HOW TO REPRESENT MEASUREMENT ERRORS? GEOMBINATORIC APPROACH

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Abstract. What is the set of possible values of a measurement error? In the majority of practical applications, an error is caused not by a single cause; it is caused by a large number of independent causes, each of which adds a small component to the total error. As a result, we get a geometric description of the area of possible values of error. In this paper, we formulate the known result (for 1-dimensional case, when this area is known to be an interval), and a hypothesis for multi-dimensional case (where it is conjectured to be a convex set).

1. FORMULATION OF A REAL-LIFE PROBLEM

Suppose that we have a measuring device that measures the values of one or several physical quantities $x_1, x_2, ..., x_k$ (e.g., current, voltage, frequency, etc). A measurement cannot be absolutely accurate, there are always some sources of error. As a result, the measured values $\tilde{x}_j$ differ from the actual ones x_j, and the errors $e_j = \tilde{x}_j - x_j$ are different from 0. For each measuring instrument, possible values of the error vector $\vec{e} = (e_1, ..., e_k)$ form an area in $k-$dimensional space.

In 1-dimensional case, it is usually an interval. In multi-dimensional case, it is usually either a parallelotope, or an ellipsoid (see, e.g., [Fuller 1987] and [Rabinovich 1993]). The main question that will be addressed in this paper is as follows: *What other areas can be thus represented?*

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2. HOW TO FORMULATE THIS PROBLEM IN GEOMETRIC TERMS?

To find an answer to the above-formulated question, we can take into consideration the fact that an error is usually caused by a large number of different factors. Therefore, this error $\vec{e}$ is a sum of the large number n of small independent component vectors: $\vec{e} = \vec{e}_1 + ... + \vec{e}_n$.

What do we mean by "small", and what do we mean by "independent"? (Let's recall that we are talking about geometry here, not about statistics, so independence has to be defined).

Let us denote the set of all possible values of a component $\vec{e}_i$ by E_i.

- It is natural to understand "small" as follows: some number $\delta > 0$ is fixed, and a component is *small* (or, to be more precise, $\delta-$*small*) if all its possible values do not exceed δ, i.e., if $\|\vec{e}_i\| \le \delta$ for all $\vec{e}_i \in E_i$.
- In order to understand independence, let us first give an example when two error components are $\vec{e}_i$ and $\vec{e}_j$ are *not* independent. For example, they may be mainly caused by the same factor and must therefore be $\alpha-$close for some small α. Then, for a given value of $\vec{e}_i$, the corresponding set of possible values of $\vec{e}_j$ is the ball of radius δ with a center in $\vec{e}_i$, and is thus different for different $\vec{e}_i$.

  So, it is natural to call the components $\vec{e}_i$ and $\vec{e}_j$ *independent* if the set of possible values of $\vec{e}_j$ does not depend on the value of $\vec{e}_i$.

  In other words, this means that all pairs $(\vec{e}_i, \vec{e}_j)$, where $\vec{e}_i \in E_i$ and $\vec{e}_j \in E_j$, are possible. Therefore, the set of all possible values of the sum $\vec{e}_i + \vec{e}_j$ coincides with the set
  $$\{\vec{e}_i + \vec{e}_j : \ \vec{e}_i \in E_i, \vec{e}_j \in E_j\},$$
  i.e., with the sum $E_i + E_j$ of the two sets E_i and E_j.

Before we turn to formal definitions, let's show that if the set of all possible values of an error is not closed, we will never be able to find that out. Indeed, suppose that E is not closed. This means that there exists a value $\vec{e}$ that belongs to the closure of E, but does not belong to E itself. In every test measurement, we measure error with some accuracy δ. Since $\vec{e}$ belongs to the closure of E, there exists a value $\vec{e}\,' \in E$ such that $\|\vec{e}\,' - \vec{e}\| \le \delta$. So, if the actual error is $\vec{e}\,'$ (and $\vec{e}\,' \in E$, and is thus a possible value of an error), we can get $\vec{e}$ as a result of measuring that error. So, no matter how precisely we measure errors, $\vec{e}$ is always possible. Therefore, we will never be able to experimentally distinguish between the cases when $\vec{e}$ is possible and when it is not.

In view of that, to add the limit point $\vec{e}$ to E or not to add is purely a matter of convenience. Usually, the limit values are added. For example, in 1-D case, we usually consider closed intervals $[-\varepsilon, \varepsilon]$ as sets of possible values of error [Fuller 1993], [Rabinovich 1993]. In view of that, we will assume that the sets E and E_i are closed.

Now, we are ready for formal definitions.

3. FORMAL DEFINITIONS AND WHAT IS CURRENTLY KNOWN

By a *sum* $A+B$ of two sets $A, B \subseteq R^k$, we understand the set $\{\vec{a}+\vec{b} : \vec{a} \in A, \vec{b} \in B\}$. For a given $\delta > 0$, a set A is called $\delta-$*small* if $\|\vec{a}\| \le \delta$ for all $\vec{a} \in A$. By a distance $\rho(A,B)$ between sets A and B, we will understand Hausdorff distance (so, for sets, terms like "$\delta-$close" will mean $\delta-$close in the sense of ρ).

Comment. For reader's convenience, let us reproduce the definition of Hausdorff distance: $\rho(A,B)$ is the smallest real number δ for which the following two statements are true:

- for every $\vec{a} \in A$, there exists a $\vec{b} \in B$ such that $\|\vec{a} - \vec{b}\| \le \delta$;
- for every $\vec{b} \in B$, there exists an $\vec{a} \in A$ such that $\|\vec{a} - \vec{b}\| \le \delta$.

We are interested in describing sets that can be represented as $E = E_1 + \ldots + E_n$ for $\delta-$small E_i. Such sets are fully described for 1-dimensional case ($k = 1$). Namely, in that case, the following results are true (see, e.g., [Kreinovich 1993]):

PROPOSITION 1. *If $E = E_1 + \ldots + E_n$ is a sum of $\delta-$small closed sets from R, then E is $\delta-$close to an interval.*

PROPOSITION 2. *If $E \subseteq R$ is a bounded set, and for every $\delta > 0$, E can be represented as a finite sum of $\delta-$small closed sets, then E is an interval.*

For reader's convenience (and in the hope that they will help to solve the open problems), the proofs are included in the Appendix (they are very simple).

4. OPEN PROBLEMS

HYPOTHESIS 1. *If $E \subseteq R^k$ is a bounded set, and for every $\delta > 0$, E can be represented as a finite sum of $\delta-$small closed sets, then E is convex.*

Comment. The inverse to this statement is trivially true: if E is a convex set, then $E = (1/n)E + ... + (1/n)E$ (n times); if we choose a sufficiently large n, we can make $(1/n)E$ as small as necessary.

HYPOTHESIS 2. *If $E = E_1 + ... + E_n$ is a sum of $\delta-$small closed sets from R^k, then E is $\delta-$close to a convex set.*

Comment. If Hypothesis 2 can be disproved, then maybe we can still prove that E is $C(k)\delta-$close to E, where $C(k)$ is some constant depending on the dimension k?

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APPENDIX: 1-D PROOFS

Proof of Proposition 1. Since each set E_i is $\delta-$small, it is bounded. Since E_i is also closed, it contains its least upper bound $\sup E_i$, and its greatest lower bound $\inf E_i$ (see, e.g., [Sprecher 1987]). Let us denote $\sup E_i$ by e_i^+, and $\inf E_i$ by e_i^-. Then, $\{e_i^-, e_i^+\} \subseteq E_i \subseteq [e_i^-, e_i^+]$. Therefore, $\underline{E} \subseteq E \subseteq \overline{E}$, where we denoted $\underline{E} = \{e_1^-, e_1^+\} + \{e_2^-, e_2^+\} + ... + \{e_n^-, e_n^+\}$, $\overline{E} = [e_1^-, e_1^+] + [e_2^-, e_2^+] + ... + [e_n^-, e_n^+] = [e^-, e^+]$,

$$e^- = \sum_{i=1}^{n} e_i^-, \text{ and } e^+ = \sum_{i=1}^{n} e_i^+.$$

Let us show that E is $\delta-$close to the interval $\overline{E}$. Since $E \subseteq \overline{E}$, every element $a \in E$ belongs to $\overline{E}$. So, it is sufficient to prove that if $b \in \overline{E}$, then b is $\delta-$close to some $a \in E$.

We will show that b is $\delta-$close to some a from the set $\underline{E}$ (which belongs to E because $\underline{E} \subseteq E$). Indeed, by definition of the sum of the sets, the set $\underline{E}$ contains, in particular, the following points:

$$a_0 = e_1^- + e_2^- + ... + e_n^-,$$

$$a_1 = e_1^+ + e_2^- + ... + e_n^-,$$

$$a_2 = e_1^+ + e_2^+ + e_3^- + ... + e_n^-,$$

$$...,$$

$$a_i = e_1^+ + e_2^+ + ... + e_i^+ + e_{i+1}^- + ... + e_n^-,$$

$$...,$$

$$a_n = e_1^+ + e_2^+ + ... + e_n^+.$$

Notice that the values a_0 and a_n coincide with the endpoints e^-, e^+ of the interval $\overline{E}$.

Each value a_i is obtained from the previous one by changing one term in the sum (namely, e_i^-) to another term that is not smaller than e_i^-, namely, to e_i^+. Therefore, $a_0 \le a_1 \le a_2 \le ... \le a_n$.

The difference between two consequent terms in this sequence is equal to $a_i - a_{i-1} = e_i^+ - e_i^-$. Since each E_i is $\delta-$small, we have $|e_i^+| \le \delta$, $|e_i^-| \le \delta$, and therefore, $|a_i - a_{i-1}| = |e_i^+ - e_i^-| \le |e_i^+| + |e_i^-| \le 2\delta$. So, the distance between any two consequent numbers in a sequence $a_0 \le a_1 \le ... \le a_n$ is $\le 2\delta$.

Now, suppose that we are given a number $b \in \overline{E} = [a_0, a_n]$. If $b = a_i$ for some i, then we can take $a = a_i = b$. So, it is sufficient to consider the case when $b \neq a_i$ for all i. In particular, in this case, $a_0 < b < a_n$. The value $a_0 - b$ is negative, the value $a_n - b$ is positive, so the sign of $a_i - b$ must change from $-$ to $+$ somewhere. Let us denote by i the value where it changes, i.e., the value for which $a_i - b < 0$ and $a_{i+1} - b > 0$. For this i, $a_i < b < a_{i+1}$. Therefore,

$$|a_i - b| + |a_{i+1} - b| = (b - a_i) + (a_{i+1} - b) = a_{i+1} - a_i \leq 2\delta.$$

The sum of two positive numbers $|a_i - b|$ and $|a_{i+1} - b|$ does not exceed 2δ. Hence, the smallest of these two numbers cannot exceed the half of 2δ, i.e., cannot exceed δ. So, either for $a = a_i$, or for $a = a_{i+1}$, we get $|a - b| \leq \delta$. Hence, E is $\delta-$close to the interval $\overline{E}$. Q.E.D.

Proof of Proposition 2. Let E be a set that satisfies the condition of this Proposition. Since E is a sum of finitely many closed sets, it is itself closed. Since E is bounded and close, it contains $\inf E$ and $\sup E$. So, $E \subseteq [\inf E, \sup E]$. Let us prove that $E = [\inf E, \sup E]$.

Indeed, let e be an arbitrary point from an interval $[\inf E, \sup E]$. Let us prove that $e \in E$. Indeed, for every natural k, we can take $\delta_k = 2^{-k}$. Since $\delta_k > 0$, E is a sum of closed δ_k-small sets. Therefore, according to Proposition 1, there exists a $e_k \in E$ such that $|e_k - e| \leq \delta_k = 2^{-k}$. So, $e = \lim e_k$, where $e_k \in E$, and e is thus a limit point for E. Since E is closed, $e \in E$. Q.E.D.