THE NECESSITY TO CHECK CONSISTENCY EXPLAINS THE USE OF PARALLELEPIPEDS IN DESCRIBING UNCERTAINTY

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Abstract. Measurements are always imprecise and therefore, in real life, we never know the exact values $x_1, ..., x_v$ of physical quantities. No matter how many measurements we make, and how many statements comprising expert knowledge we use, we still end up with a set $X \subseteq R^v$ of possible values of $\vec{x} = (x_1, ..., x_v)$. Each measurement result brings an additional restriction on $\vec{x}$. The more measurements we make, the more complicated is the shape of this set X and therefore, the more difficult it is to process X. Therefore, it is necessary to approximate such sets X by simpler sets.

In many cases, parallelepipeds are used for such approximations (in particular, intervals for $v = 1$, parallelograms for $v = 2$, etc). Why?

In this paper, we show that their usage can be justified by the necessity to check consistency of knowledge. A natural formalization of this consistency demand leads us to a geombinatoric property (2-Helly). It is known that for convex compact sets M,

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[1] This work was partially done when the author was a Visiting Researcher with the Department of Mathematics, University of Texas at El Paso.

[2] This work was supported by NSF grants No. CDA-9015006 and No. EEC-9322370, and Grant No. PF90–018 from the General Services Administration (GSA), administered by the Institute for Manufacturing and Materials Management. The authors are thankful to Michael Gelfond, M. Amine Khamsi, and Olga Kosheleva for valuable discussions, and to Chih-Han Sah for important reprints.

this property uniquely determines the class of parallelepipeds. For non-convex sets, the problem is still open.

1. FORMULATION OF THE PHYSICAL PROBLEM

Prediction is one of the main goals of science. One of the main goals of science is to predict what will happen in the world. In other words, to predict the values of different physical quantities. Traditional prediction algorithms take as input the current values $x_1, ..., x_v$ of some physical variables, and use these value to predict the future state of the Universe. For example, in weather prediction, we take as input the values $x_1, ..., x_v$ of the meteorological parameters in different points.

Measurements are needed for prediction. There are two main ways to find the values of x_i:

- by directly measuring them;
- by using the ability of an expert to give a reasonable estimate.

Measurements are always approximate. Measurements are never 100% accurate. Expert estimate are also only approximate. As a result, we never know the exact values of the desired physical quantities $x_1, ..., x_v$. There are usually several different possibilities that are all consistent with all our measurements and expert estimates.

For example, if the measured value $\tilde{x}_1$ of a temperature x_1 is 35, and the accuracy of this measurement is ± 5, this means that the actual value of x_1 can be equal to any number from the interval $[30, 40]$.

For these different possibilities, the predicting algorithm will lead to slightly different results. Therefore, instead of a *single* predicted result, we get the *set* of possible future values.

It is necessary to approximate the sets that describe uncertainty. Each measurement bring an additional restriction on the set X of all possible values of $\vec{x} = (x_1, ..., x_v)$. As a result, the more measurements we take, the more complicated the shape of this set X can be. But we need to process this set in order to get the set of possible future values, and processing odd-shaped sets is computationally very complicated. So, we need to *approximate* these sets by sets belonging to some pre-chosen family of simple sets (e.g., enclose X by a ball, or by an ellipsoid, or by a parallelepiped). What family should we choose?

2. THE NECESSITY TO CHECK CONSISTENCY

In physical terms: Measuring devices can go wrong; expert estimates can be wrong. As a result, different conditions on $\vec{x}$ may turn out to be inconsistent. So, before we start processing, it would be nice to check the existing knowledge for possible inconsistencies.

In view of this necessity, it is desirable to choose the family of approximating sets in such a way that this check will not be computationally very complicated.

Motivations for the (following) precise definition. Let's formulate this necessity in mathematical terms. Assume that we have several pieces of knowledge about $\vec{x}$. Each piece of knowledge can be formulated as a set of all the vectors $\vec{x}$ that are consistent with this particular knowledge. So, we instead of saying that we have several pieces of knowledge, we can say that we have several sets $A, B, ..., C \subseteq R^v$. Consistency means that it is possible that a certain value $\vec{x} \in R^v$ satisfies all these properties, i.e., belongs to all these sets $A, B, ..., C$ (in other words, that $A \cap B \cap ... \cap C \neq \phi$).

How can we actually check consistency? Knowledge usually comes piece after piece, so a typical situation is as follows:

- We already have a consistent knowledge base. In other words, we have the pieces of knowledge represented by sets $A_1, ..., A_s$, and these pieces of knowledge are consistent.
- Then, a new piece of knowledge arrives, described by a set A.

A natural way to check consistency is to check whether A is consistent with each of the existing sets A_i. The consistency between A and each of A_i is, of course, necessary for the new knowledge base $\{A_1, ..., A_s, A\}$ to be consistent. It is easy to see, however, that this comparison is not always sufficient (the examples of why it is not always so can be easily extracted from the following text). So, we arrive at the following definition:

Definition 1. We say that a family $\mathcal{S}$ of sets *allows checking consistency* if the following is true: For every s, and for every tuple of sets $A_1 \in \mathcal{S}, ..., A_s \in \mathcal{S}, A \in \mathcal{S}$, for which:

- sets $A_1, ..., A_s$ are consistent (i.e., $A_1 \cap ... \cap A_s \neq \phi$) and
- A is consistent with all A_i (i.e., $A \cap A_i \neq \phi$ for all $i = 1, ..., s$),

all $s+1$ sets $A_1, ..., A_s, A$ are consistent (i.e., $A_1 \cap ... \cap A_n \cap A \neq \phi$).

3. ABILITY TO CHECK CONSISTENCY IS EQUIVALENT TO 2-HELLY PROPERTY FROM COMBINATORIAL GEOMETRY

Let us show that this property can be reformulated in terms of *combinatorial geometry.* Indeed, this property is very similar to the following 2$-$*Helly* property (see, e.g., [Boltiansky 1981], [Jawhari et al 1986]): *if every two sets from $\mathcal{S}$ have a common point, then any finite number of sets from $\mathcal{S}$ also have a common point* (the name of this property came, of course, from the well-known Helly theorem).

It is easy to prove that these two properties are actually equivalent:

PROPOSITION 1. *A family of sets $\mathcal{S}$ allows checking consistency iff it satisfies a 2$-$Helly property.*

(for reader's convenience, all the proofs are placed into the last Section).

Translation invariance: additional condition on the desired family of sets. We are interested in measurements, so it is natural to assume that if M is a reasonable approximation to sets that describe the uncertainty of our knowledge, then a set $M + \vec{a}$ (that is obtained from M by a translation) is also a reasonable approximation. For example, assume that we are measuring time, and as a result we get 35 ± 5 (i.e., a set $M = [30, 40]$). If we now change the starting point for measuring time, e.g., take -5 as the new starting point, then the same result will be expressed as

$M + 5 = 40 \pm 5 = [35, 45]$. This new set $M + 5$ must also be a reasonable approximation.

Definition 2. We say that a family $\mathcal{S}$ of sets is *translation-invariant* if for every $M \in \mathcal{S}$, and for every $\vec{a} \in R^v$, the set $M + \vec{a}$ also belongs to $\mathcal{S}$ (this set $M + \vec{a}$ is called a *translate* of M).

Sets from translation-invariant families that allow checking consistency can be easily characterized by the following condition:

PROPOSITION 2. *For a set $M \subseteq R^v$, the following two conditions are equivalent to each other:*
i) M is an element of some translation-invariant family $\mathcal{S}$ that allows checking consistency;
ii) translates of M satisfy the $2-$Helly property (i.e., if the sets $M_1, ..., M_s$ are obtained from M by translations, and every pair of these sets M_i has a non-empty intersection, then $M_1 \cap ... \cap M_s \neq \phi$).

4. WHAT IS KNOWN?

It is known ([Szökefalvi-Nagy 1954]; [Boltiansky 1981], p. 237) that the *translates of a compact convex set M satisfy a $2-$Helly property if and only if M is a parallelepiped.*

So, a convex compact set M can be a reasonable representation of uncertainty iff M is a parallelepiped. Thus, the above-cited result of Szökefalvi-Nagy *explains why parallelepipeds are often used to describe uncertainty.*

5. AN OPEN PROBLEM

Szökefalvi-Nagy's result is based on the assumption that M is convex. However, we often have knowledge that does not correspond to a convex set. E.g., if we have measured the value of the velocity as 10 ± 1 (i.e., the possible values form an interval $[9, 11]$), and this is the only information that we have about the motion, then the set of all possible values of the velocity components v_1, v_2, v_3 forms a (non-convex) "slice" between two spheres (of radii 9 and 11).

So, we arrive at the following open problem:

OPEN PROBLEM. *To describe sets for which translates satisfy* $2-$*Helly property.*

6. PRELIMINARY ANALYSIS AND TWO RESULTING HYPOTHESES

First hypothesis. First of all, there exist non-convex compact sets with this property. For example, it is easy to check that translates of $M = \{0, 1\} \subseteq R^1$ satisfy $2-$Helly property. However, all such examples that we have constructed so far are disconnected. Therefore, we can formulate the following hypothesis:

HYPOTHESIS 1. *If translates of a compact connected set* M *have a* $2-$*Helly property, then* M *is a parallelepiped.*

Second hypothesis. Another hypothesis stems from the following 1-dimensional result:

Definition 3. We say that a set $M \subseteq R^v$ is *central-symmetric* if $M = \vec{a} - M$ for some $\vec{a} \in R^v$.

Comment. This means that M becomes literally central symmetric ($M = -M$) if we choose an appropriate coordinate origin.

PROPOSITION 3. *If translates of a compact set* $M \subseteq R^1$ *have a* $2-$*Helly property, then* M *is central symmetric.*

Taking into consideration that parallelepipeds are also central symmetric, we arrive at the following hypothesis:

HYPOTHESIS 2. *If translates of a compact set* M *satisfy a* $2-$*Helly property, then* M *is central symmetric.*

Comment. To help to solve our open problem, let's reformulate it in terms close to those of a well-developed field of math: namely, of homological algebra.

Our hope that this reformulation may be useful is justified by the fact that group cohomologies have been successfully used in another geombinatoric problem: equidecomposability of polyhedra (see, e.g., [Jessen 1968], [Jessen et al 1968], [Kosheleva 1980], [Dupont et al 1982], [Dupont et al 1988], [Sah 1989], [Dupont et al 1990]).

7. REFORMULATION OF THE PROBLEM IN TERMS CLOSE TO HOMOLOGICAL ALGEBRA

Definition 4. Assume that a set $M \subseteq R^v$ is given, and a positive integer s is fixed. For every n, $n-$*cochains* will be defined as (completely) antisymmetric functions f from $\{1, ..., s\}^n$ to a certain subset $M_n \subseteq R^v$. A *coboundary operator* δ^n transforms an $n-$cochain into an $(n+1)-$cochain as follows:

$$(\delta^n f)(i_0, ..., i_n) = \sum_{k=0}^{n} (-1)^k f(i_0, ..., i_{k-1}, \hat{i}_k, i_{k+1}, ..., i_n),$$

where $\hat{i}_k$ means that we are skipping $k-$th variable. We define $M_0 = M_1 = 1$, and $M_{n+1} = M_n - M_n + ...$ $(n+1$ times), so that $M_2 = M_1 - M_1 = M - M$.

Comment. This definition is similar to standard definitions from group cohomologies (see, e.g., [MacLane 1963] or [Itô 1987], Vol. 1, pp. 749-761), with the only difference that in our case, the range of the functions defined as cochains is not a group, it is a *subset* of an additive group R^v. Because of that, we have to be careful in our definition to guarantee that the coboundary of a $n-$cochain will be an $(n+1)-$cochain. This is indeed guaranteed by our choice of M_n. Similarly to traditional cohomologies, it is easy to check that δ^n is really a coboundary operator:

PROPOSITION 4. *For every $n-$cochain f, $\delta^{n+1}(\delta^n f) = 0$.*

Definition 5. An $n-$cochain is called an $n-$*coboundary* if it is a coboundary $\delta^{n-1} g$ of some $(n-1)-$cochain g. An $n-$cochain is called an $n-$*cocycle* if $\delta^n f = 0$.

Comment. From Proposition 4, it follows that *every $n-$coboundary is an $n-$cocycle.* It turns out that the inverse statement is true if and only if M possesses the 2−Helly property. This gives us the desired reformulation of 2−Helly property is cohomological terms:

PROPOSITION 5. *For every set $M \subseteq R^v$, the following two conditions are equivalent to each other:*
i) translates of M satisfy the 2−Helly property;
ii) every 2−cocycle is a 2−coboundary.

Comment. In traditional homology theory, both the set of all $n-$cocycles Z^n and the set of all $n-$coboundaries B^n are abelian groups. Since $B^n \subseteq Z^n$, in this traditional theory, the condition *ii)* can be reformulated as $H^2 = 0$, where the factor-group $H^2 = Z^2/B^2$ is called the *second cohomology group.*

8. PROOFS

Proof of Proposition 1. Assume that $\mathcal{S}$ allows checking consistency, and the sets $A_1, ..., A_s$ from $\mathcal{S}$ have pairwise non-empty intersections. To prove that $A_1 \cap .. \cap A_s \neq \phi$, let's start with a consistent class $\{A_1, A_2\}$. Since $\mathcal{S}$ allows checking consistency, and A_3 is consistent with both A_1 and A_2, we can conclude that $A_1 \cap A_2 \cap A_3 \neq \phi$. Now, likewise, we can add A_4, and conclude that $A_1 \cap A_2 \cap A_3 \cap A_4 \neq \phi$. Adding the sets A_i one by one, we finally conclude that $A_1 \cap ... \cap A_s \neq \phi$.

Vice versa, assume that 2−Helly property is true, $A_1 \cap ... \cap A_s \neq \phi$, and $A_i \cap A \neq \phi$ for all i from 1 to s. Then, $A_i \cap A_j \neq \phi$ for all i, j and therefore, due to 2−Helly property, $A_1 \cap ... \cap A_s \cap A \neq \phi$. Q.E.D.

Proof of Proposition 2.

i)→ *ii)* If M belongs to a translation-invariant family $\mathcal{S}$ that satisfies 2−Helly property, then all translates of M belong to $\mathcal{S}$ and therefore, satisfy the same property.

ii)→ *i)* As the desired $\mathcal{S}$, we can take the set of all translates of M. Q.E.D.

Proof of Proposition 3. Let M be a compact subset of R^1 whose translates have 2−Helly property. Let's prove that it is central symmetric. Indeed, since the set $M \subset R$ is compact, it has an upper bound m^+ and a lower bound m^- that both belong to M. Let us prove that for $m = m^+ + m^-$, we have $M = m - M$.

Indeed, assume that $x \in M$. We must prove that $m - x \in M$. To prove that, let's consider the following three translates of M: $M_1 = M$, $M_2 = M + (x - m^-)$, and $M_3 = M + (m^+ - m^-)$. Each pair has an element in common. Indeed:

- $x \in M_1 = M$ and $x = m^- + (x - m^-) \in M + (x - m^-) = M_2$;
- $m^+ \in M_1 = M$ and
$$m^+ = m^- + (m^+ - m^-) \in M + (m^+ - m^-) = M_3;$$
- $m^+ + x - m^- = m^+ + (x - m^-) \in M + (x - m^-) = M_2$ and $m^+ + x - m^- = x + (m^+ - m^-) \in M + (m^+ - m^-) = M_3$.

Therefore, due to 2−Helly property, there exists a number y that belongs to all three sets M_i. In particular, $y \in M_1 \cap M_3$. But $M_1 = M \subseteq [m^-, m^+]$ and $M_3 = M + (m^+ - m^-) \subseteq [m^-, m^+] + (m^+ - m^-) = [m^+, m^+ + (m^+ - m^-)]$. So, all points from M_1 are $\le m^+$, and all points from M_3 are $\ge m^+$. The only common point of these two sets is therefore the point m^+.

Hence, $y = m^+$. From $m^+ = y \in M_2 = M + (x - m^-)$, we can conclude that $m^+ - (x - m^-) \in M$. But $m^+ - (x - m^-) = (m^+ + m^-) - x = m - x$. So, $m - x \in M$. Q.E.D.

Proof of Proposition 5.

i)→ ii) Assume that translates of M satisfy the 2−Helly property. Let's prove that every 2-cocycle is a 2-coboundary. Indeed, let f be a 2-cocycle. A 2-cocycle is a 2-cochain for which $\delta^2 f = 0$. The fact that f is a 2-cochain means that for some integer s, we have s^2 values $f(i,j) \in M_2 = M - M$, $1 \le i \le s$, $1 \le j \le s$ such that $f(i,j) = -f(j,i)$. The condition $\delta^2 f = 0$ means that

$$f(j,k) - f(i,k) + f(i,j) = 0 \tag{1}$$

for all i, j, and k.

We want to prove that $f = \delta^1 g$ for some 1-cochain g. In other words, we want to prove that there exist values $g(i) \in M$ such that

$$f(i,j) = g(i) - g(j). \qquad (2)$$

To prove the existence of such g, we will do the following:
1) choose some sets M_i, $1 \le i \le s$;
2) prove that $M_i \cap M_j \neq \phi$ for all i, j;
3) conclude from here that the intersection of all M_i is not empty;
4) and find $g(i)$ from this intersection.

As M_i, we take $M - f(i,1)$. Let's fix $i \neq j$ and prove that $M_i \cap M_j \neq \phi$. Due to (1) for $k = 1$, we have $f(j,1) - f(i,1) + f(i,j) = 0$, and therefore,

$$f(i,j) = f(i,1) - f(j,1). \qquad (3)$$

Since $f(i,j) \in M - M$, by definition of $M - M$, there exist values $a(i)$ and $a(j)$ such that $f(i,j) = a(i) - a(j)$. Due to (3), we have $f(i,1) - f(j,1) = a(i) - a(j)$, and hence, $a(i) - f(i,1) = a(j) - f(j,1)$. From $a(i) \in M$, we conclude that $a(i) - f(i,1) \in M - f(i,1) = M_i$. Similarly, $a(j) - f(j,1) \in M_j$. So, M_i and M_j indeed have a common element.

Therefore, because of 2−Helly property of M, all M_i have a common element. Let us denote this element by c. Let's prove that $f = \delta^1 g$ for $g(i) = c + f(i,1)$.

First we prove that g is a 1−cochain. From $c \in M_i = M - f(i,1)$, we conclude that $g(i) = c + f(i,1) \in M$, so g is indeed a 1−cochain.

To prove that f is a coboundary of g, we must prove (2). This immediately follows from the equation (3) and the definition of g.

So, every 2−cocycle is a 2−coboundary. Q.E.D.

ii)$\to$ *i)* Assume that every $2-$cocycle is a $2-$coboundary, and that the sets $M_i = M + \vec{a}_i$ have pairwise non-empty intersection. Let's prove that there exists a point that is common to all M_i. To prove that, let's take a function

$$f(i,j) = \vec{a}_j - \vec{a}_i. \tag{4}$$

First, let us prove that f is a $2-$cocycle. Indeed, this function is evidently antisymmetric, and it satisfies equation (1). So, to prove that f is a $2-$cocycle, it is sufficient to prove that $f(i,j) \in M_2 = M - M$ for all i and j. Indeed, M_i and M_j have a common point, let's denote it z. From $z \in M_i = M + \vec{a}_i$, it follows that $z - \vec{a}_i \in M$. Similarly, $z - \vec{a}_j \in M$. Hence,

$$(z - \vec{a}_i) - (z - \vec{a}_j) = \vec{a}_j - \vec{a}_i \in M - M.$$

So, f is a $2-$cocycle.

Therefore, f is a $2-$coboundary, i.e., there exist $g(i) \in M$ for which (2) is true for all i and j. Substituting our definition of f (equation (4)) into (2), we conclude that $\vec{a}_j - \vec{a}_i = g(i) - g(j)$, which is equivalent to $g(i) + \vec{a}_i = g(j) + \vec{a}_j$. So, for the point $t = g(i) + \vec{a}_i$, we have $t \in M + \vec{a}_i = M_i$ for all i. Hence, $M_1 \cap \ldots \cap M_s \neq \phi$. Q.E.D.

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