi)|d\xi$ and $\int|G(\xi)|d\xi$ are both finite, so that Fourier transforms of both functions are well defined.

To find this λ , let us recall that the total density $\int\rho_{T_0}(\nu)d\nu$ of Planck's distribution is equal to kT_0^4 . Therefore, the total density of the density distribution $\rho(\nu)=\int\phi(T)\rho_T(\nu)dT$ is equal to

$$\int\rho(\nu)d\nu=\int\int\phi(T)\rho_T(\nu)dT d\nu=\int\phi(T)dT(\int\rho_T(\nu)d\nu)=\int k\phi(T)T^4 dT.$$

Since we assumed that $\rho(\nu)=\rho_{T_0}(\nu)$, we conclude that $\int\phi(T)T^4 dT=kT_0^4<\infty$. Here, $\phi(T)>0$. Now, $\tilde{\phi}(T)=\phi(T)-\delta(T-T_0)$, hence $|\tilde{\phi}(T)|\leq\phi(T)+\delta(T-T_0)$, and $\int|\tilde{\phi}(T)|T^4 dT\leq\int\phi(T)T^4 dT+\int\delta(T-T_0)T^4 dT=2kT_0^4<\infty$. If we substitute $T=h/(k\mu)$ and $dT=-h/(k\mu^2)d\mu$ into this inequality, we conclude that $\int|\tilde{\phi}(T)|T^4 dT=(h/k)^6\int|\tilde{\Phi}(\mu)|\mu^{-2}\mu^{-4}d\mu$, hence, from $\int|\tilde{\phi}(T)|T^4 dT<\infty$, we deduce that $\int|\tilde{\Phi}(\mu)|\mu^{-6}d\mu<\infty$. In terms of $A(\mu)=\mu^{-2}\tilde{\Phi}(\mu)$, this inequality takes the form $\int|A(\mu)|\mu^{-4}d\mu<\infty$. Finally, when we introduce a new variable ξ , so that $\mu=\exp(\xi)$, $d\mu=\exp(\xi)d\xi$, $\mu^{-4}=\exp(-4\xi)$, and $A(\mu)=a(\xi)$, then we can conclude that $\int|a(\xi)|\exp(-4\xi)\exp(\xi)d\xi=\int|a(\xi)|\exp(-(1-4)\xi)d\xi<\infty$.

So, for $\lambda=4$, $\int|F(\xi)|d\xi<\infty$.

Let us prove that for this λ , $\int|G(\xi)|d\xi<\infty$. Actually, this is true for any $\lambda>1$. Indeed, this integral can be represented as sum of two integrals: over $(0,+\infty)$, and over $(-\infty,0)$. For

$$G(\xi)=\frac{\exp(\lambda\xi)}{\exp(\exp(\xi))-1},$$

when $x\rightarrow+\infty$, the numerator tends to ∞ much slower than the denominator, so the integral over $(0,\infty)$ is finite. When $\xi\rightarrow-\infty$, then $\exp(\xi)\rightarrow 0$, hence $\exp(\exp(\xi))\sim 1+\exp(\xi)$, and $\exp(\exp(\xi))-1\sim\exp(\xi)$, hence $|G(\xi)|\sim\exp(\lambda\xi)/\exp(\xi)=\exp((\lambda-1)\xi)$.

Since $\lambda - 1 > 0$, the integral $\int_{-\infty}^0 \exp((\lambda - 1)\xi) d\xi$ is finite. Therefore, the integral of $|G(\xi)|$ over $(-\infty, 0)$ is also finite. So, the entire integral $\int |G(\xi)| d\xi$, as a sum of two finite integrals, is finite as well.

For $\lambda = 4$, both integrals are finite, hence, Fourier transform is well defined for both functions $F(\xi)$ and $G(\xi)$. So, from the fact that the convolution of F and G is equal to 0, we conclude that the product of their Fourier transforms is equal to 0: $\hat{F}(-\omega)\hat{G}(\omega) = 0$. According to ³, the Fourier transform $\hat{G}(\omega)$ of $G(\xi)$ is equal to $\Gamma(\lambda + i\omega)\zeta(\lambda + i\omega)$. It is known that neither the function $\Gamma(z)$, nor the Riemann function $\zeta(z)$ have roots with $Re(z) > 1$. Therefore, we conclude that $\hat{G}(\omega) \neq 0$ for all ω . So, from $\hat{F}(-\omega)\hat{G}(\omega) = 0$, we can conclude that $\hat{F}(-\omega) = 0$ for all ω . So, the Fourier transform of $F(\xi)$ is equal to 0. Therefore, $F(\xi) = 0$ for all ξ . But $F(\xi) = a(\xi) \exp((1 - \lambda)\xi)$, and the term $\exp((1 - \lambda)\xi)$ is always positive. Hence, $a(\xi) = 0$ for all ξ . So, $A(\mu) = a(\ln \mu) = 0$ for all μ . Since $A(\mu) = \mu^{-2}\tilde{\Phi}(\mu)$, we can conclude that $\tilde{\Phi}(\mu) = \mu^2 A(\mu) = 0$ for all μ . From $\tilde{\Phi}(\mu) = \tilde{\phi}(h/(k\mu))$, we conclude that $\tilde{\phi}(T) = \tilde{\Phi}(h/(kT)) = 0$ for all T . So, $\phi(T) - \delta(T - T_0) = 0$, and $\phi(T) = \delta(T - T_0)$. Q.E.D.

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