Symmetric Linear Systems with Perturbed Input Data

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Dedicated to Prof. Dr.–Ing. Hans–Wolfgang Stolle on the occasion of his 65th birthday

Abstract. We present a method for constructing a set of inequalities which describe completely the so–called *symmetric solution set* $S_{sym} := \{x \in \boldsymbol{R}^n |\ Ax = b,\ A = A^T \in [A],\ b \in [b]\}$. Here, $[A]$ is an $n \times n$ interval matrix satisfying $[A] = [A]^T$ and $[b]$ is an interval vector with n components.

1 Introduction

In many interval analytic algorithms one is faced with the problem of enclosing the solution set

$$S := \{x \in \boldsymbol{R}^n |\ Ax = b,\ A \in [A],\ b \in [b]\}$$

where A varies in a given $n \times n$ interval matrix $[A]$ and where $[b]$ is a given interval vector with n components. Often, for the underlying problem it is sufficient to know interval bounds for the so–called *symmetric solution set*

$$S_{sym} := \{x \in \boldsymbol{R}^n |\ Ax = b,\ A = A^T \in [A],\ b \in [b]\}$$

where now only symmetric matrices A from $[A]$ are allowed as coefficient matrices. For S_{sym} we always assume $[A] = [A]^T$ in contrast to S, for which $[A]$ is not subject to such a restriction. The set S_{sym} can be interpreted as the solution set of symmetric linear systems with perturbed input data. It is obvious that S_{sym} is a subset of S and that interval bounds for S are trivially bounds for S_{sym}. In [1] and [4] it was illustrated, however, that S_{sym} may be bounded by a smaller interval vector than it is necessary for S. Therefore, it is interesting to know how S_{sym} looks like. In [3] (and, for the two–dimensional case, already in [2]) it was shown that the boundary of S_{sym} can completely be described by pieces of hyperplanes and by pieces of quadrics in $\boldsymbol{R}^n$. In our present contribution we will outline the ideas for the proof of this property by a slightly modified way as compared with that in [3].

2 Characterization of S_{sym} by inequalities

Let $x = (x_i) \in \boldsymbol{R}^n$ be given and fixed. By definition, the property '$x \in S_{sym}$' is equivalent to the existence of some $A = (a_{ij}) = A^T \in [A] = [A]^T,\ b \in [b]$ such that $Ax = b$. This, in turn, can be equivalently expressed by the assertion

$$\exists\, a_{ij} \in \boldsymbol{R} \text{ such that } \left\{ \begin{array}{cl} \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, & i,j = 1,\dots,n \\ a_{ij} = a_{ji}, & i,j = 1,\dots,n \\ \underline{b}_i \le \sum_{j=1}^n a_{ij}x_j \le \overline{b}_i, & i = 1,\dots,n \end{array} \right\}. \quad (1)$$

For $i,j = 1,\dots,n$ let

$$a_{ij}^- := \left\{ \begin{array}{ll} \underline{a}_{ij}, & \text{if } x_ix_j \ge 0 \\ \overline{a}_{ij}, & \text{if } x_ix_j < 0 \end{array} \right. , \qquad a_{ij}^+ := \left\{ \begin{array}{ll} \overline{a}_{ij}, & \text{if } x_ix_j \ge 0 \\ \underline{a}_{ij}, & \text{if } x_ix_j < 0 \end{array} \right. ,$$

$$b_i^- := \left\{ \begin{array}{ll} \underline{b}_i, & \text{if } x_i \ge 0 \\ \overline{b}_i, & \text{if } x_i < 0 \end{array} \right. , \qquad b_i^+ := \left\{ \begin{array}{ll} \overline{b}_i, & \text{if } x_i \ge 0 \\ \underline{b}_i, & \text{if } x_i < 0 \end{array} \right. .$$

Then (1) is equivalent to

$$\left.\begin{array}{l} x\in S \quad \wedge \quad \exists\, z_{ij}\in \boldsymbol{R} \text{ such that} \\ \qquad \left\{\begin{array}{ll} a_{ij}^- x_i x_j \le z_{ij} \le a_{ij}^+ x_i x_j, & i,j=1,\dots,n, \\ \qquad z_{ij} = z_{ji}, & i,j=1,\dots,n, \\ b_i^- x_i \le \sum_{j=1}^n z_{ij} \le b_i^+ x_i, & i=1,\dots,n. \end{array}\right. \end{array}\right\} \tag{2}$$

We want to prove this equivalence. The implication '(1) $\Rightarrow$ (2)' is immediately seen by setting $z_{ij} := a_{ij}x_i x_j$. In order to show the converse let (2) be true. We will construct $A\in \boldsymbol{R}^{n\times n}$ such that $A = A^T \in [A]$ and $Ax \in [b]$. To this end, for each $i = 1,\dots,n$ let j run from 1 to n. Consider now a fixed index pair i_0, j_0 and define $a_{i_0 j_0}$ according to the following procedure:

Case 1: $x_{i_0} = 0$.
Since $x \in S$ by (2), there are real numbers $a^*_{i_0 j}$ for $j = 1,\dots,n$ such that

$$\underline{a}_{i_0 j} \le a^*_{i_0 j} \le \overline{a}_{i_0 j} \tag{3}$$

and

$$\underline{b}_{i_0} \le \sum_{j=1}^n a^*_{i_0 j} x_j \le \overline{b}_{i_0}. \tag{4}$$

If $x_{j_0} \ne 0$ then $a_{i_0 j_0} := a^*_{i_0 j_0} =: a_{j_0 i_0}$; if $x_{j_0} = 0$ then $a_{i_0 j_0} := \underline{a}_{i_0 j_0}$.

Case 2: $x_{i_0} \ne 0$.
If $x_{j_0} \ne 0$ then $a_{i_0 j_0} := \dfrac{z_{i_0 j_0}}{x_{i_0} x_{j_0}}$; if $x_{j_0} = 0$ then $a_{i_0 j_0}$ is already defined by the preceding case, when the roles of i_0 and j_0 are exchanged.

In this way a matrix A is determined which is symmetric and which apparently fulfills the first double inequality of (1). In order to prove

$$\underline{b}_i \le \sum_{j=1}^n a_{ij} x_j \le \overline{b}_i \quad \text{for } i = 1,\dots,n, \tag{5}$$

we consider a fixed index i_0, and we distinguish two cases:

Case 1: $x_{i_0} = 0$.
We first show

$$a^*_{i_0 j} x_j = a_{i_0 j} x_j \quad \text{for } j = 1,\dots,n. \tag{6}$$

If $x_{j_0} \neq 0$ then $a^*_{i_0j_0} = a_{i_0j_0}$ by definition, and (6) is true for $j = j_0$. If $x_{j_0} = 0$ then (6) holds trivially for $j = j_0$. Therefore,

$$\sum_{j=1}^{n} a^*_{i_0j} x_j = \sum_{j=1}^{n} a_{i_0j} x_j,$$

and (4) implies (5) for $i = i_0$.

Case 2: $x_{i_0} \neq 0$.

Here, we will use the double inequality

$$b^-_{i_0} x_{i_0} \leq \sum_{j=1}^{n} z_{i_0j} \leq b^+_{i_0} x_{i_0} \qquad (7)$$

from (2). If $x_{j_0} \neq 0$ then

$$z_{i_0j_0} = a_{i_0j_0} x_{i_0} x_{j_0} \qquad (8)$$

by the definition of $a_{i_0j_0}$. If $x_{j_0} = 0$ then the first double inequality of (2) implies $z_{i_0j_0} = 0$, and (8) holds again. Replace now z_{i_0j} in (7) by $a_{i_0j} x_{i_0} x_j$ and divide this inequality by x_{i_0}. If $x_{i_0} > 0$ then $b^-_{i_0} = \underline{b}_{i_0}$, $b^+_{i_0} = \overline{b}_{i_0}$ by definition, and (5) holds. If $x_{i_0} < 0$ then the division by x_{i_0} reverses the inequality signs in (7), but since $b^-_{i_0} = \overline{b}_{i_0}$, $b^+_{i_0} = \underline{b}_{i_0}$ the assertion (5) follows again for $i = i_0$.

Thus the implication '(2) $\Rightarrow$ (1)' is also true.

The inequalities (2) form the starting point for the construction of inequalities characterizing S_{sym}. The idea consists in transforming (2) in at most $\dfrac{n(n+1)}{2}$ steps equivalently into a set of inequalities which do not contain any z_{ij}. In each step (at least) one of the $z_{ij} = z_{ji}$, $i \leq j$, is completely eliminated. How this can be achieved is illustrated by the elimination step for z_{12}:

Let (2) hold. Then it is equivalent to

$$
\left.
\begin{array}{l}
x \in S \quad \wedge \quad \exists\, z_{ij} \in \boldsymbol{R} \text{ such that} \\
\qquad \left\{
\begin{array}{l}
a_{12}^{-} x_1 x_2 \le z_{12}, \\
b_1^{-} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j} \le z_{12}, \\
b_2^{-} x_2 - \sum_{j=2}^{n} z_{2j} \le z_{21} = z_{12}, \\
z_{12} \le a_{12}^{+} x_1 x_2, \\
z_{12} \le b_1^{+} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j}, \\
z_{12} \le b_2^{+} x_2 - \sum_{j=2}^{n} z_{2j}, \\
\text{remaining (in)equalities of (2) \quad (unchanged!)}
\end{array}
\right.
\end{array}
\right\} \tag{9}
$$

Note that none of the 'remaining (in)equalities of (2)' in (9) contains z_{12}. A real number z_{12}, which fulfills the first six inequalities in (9), exists if and only if the maximum of the first three left–hand sides in (9) is less than or equal to the minimum of the right–hand sides of the next three inequalities, i. e.,

$$
\begin{array}{l}
\max\{a_{12}^{-} x_1 x_2,\ b_1^{-} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j},\ b_2^{-} x_2 - \sum_{j=2}^{n} z_{2j}\} \\
\le \min\{a_{12}^{+} x_1 x_2,\ b_1^{+} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j},\ b_2^{+} x_2 - \sum_{j=2}^{n} z_{2j}\}.
\end{array} \tag{10}
$$

This, in turn, holds if and only if one requires that each element of the set for the maximum in (10) is less than or equal to each element of the set for the minimum in (10), i. e.,

$$
\left.
\begin{array}{l}
a_{12}^{-} x_1 x_2 \le b_1^{+} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j}, \\
a_{12}^{-} x_1 x_2 \le b_2^{+} x_2 - \sum_{j=2}^{n} z_{2j}, \\
b_1^{-} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j} \le a_{12}^{+} x_1 x_2, \\
b_1^{-} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j} \le b_2^{+} x_2 - \sum_{j=2}^{n} z_{2j}, \\
b_2^{-} x_2 - \sum_{j=2}^{n} z_{2j} \le a_{12}^{+} x_1 x_2, \\
b_2^{-} x_2 - \sum_{j=2}^{n} z_{2j} \le b_1^{+} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j},
\end{array}
\right\} \tag{11}
$$

where we omitted the three trivial inequalities

$$
\begin{array}{rcl}
a_{12}^{-} x_1 x_2 & \le & a_{12}^{+} x_1 x_2, \\
b_1^{-} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j} & \le & b_1^{+} x_1 - z_{11} - \sum_{j=3}^{n} z_{1j}, \\
b_2^{-} x_2 - \sum_{j=2}^{n} z_{2j} & \le & b_2^{+} x_2 - \sum_{j=2}^{n} z_{2j}.
\end{array}
$$

Summarizing we have shown that the property '$x \in S_{sym}$' is equivalent to

$$\left.\begin{array}{l} x \in S \quad \wedge \quad \exists\, z_{ij} \in \boldsymbol{R} \text{ such that} \\ \qquad \left\{\begin{array}{l} a_{12}^- x_1 x_2 \le b_1^+ x_1 - z_{11} - \sum_{j=3}^n z_{1j}, \\ a_{12}^- x_1 x_2 \le b_2^+ x_2 - \sum_{j=2}^n z_{2j}, \\ b_1^- x_1 - z_{11} - \sum_{j=3}^n z_{1j} \le a_{12}^+ x_1 x_2, \\ b_1^- x_1 - z_{11} - \sum_{j=3}^n z_{1j} \le b_2^+ x_2 - \sum_{j=2}^n z_{2j}, \\ b_2^- x_2 - \sum_{j=2}^n z_{2j} \le a_{12}^+ x_1 x_2, \\ b_2^- x_2 - \sum_{j=2}^n z_{2j} \le b_1^+ x_1 - z_{11} - \sum_{j=3}^n z_{1j}, \\ \text{remaining (in)equalities of (2)} \quad \text{(as in (9))}, \end{array}\right. \end{array}\right\} \qquad (12)$$

where z_{12} does no longer occur. This process of eliminating z_{ij} can be continued until all z_{ij} have disappeared.

Taking a closer look to the resulting inequalities shows that for each $i \in \{1, \ldots, n\}$ there are 2 inequalities which can be divided by $x_i \neq 0$ such that no fractions occur. For example, if the first inequality of (12) is combined successively with the inequalities $a_{1j}^- x_1 x_j \le z_{1j}$ one obtains the final inequality $\sum_{j=1}^n a_{1j}^- x_1 x_j \le b_1^+ x_1$ which can be divided by $x_1 \neq 0$. In an analogous way one thus gets

$$\sum_{j=1}^n \hat{a}_{ij}^- x_j \le \overline{b}_i,\ i = 1, \ldots, n, \quad \text{and} \quad \sum_{j=1}^n \hat{a}_{ij}^+ x_j \ge \underline{b}_i,\ i = 1, \ldots, n, \qquad (13)$$

with $\hat{a}_{ij}^- := \begin{cases} \underline{a}_{ij}, & \text{if } x_j \ge 0 \\ \overline{a}_{ij}, & \text{if } x_j < 0 \end{cases}, \qquad \hat{a}_{ij}^+ := \begin{cases} \overline{a}_{ij}, & \text{if } x_j \ge 0 \\ \underline{a}_{ij}, & \text{if } x_j < 0 \end{cases}.$

The inequalities (13) are just those which characterize S (cf. [2]). They can either be omitted from the list of inequalities, if $x \in S$ remains there as in (12), or '$x \in S$' can be cancelled when (13) is used instead. This last remark holds also if some of the components of x are zero.

The following statements are obvious:

(i) The coefficients a_{ij}^-, a_{ij}^+, $\hat{a}_{ij}^-$, $\hat{a}_{ij}^+$, b_i^-, b_i^+ do not change if x varies in any fixed orthant.

(ii) If $\hat{a}_{ij}^-$, $\hat{a}_{ij}^+$ are fixed and if they are no longer be thought to be influenced by x_j (in contrast to their definition!) then each inequality (13) describes a half space in $\boldsymbol{R}^n$ when x varies in $\boldsymbol{R}^n$.

Under similar assumptions on a_{ij}^-, a_{ij}^+, b_i^-, b_i^+ the remaining inequalities resulting from (12) describe sets the boundaries of which are quadrics in $\boldsymbol{R}^n$.

We have thus shown:

Theorem

Let $[A] = [A]^T$ be an $n \times n$ interval matrix and let $[b]$ be an interval vector with n components. Then for any orthant $O \subseteq \boldsymbol{R}^n$ the set $S_{sym} \cap O$ can be represented as an intersection of finitely many closed sets the boundaries of which are quadrics or hyperplanes. The inequalities characterizing these hyperplanes and quadrics can be derived from the elimination process described above, or they are of the form $x_i = 0$. □

Examples which illustrate this theorem can be found in [2] and [3]. One of them reads

$$[A] := \begin{pmatrix} 1 & [1,2] \\ [1,2] & [-1,0] \end{pmatrix}, \quad [b] := \begin{pmatrix} 4 \\ [1,2] \end{pmatrix}.$$

Here, the set S is the convex hull of the points $(\frac{1}{2}, \frac{7}{4})$, $(\frac{1}{2}, \frac{7}{2})$, $(\frac{8}{3}, \frac{2}{3})$ and $(3,1)$, while the boundary of S_{sym} is formed by the straight line between $(\frac{1}{2}, \frac{7}{4})$ and $(\frac{8}{5}, \frac{6}{5})$, by the straight line between $(1,3)$ and $(3,1)$, by the part of the parabola $x_2 = 4 - (x_1 - 2)^2$ between $(\frac{1}{2}, \frac{7}{4})$ and $(1,3)$ and by the part of the circle $(x_1 - 2)^2 + (x_2 + 1)^2 = 5$ between $(\frac{8}{5}, \frac{6}{5})$ and $(3,1)$.

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