CAUSALITY EXPLAINS WHY SPATIAL AND TEMPORAL TRANSLATIONS COMMUTE: A REMARK

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Abstract. *Under reasonable assumptions, it is proven that if a space-time has symmetries of translation type, then these symmetries form a commutative group.*

Historically, the first physics-oriented non-planar geometric models of space-time (proposed by Einstein) were smooth manifolds. However, it turned out that in the general case, physical space-times cannot be described by smooth manifolds: first, due to the equations of general relativity, they have *singularities* [Misner 1973], and second, due to quantum effects, space-times are locally non-smooth. Therefore, a more general mathematical description of space-time is needed. A natural idea is to use for describing space-time a structure that is more physically fundamental than the structure of a smooth manifold. The most fundamental structure related to space-time is the structure of *causality*; therefore, in [Buseman 1967], [Kronheimer 1967], and [Pimenov 1970], it was suggested to describe a space-time as an *ordered set* $(M, \le)$, with $a \le b$ meaning that an event a can causally influence an event b. In Newtonian physics, if two events a and b are simultaneous, they can influence each other; in other words, we have $a \le b$ and $b \le a$ and therefore, $\le$ is not an order. Since Einstein, however, it is believed that such instantaneous action is impossible. In view of that, we will assume that $\le$ is an order.

Definition 1. *By a space-time, we mean an ordered set $(M, \le)$ that has at least one pair (a, b). A 1-1 mapping $g : M \to M$ is called a symmetry if it preserves causality, i.e., if for every a and b, $a \le b$ iff $g(a) \le g(b)$.*

Comments.

- It is easy to see that symmetries form a group.

- Observers inside the space-time use numbers to describe the events. A function $x : M \to R$ that assigns numbers to events will be called a *coordinate*. Since we consider symmetric space-times (with symmetries generalizing translations in space-time), it is reasonable to consider only *inertial* coordinates, i.e., coordinates in which "translations" from the group G act as shifts $x \to x + \text{const}$. Of special interest are *temporal coordinates* t, i.e., coordinates in which if a can causally influence b, then $t(a) \le t(b)$.

  In special relativity, it is not only true that every time when $a \le b$, we have $t(a) \le t(b)$ for all temporal coordinates t, but the inverse is also true: the only reason why for some pairs of events a cannot influence b is that in some inertial coordinates, the time of b *precedes* the time of a. It is reasonable to make a similar assumption for our general case as well. Let us formulate it in mathematical terms.

Definition 2. *Let $(M, \le)$ be a space-time, and let G be a group of symmetries of M.*

- *By a coordinate, we understand a function $x : M \to R$.*
- *A coordinate x is called inertial if for every $g \in G$ there exists a number $s_x(g)$ such that $x(g(a)) = x(a) + s_x(g)$ for all a.*
- *A coordinate x is called temporal if $x(a) \le x(b)$ whenever $a \le b$.*
- *A pair (M, G) is called natural if the following condition holds:*

  For every a, b, if $t(a) \le t(b)$ for all inertial temporal coordinates t, then $a \le b$.

PROPOSITION. *Let $(M, \le)$ be a space-time, and let G be a group of symmetries of M. If (M, G) is a natural pair, then the group G is commutative (i.e., $g_1 g_2 = g_2 g_1$ for all $g_i \in G$).*

Proof.

1^o. Let us now prove that for every inertial coordinate x and for every $g_1, g_2 \in G$, we have $s_x(g_1 g_2) = s_x(g_1) + s_x(g_2)$.

Indeed, for every x and for every $a \in M$, we have $x(g_1(g_2(a))) = x(g_2(a)) + s_x(g_1) = x(a) + s_x(g_2) + s_x(g_1)$. On the other hand, $s(g_1(g_2(a))) = s(g_1 g_2(a)) = s_x(g_1 g_2) + g(a)$. Equating the resulting two expressions for $s(g_1(g_2(a)))$, we get the desired equality.

2^o. Let us finally prove that for every a, we have $g_1 g_2 = g_2 g_1$.

Indeed, for every a and for every temporal inertial coordinate t, we have $t(g_1 g_2(a)) = s_t(g_1) + s_t(g_2) + t(a)$ and similarly have $t(g_2 g_1(a)) = s_t(g_1) + s_t(g_2) + t(a)$. Hence, for every t, $t(g_1 g_2 a) = t(g_2 g_1 a)$ and therefore, $t(g_1 g_2 a) \ge t(g_2 g_1 a)$. Since the pair (M, G) is assumed to be natural, it follows that $g_1 g_2(a) \ge g_2 g_1(a)$. Similarly, $g_2 g_1(a) \ge g_1 g_2(a)$. Since $\le$ is an order, we conclude that $g_1 g_2(a) = g_2 g_1(a)$ for all a, i.e., that $g_1 g_2 = g_2 g_1$. Q.E.D.

Comment. This simple proof was influenced by the results presented (for topological groups) in [Gładysz 1962], [Gładysz 1964], and [Charin 1966], p. 139.

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