HOW TO MEASURE ARBITRARY DISTANCES USING A GIVEN STANDARD LENGTH (I.E., A STICK WITH TWO MARKS ON IT): IT IS NECESSARY, IT IS THEORETICALLY POSSIBLE, IT IS FEASIBLE

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Abstract. It is known that every mapping that preserves unit distance is an isometry. This result has an immediate practical application: it shows that if we have a standard unit of length (i.e., a stick with two marks on it) that enables us to check whether a given distance is equal to 1 unit or not, then we can (in principle) measure an arbitrary distance. This unit-distance preserving theorem proves that such measurement is potentially possible. However, the actual procedures that stem from the existing proofs are too lengthy and thus, not practically feasible (for example, according to these procedures, we need at least 10^6 applications of the original standard to determine distance with accuracy 10^{-6}). In this paper, we describe a *feasible* procedure that measures an arbitrary distance using a given standard unit of length.

1. IT IS NECESSARY

For many decades, a standard of length (a standard meter) was a distance between two marks on a stick stored in a special location. Nowadays, a standard length is defined as a distance between the two consequent wavepeaks of a certain electromagnetic wave (actually, a meter is defined as a certian multiple of this distance). We can reproduce this unit of length and get the exact unit, but in real life, we must also measure arbitrary distances that are not necessarily equal to one unit. To get precise measurements, it is

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necessary to be able to use the unit of length for measuring arbitrary distances.

2. IT IS THEORETICALLY POSSIBLE

By applying a stick with two marks, we can check for every two points in a 3-D space whether the distance between these two points is equal to 1 or not. So, in order to decide whether a stick with two marks is theoretically sufficient for measuring arbitrary distances, we must figure out whether knowing all pairs of points (x, y) with $d(x, y) = 1$ will determine all the distances uniquely.

A simple way to get distances between any two points is to determine the coordinates of all the points. With respect to coordinates, our question can be reformulated as follows: if we know all the pairs (x, y) for which $d(x, y) = 1$, will we be able to reconstruct coordinates uniquely? Non-uniqueness would mean that in addition to the standard Cartesian coordinates, it is possible to assign other coordinates to points from R^3 and still get the same pairs with distance 1. If we denote the mapping from the standard coordinates to the new ones by $f : R^3 \to R^3$, then in terms of such mappings, the questions is: to describe all the functions $f : R^3 \to R^3$ for which $d(x, y) = 1$ iff $d(f(x), f(y)) = 1$.

Such function have been described in (Beckman et al 1953): they are isometries (hence, linear mappings). Therefore, no matter what coordinates we use, the distances are uniquely determined by the set of all pairs (x, y) for which $d(x, y) = 1$. In other words, it is *theoretically possible* to uniquely determine the distance between any two points if we have a stick with two marks.

3. IS IT FEASIBLE?

Formulation of the problem. The above result shows, crudely speaking, that if we make sufficiently many applications of the stick with two marks, then we will be able to measure an arbitrary distance. The natural question is: is the resulting procedure feasible? I.e., how many times do we need to apply the stick with two marks to actually measure a distance between the two given points with a given accuracy?

The existing procedures are not feasible. The procedures that have been presented in the existing proofs of the above-mentioned theorem are not very feasible: for example, a procedure from (Townsend 1970) requires $\geq m+n$ applications of a unit distance to measure the distance of m/n. So, according to this procedure, we need ≥ 1000 applications of a standard to measure a distance (e.g., 0.739=739/1000) with an accuracy 10^{-3}, and similarly, at least 10^6 applications of a standard to measure distances with an accuracy 10^{-6}. This is too much.

Other proofs and ideas from (Zvengrowski 1965), (Greenwell et al 1976), (Greenwell et al 1993), (Johnson 1994), (Kuz'minykh 1994; see also references therein) also (to the best of our knowledge) do not automatically lead to feasible measuring procedures.

The reason why the existing procedures are not feasible is because they have been invented as theoretical proofs, and not as practical procedures. So, the question remains: is a feasible procedure possible?

What we are planning to do. We want to present a feasible procedure that enables us to measure an arbitrary distance using the stick with two marks.

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4. MOTIVATIONS FOR THE FOLLOWING DEFINITIONS

To design a feasible procedure, let us describe what elementary steps we can apply.

- First, we have the original stick. We can duplicate it, and we can apply the original stick or its copy to an arbitrary pair of points x and y, and thus check whether $d(x,y) < 1$, $d(x,y) = 1$, or $d(x,y) > 1$. In general, if we have already managed to design an auxiliary standard of length α, then we will be able, given x and y, to check whether $d(x,y) < \alpha$, $d(x,y) = \alpha$, or $d(x,y) > \alpha$. It is natural to call this operation *comparison*.

- Second, in addition to the original solid body with two marks, we can form a new solid body with new marks, that will be used as a new standard (i.e., distances between these marks will be also used for comparison).
  - The first example of such a procedure is a simple copying of the original standard. If a standard has more than two marks, and we do not need all of them, then we can copy only some of them.
  - As a more complicated example, let us consider designing a *triangle* (x, y, z) with distances $d(x, y) = d(y, z) = d(z, x) = 1$. Physically, we can design a triangle, e.g., if we fix the original two marks as x and y, and then, using copies of the original standard, try to find z for which $d(x, z) = d(y, z) = 1$. If $d(x, z) > 1$, we bring z closer to x; if $d(z, x) < 1$, we move z further away from x; similarly, with respect to y.

  For this procedure to lead to a new *standard*, we need it to be reproducible. In other words, we want to list the conditions on the distances between the marks in such a way that all the distances between all the marks will be uniquely determined by these conditions. It is natural to call this step *design of an auxiliary standard.*

Now, we are ready for the formal definitions.

5. DEFINITIONS AND THE MAIN RESULT

Definition 1.

- By a *standard* S, we mean a finite sequence $(x_1, ..., x_n)$ of different points from R^3.
- We say that a standard is *linear* if it is contained in a straight line.
- We say that a sequence $(y_1, ..., y_n)$ is a *copy* of S if $d(y_i, y_j) = d(x_i, x_j)$ for all i and j.
- Let $S = (x_1, ..., x_n)$ be a standard. By an *instruction for partial copying*, we mean a finite sequence $I = (i_1, ..., i_m)$ of different integers from 1 to n. We say that a standard $(x_{i_1}, ..., x_{i_m})$ is obtained from S by *applying instruction* I.

Definition 2. By a *checking condition* for an $m-$element sequence $(y_1, ..., y_m)$, we mean a pair (S, I) consisting of a standard $S = (x_1, ..., x_n)$ and a sequence $I = (i_1, ..., i_n)$ of n different integers $\le m$. We say that this condition is *satisfied* if the subsequence $(y_{i_1}, ..., y_{i_n})$ is a copy of S.

Definition 3. We say that a sequence of checking conditions *determines a new standard* if the following two statements are true:

- there exists a sequence that satisfies all these conditions, and
- all sequences that satisfy these conditions are copies of each other.

Definition 4. By a *comparison result*, we mean a pair (S, s), where:

- $S = (x_1, x_2)$ is a 2-point standard, and
- s is one of the symbols $<$, $=$, or $>$.

We say that a non-negative real number d is *consistent* with the comparison result (S, s) if the relation between d and $d(x_1, x_2)$ is described by the symbol s (i.e., d is consistent with $((x_1, x_2), <)$ iff $d < d(x_1, x_2)$, etc).

Definition 5. We say that a sequence of comparison results is *consistent* if there exists a real number d that is consistent with all of them.

Comment. For each comparison result, the set of all real numbers d that are consistent with this result forms either an semi-open interval $[0, d(x_1, x_2))$, or a one-point set $\{d(x_1, x_2)\}$, or an infinite open interval $(d(x_1, x_2), \infty)$. Therefore, for every consistent set of comparison results, the set of all real numbers that are consistent with all of them form an intersection of such sets, i.e., an interval (open or semi-open, finite or infinite, and maybe, degenerate).

Definition 6. By a *state of the measurement*, we mean a pair $(\mathcal{S}, \mathcal{R})$, where:

- $\mathcal{S}$ is a finite set of standards, and
- $\mathcal{R}$ is a finite set of comparison results made with standards from $\mathcal{S}$.

Comment.

- $\mathcal{S}$ describes all the standards (main and auxiliary) that have been designed so far, and
- $\mathcal{R}$ describes the results of comparisons that have been applied so far.

Definition 7. By a *distance measuring procedure*, we mean the mapping that takes as input a state $(\mathcal{S}, \mathcal{R})$ of the measurement, and returns one of the following four outputs:

- a 2-point standard from $\mathcal{S}$;
- a sequence of checking conditions (with standards from $\mathcal{S}$) that determines a new standard;
- a pair consisting of a standard $S \in \mathcal{S}$ and an instruction I for its partial copying;
- the message "stop".

Depending on the output, the *result of applying one step of this procedure* to points $x, y \in R^3$ and state $(\mathcal{S}, \mathcal{R})$ is defined as follows:

- If the output is a 2-point standard $S = (x_1, x_2)$, then the new state is obtained by adding the comparison result (S, s) to the set $\mathcal{R}$ of comparison results, where s is defined as follows:
  - s is $<$ if $d(x, y) < d(x_1, x_2)$;
  - s is $=$ if $d(x, y) = d(x_1, x_2)$;
  - s is $>$ if $d(x, y) > d(x_1, x_2)$;
- If the output is a set of checking conditions that defines a new standard, then we add the new standard defined by these conditions to $\mathcal{S}$.
- If the output is a pair (S, I) of a standard $S \in \mathcal{S}$ and an instruction I for its partial copying, then we add to $\mathcal{S}$ the new standard that is obtained from S by applying instruction I.
- If the output is the message "stop", then we return the interval consisting of all the real numbers d that are consistent with all the comparison results from $\mathcal{R}$.

By the *application of the measuring procedure* to points $x, y \in R^3$, we mean the following sequence of measuring states and intervals:

- Initially, the state is (S_0, ϕ), where S_0 is a 2-point set with unit distance between the points.
- If $k-$th state is defined, then $(k+1)-$th state (or interval) is

obtained as the result of applying one step of the measuring procedure to $k-$th state.

If the application of the measuring procedure ends in an interval, then we say that this procedure *terminates*, and call the resulting interval the *result* of applying the measuring procedure to points x and y.

End of Definition 7.

Comment. By definition of a distance measuring procedure, the resulting interval always contians the actual distance $d(x, y)$.

Definition 8. Let $\varepsilon > 0$ be a real number. We say that a measuring procedure *measures distances* $\leq D$ *with accuracy* ε if for every $x, y \in R^3$, for which $d(x, y) \leq D$, this procedure terminates, and the result of applying this procedure is an interval of length $\leq \varepsilon$.

PROPOSITION 1. *There exists a constant* $C > 0$ *such that for every integer* $n > 0$, *there exists a measuring procedure that measures distances* $\leq 2^m$ *with accuracy* 2^{-n} *and terminates in* $\leq C(m + n)$ *steps.*

Comments.

1. This result can be reformulated as follows: Suppose that we want to describe a distance $d(x, y)$ that is $\leq 2^m$ with accuracy 2^{-n}. In binary representation of numbers, this means that the desired result is a binary real number with $m + n$ binary digits (m before the binary point and n after). Our Proposition says that it is possible to get these $m + n$ bits in $\leq C(m + n)$ steps: no more than C steps per bit. To get the measuring result with the accuracy of $10^{-6} \approx 2^{-20}$, we thus only need $\leq C \cdot 20$ steps. As we will see from the proof, $C \approx 10$, so this is quite feasible. So, *it is not only theoretically possible, but it is also feasible to measure an arbitrary distance using only a standard meter (i.e., a stick with two marks).*

2. The following result shows that this linear bound on the number of steps is as feasible as we can get:

PROPOSITION 2. *There exists a constant $c > 0$ such that for every integer $n > 0$, for every measuring procedure that measures distances with accuracy 2^{-n}, there exists a distance $d(x, y) \le 2^m$, for which the procedure does not terminate in $c(m + n)$ steps.*

Comment. Similar results are true if we take R^2 instead of R^3 in our definitions.

6. PROOFS

Proof of Proposition 1. Before we describe the desired measuring procedure, let us describe the auxiliary procedures that it will consist of.

1. First, let us describe the procedure that, given a 2-point standard $S = (x_1, x_2)$ of length a, returns a 3-point linear standard (y_1, y_2, y_3) with $d(y_1, y_2) = d(y_2, y_3) = a$ and $d(y_1, y_3) = 2a$, and a 2-point standard (y_1, y_3) with $d(y_1, y_3) = 2a$. This will be done in three steps:

- First, we design a standard consisting of two regular tetrahedra with a common face: $S_1 = (x_1, ..., x_5)$ with the following checking conditions:
  - $d(x_1, x_2) = d(x_1, x_3) = d(x_1, x_4) = d(x_2, x_3) = d(x_2, x_4) = d(x_3, x_4) = a$ (meaning that (x_1, x_2, x_3, x_4) is a regular tetrahedron), and
  - $d(x_5, x_2) = d(x_5, x_3) = d(x_5, x_4) = d(x_2, x_3) = d(x_2, x_4) = d(x_3, x_4) = a$ (meaning that (x_5, x_2, x_3, x_4) is a regular tetrahedron).

  This configuration uniquely determines all the distances and is therefore a new standard. It is known that $d(x_1, x_5) = \sqrt{8/3} \cdot a$.
- By partial copying of S_1, we can get a new 2-point standard $S_2 = (z_1, z_2)$ with $d(z_1, z_2) = \sqrt{8/3} \cdot a$.
- Now, we can design a new standard $S_3 = (t_1, ..., t_9)$ by using the following checking conditions:
  - $d(t_i, t_{i+1}) = a$ for $1 \le i \le 5$, and $d(t_6, t_1) = a$;
  - $d(t_i, t_7) = a$ for $1 \le i \le 6$;
  - $d(t_i, t_8) = d(t_i, t_9) = \sqrt{8/3} \cdot a$ for $1 \le i \le 6$.

Let us check that these conditions satisfy Definition 3.

This set of checking conditions is consistent. Indeed, we can pick an arbitrary point t_7, pick a plane that contains t_7, and in that plane, place 6 equilateral triangles of size a with t_7 as one of the vertices side-to-side: $t_1t_2t_7$, $t_2t_3t_7$, ..., $t_6t_1t_7$. Then, we place points t_8 and t_9 on two sides of the line going through t_7 and orthogonal to the chosen plane so that the distance between t_8 and each of the points $t_i, 1 \le i \le 6$, is exactly $\sqrt{8/3} \cdot a$ (due to Pythagoras theorem, we must place t_8 and t_9 at a distance $\sqrt{5/3} \cdot a$ from t_7).

This set of checking conditions determines all the distances uniquely. Indeed:

- Since $d(t_8, t_i) = d(t_9, t_i) = \sqrt{8/3} \cdot a$ for $i = 1, ..., 6$, the points $t_i, 1 \le i \le 6$ belong to the plane $\{z \mid d(t_8, z) = d(t_9, z)\}$.
- From the fact that $d(t_i, t_7) = a$ for $i \le 6$, it follows that the first six points t_i belong to a sphere of radius a with a center in t_7.
- So, the first six points t_i belong to the intersection of the sphere and the plane, i.e., to a circle.
- Thus, we have 6 points on a circle, and the distance between each point and the next one is exactly a. Hence, these 6 points form a regular sexagon with size a. Therefore, the radius of this circle is equal to a. One can easily see that the only point whose distance form all $t_1, ..., t_6$ is a is the center of this circle. Therefore, t_7 is this center.
- It is also easy to see that the points that are equidistant from all these six points belong to the symmetry axis (i.e., to the line that is orthogonal to the plane in which $t_1, ..., t_6$ are located, and passes through t_7). On this axis, there are exactly two different points t_i whose distance from t_1 is exactly $\sqrt{8/3} \cdot a$. So, we have a unique location for t_8 and t_9.

As a result, all the distances are determined uniquely, so, we can define a new standard S_2.

- Finally, we get the desired standards (y_1, y_2, y_3) and (y_1, y_3)

by partial copying from S_2: namely, we take $y_1 = t_1$, $y_2 = t_7$, and $y_3 = t_4$.

2. Let us now show how, given a 2-point standard $S = (x_1, x_2)$ of length a, we can design a 2-point standard (z_1, z_2) with $d(z_1, z_2) = a/2$. This is done similarly to the usual proofs:

- First, we design a standard $S_1 = (y_1, y_2, y_3)$ with $d(y_1, y_2) = d(y_2, y_3) = a$ and $d(y_1, y_3) = 2a$.
- Then, we design a new standard $S_2 = (t_1, t_2, t_3, t_4, t_5)$ with the following three checking conditions:
  - (t_1, t_2, t_3) is a copy of S_1;
  - (t_1, t_4, t_5) is a copy of S_1; and
  - (t_3, t_5) is a copy of S.

  It is easy to see that all the distances in S_2 are determined uniquely, and that $d(t_2, t_4) = a/2$.
- Finally, we make a partial copy of the standard S_2, retaining only the points t_2 and t_4.

3. Now, we are ready to describe the preliminary part of the desired measuring procedure (that must be done before we start any comparisons):

- First, we apply the construction described in Part 2 of this proof m times, and generate standards with length 1/2, 1/4, .., 2^{-m}. Each generation requires a constant number of steps, so, the total number of steps for this generation is $\le Cm \le C(m+n)$.
- Second, we generate $m + n$ linear 3-point standards (like in Part 1 of this proof) with distances $d(y_1, y_2) = 2^{-m}, 2^{-(m-1)}, ..., 1, 2, ..., 2^n$. To do that, we consequently apply the method of Part 1 for $a = 2^{-m}, ..., 1, ..., 2^n$. Totally, we need $n + m$ applications of this procedure, and each applications consists of finitely many (actually, four) steps. So, the total number of steps is also $\le C(m+n)$.

4. Finally, let us describe the part that depends on the comparisons. This part will implement the *bisection* algorithm. Namely, it will consist of $m + n$ big steps. On $p-$th big step, we start with a linear sequence of points (x_1, x_2, x_3) for which $d(x_1, x_2) \le$

$d(x, y) \le d(x_1, x_3)$ and $d(x_2, x_3) = 2^{m-p}$. On the next step, we compare $d(x, y)$ with the midpoint $(1/2)(d(x_1, x_2) + d(x_1, x_3))$ of the interval in which this distance was located at the beginning of the big step, and, depending on the results of this comparison, we design a new standard.

For $p = m+n$, we will thus have the desired $2^{-n}-$size interval that contains $d(x, y)$.

Each big step will be performed as follows:

- Initially, we take a 2-point standard with $d(x_1, x_3) = 2^m$. This standard satisfies the desired condition if we take $x_2 = x_1$.
- Suppose now that after $p-$th big step, we have a standard $S = (x_1, x_2, x_3)$ for which $d(x_1, x_2) \le d(x, y) \le d(x_1, x_3)$ and $d(x_2, x_3) = 2^{m-p}$, and which is linear. Then, the next big step consists of the following sequence of steps:
  - Use a linear standard $S_1 = (y_1, y_2, y_3)$ with $d(y_1, y_2) = d(y_2, y_3) = (1/2)2^{m-p}$ (designed in the preliminary part of this measuring procedure) to design a new standard $S_2 = (x_1, x_2, x_4, x_3)$ for which (x_1, x_2, x_3) is a copy of S, and (x_2, x_4, x_3) is a copy of S_1.
  - Make a partial copy (x_1, x_4) of the standard S_2, and compare $d(x, y)$ with $d(x_1, x_4)$. If $d(x, y) \le d(x_1, x_4)$, then return a partial copy (x_1, x_2, x_4) of S_2 as a result of the next big step. If $d(x, y) < d(x_1, x_4)$, then we return a partial copy (x_1, x_4, x_3) of S_2 as the desired big step's result.

  It is easy to see that the desired conditions are satisfied for this new 3-point standard (z_1, z_2, z_3) (i.e., that $d(z_1, z_2) \le d(x, y) \le d(z_1, z_3)$ and $d(z_2, z_3) = 2^{m-(p+1)}$).

5. Each big step consists of finitely many steps, and the total number of big steps is $m+n$. Therefore, the total number of steps that form this part of the measuring procedure is $\le C(m+n)$. Combining this inequality with the similar estimate from Part 3 of this proof, we conclude that the total number of steps is $\le C(m+n)$. Q.E.D.

Proof of Proposition 2. Suppose that we are given a measuring procedure that measures distances with accuracy 2^{-n}. Let us denote the largest number of steps that this procedure requires for all distances $\le 2^m$ by K. If $K=\infty$, then we have proved our Proposition. So, it is sufficient to consider the case when $K<\infty$.

To prove the proposition, we must show that $K \ge c(m+n)$ for some c. Indeed, for a fixed measuring procedure, on each step, we have at most 3 different possible outputs (namely, if this step is a comparison, we have three possible outputs, and in all other cases, we have only one possibility). So, after the first step, we have at most 3 different states. Similarly, for each of these ≤ 3 states, on the second step, we have at most 3 different outputs, so, the total number of different states after 2 steps of the measuring procedure is $\le 3 \cdot 3 = 3^2$ possible states. Similarly, after k steps, we have at most 3^k possible states. Hence, in all the cases in which the procedure ended in k steps, we have at most 3^k different resulting intervals. We know that the procedure ends in $\le K$ steps, so, it can end in 0, 1, 2, ..., K steps. Combining all these cases, we conclude that this procedure can return $\le 3^0 + 3^1 + ... + 3^K$ intervals. This expression is a geometric progression, and its sum $(3^{K+1}-1)/(3-1)$ is bounded by $(3/2)\cdot 3^K$. So, we have $\le (3/2)\cdot 3^K$ different intervals as a result of the procedure.

Each interval contains the original distance and has length $\le 2^{-n}$. Therefore, if one and the same interval is returned for two different distances, this means that both distances belong to this interval, and therefore, the difference between these distances does not exceed the length 2^{-n} of this interval. So, if we take the distances $0, 2\cdot 2^{-n}, 4\cdot 2^{-n}, ..., 2^m$ whose differences are $> 2^{-n}$, then we can be sure that these distances result in different intervals. The total number of these distances is $2^m/(2\cdot 2^{-n}) = 2^{m+n-1}$. Therefore, we have $\ge 2^{m+n-1}$ different intervals.

So, $2^{m+n-1} \le$ number of different intervals $\le (3/2)\cdot 3^K$. Therefore, $2^{m+n-1} \le (3/2)\cdot 3^K$. Taking logarithms of both sides, we conclude that $K \ge (\ln(2)/\ln(3))(m+n-1) - \ln(3/2)/\ln(3)$, from which desired inequality easily follows. Q.E.D.

Proof of the Comment (about R^2). For R^2, the proof is similar, with the only difference that we can immediately start with the 7-point planar construction (regular sexagon and a point in the middle) that has lead us to the linear standard with three equidistant marks.

7. OPEN PROBLEM

It is know that every mapping that preserves unit distance is an isometry. In practical terms, this theorem says that if we are able to check whether a distance between two given points is equal to 1 or not, then, in principle, we can uniquely determine an arbitrary distance. This is a real-life problem, because in real distance measurements, the standard length is defined as a distance between two given points (marks on a stick in the old days; the distance between the wave peaks, according to the current standard), and the problem of how to measure arbitrary distances is left to the user.

Potentially, we can use procedures outlined in the existing proofs of the unit-distance preserving theorem. However, these procedure take too long to be practically feasible (e.g., they require 10^6 comparisons with a standard to get the result with an accuracy 10^{-6}). So, a natural question arises: can we have a feasible procedure for such a measurement?

In this paper, we have shown that a feasible procedure is possible. Moreover, we have designed a procedure that was proved to be asymptotically optimal; to be more precise, it was proved to be optimal modulo a multiplication constant. So, the next natural question is: *to design a procedure for measuring an arbitrary distance using a standard meter that is truly* (and not just asymptotically) *optimal, i.e., a procedure that will require the smallest possible number of steps.*

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