Why Are Symmetries a Universal Language of Physics? (on the unreasonable effectiveness of symmetries in physics)

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Abstract

In this paper, we prove that in some reasonable sense, every possible physical law can be reformulated in terms of symmetries. This result explains the well-known success of group-theoretic approach in physics.

1 Formulation of the Problem

Traditional physics used differential equations to describe physical laws. Modern physical theories (starting from quarks) are often formulated not in terms of differential equations (as in Newton's days), but it terms of the corresponding *symmetry groups* (see, e.g., [3, 9, 8]). Moreover, it turned out that theories that have been originally proposed in terms of differential equations, including general relativity (and scalar-tensor version of gravity theory), quantum mechanics, the electrodynamics, etc., can be reformulated in terms of symmetries [4, 1, 2].

Symmetry approach has been very successful in physics. Its success raises two natural questions:

- How universal is this approach? Can we indeed (as some physicists suggest) express any new idea in terms of symmetries.
- Is the fact that many (if not all) physical theories can be expressed in terms of symmetries a general mathematical result or a peculiar feature of our physical world?

2 What We Are Planning to Do

In this paper, we analyze this problem from a mathematical viewpoint. We will show that, in some reasonable sense, every possible physical law can be expressed in terms of symmetries.

Comment. This result was first announced in [5].

In order to prove the corresponding theorem, we will have to formalize what "law" is and what "symmetry" is.

3 Motivations of the Following Definitions

3.1 What do we mean by "law"?

In order to describe what we mean by a physical law, let us first describe what it means that an object does not satisfy any law at all. For example, let us consider the direction of a linear-shaped molecule. If this molecule is magnetic, then its orientation must follow the magnetic field. If the molecule is an electric dipole, then it must follow the electric field, etc. In all these cases, there is a physical law that controls the orientation of the molecule.

It is also possible that the molecule has neither of these orienting properties (or it has, but the corresponding fields are absent). In this case, there is no physical law to control its orientation; therefore, the orientation is not described by any law, it is *random*.

We still have to formalize what "random" means but so far, we hope, the conclusion sounds reasonable: if something does not satisfy any physical law, then it is random.

By inverting this informal statement, we can conclude that an object satisfies a physical law if and only if it is not random. To make this definition precise, we must define what "random" means.

3.2 What do we mean by "random"?

The formal definition of "random" was proposed by Kolmogorov's student P. Martin-Löf [7]; for a current state of the art, see, e.g., [6]. (We will be using a version of this definition proposed by P. Benioff.)

To describe this definition, let us recall how physicists use the assumption that something is random. For example, what can we conclude if we know that the sequence of heads and tails obtained by tossing the coin is random? One thing we can conclude is that the fraction of heads in this sequence tends to 1/2 as the number of tosses tends to infinity. What is the traditional argument behind this conclusion? In mathematical statistics, there is a mathematical theorem saying that w.r.t. the natural probability measure on the set of all infinite sequences, for *almost all* sequences, the frequency of heads tends to 1/2.

In more mathematical terms, this means that the probability measure $\mu(S)$ of the set S of all sequences for which the frequency does not tend to $1/2$ is 0.

Because the property P holds for almost all sequences ω, sequences that do not satisfy this property are (in some sense) *exceptional*. Because we have assumed that a given sequence ω is *random*, it is, therefore, not exceptional, and hence, this sequence ω must satisfy the property P.

The informal argument that justifies this conclusion goes something like that: if ω does not satisfy the property P, this means that ω possesses some property (not P) that is very rare (is almost never true), and therefore, ω is not truly random.

All other existing applications of statistics to physics follow the same pattern: we know that something is true for almost all elements, and we conclude that it is true for an element that is assumed to be random; in this manner, we estimate the fluctuations, apply random processes, etc. So, a random object is an object that satisfies all the properties that are true for almost all objects (almost all with respect to some reasonable probability measure). To give a definition of randomness, we must somewhat reformulate this statement.

To every property P that is true almost always, we can put into correspondence a set S_P of all objects that do not satisfy P; this set has, therefore, measure 0. An object satisfies the property P if and only if it does not belong to the set S_P. Vice versa, if we have a definable set S of measure 0, then the property "not to belong to S" is almost always true.

In terms of such sets, we can reformulate the above statement as follows: if an object is random, then it does not belong to any definable set of measure 0. So, if an object does not belong to any definable set of measure 0, we can thus conclude that it has all the properties that are normally deduced for random objects, and therefore, it can reasonably be called random.

Thus, we arrive at the following definition: an object is random if and only if it does not belong to any definable set S of measure 0 $\mu(S) = 0$, with respect to some natural probability measure μ. This, in effect, is the definition proposed by Martin-Löf.

Example. In the above example, the state of possible orientations can be represented as a unit sphere. The natural probability measure on this sphere is as follows: $\mu(A)$ is the area of the set A divided by the total area (4π) of the sphere.

3.3 What do we mean by "symmetry"?

A symmetry is a transformation on the set of all objects. For example, for orientations, typical symmetries are rotations.

A symmetry is usually assumed to be invertible and therefore, it must be a one-to-one function.

A symmetry must also be *non-trivial* in the sense that being symmetric must be a very informative property (i.e., only very few elements must be symmetric).

In other words, almost all elements must not be invariant with respect to a symmetry.

It is also natural to assume that a symmetry transformation preserves the a priori (natural) probability measure. This is not absolutely necessary; however, our goal is to prove that every physical law can be expressed in terms of symmetries. Therefore, if we managed to prove that it can expressed in terms of symmetries that preserve the a priori measure, we have proven what we intended to.

Now, we are ready for the formal definitions.

4 Definitions

- *Let a language L be fixed (e.g., the language of set theory). We say that a set X is definable if in the language L there is a formula $P(Z)$ with one free set variable Z that is true for only one object: this set X.*

- *Let U be a set with a probability measure μ. We say that an element $u \in U$ is random* w.r.t. *μ if it does not belong to any definable set of $\mu-$measure 0. We say that an element u satisfies some law if it is not random.*

- *By a* symmetry *S, we mean a 1-1, measure preserving, definable mapping $S : U \to U$ for which $\mu\{u \mid S(u) = u\} = 0$ (i.e., almost always $S(u) \neq u$).*

- *A probability space (U, μ) is* non-trivial *if it has at least one symmetry S_0.*

- *We say that an element u is* invariant *(or* symmetric*)* w.r.t. *S if $S(u) = u$.*

5 A Simple Physical Example Illustrating the Definitions

5.1 Example in Physical Terms

To illustrate how the above mathematical definition is related to real physical symmetries, let's use the following simple physical example: Let us describe the fact that a given spherical molecule has no dipole moment $\vec{d}$.

5.2 Reformulation in Terms of Symmetries: Physicists' Viewpoint

From the physical viewpoint, this fact can be easily reformulated in terms of symmetries: namely, it means that the dipole moment vector $\vec{d}$ is invariant w.r.t. arbitrary rotations around the center of the molecule.

5.3 Reformulation in Terms of the Above Mathematical Definition

In this case, the set U is clearly the set of all possible 3-D vectors. If we choose a coordinate system in such a way that the center of the molecule is at a point $\vec{0} = (0,0,0)$, then this set is in 1-1 correspondence with the 3-D space R^3.

To apply our definition, we must choose a probability measure μ on the set $U = R^3$ that is invariant w.r.t. all rotations around 0. The easiest way to do that is to assume that the components d_x, d_y, and d_z of the vector $\vec{d}$ are independent Gaussian random variables with one and the same standard deviation σ. The probability density $\rho(\vec{d})$ of the resulting measure is equal to

$$\text{const} \cdot \exp(-\frac{d_x^2 + d_y^2 + d_z^2}{2\sigma^2}),$$

and is, therefore, invariant w.r.t. rotations. Let us check that our definition of symmetry is satisfied.

First, each rotation is measure preserving (this is how we chose a measure). Second, each rotation S around 0 is a rotation around a line. The set of all points u that are invariant under this rotation S coincides with this line and has, therefore, measure 0. Thus, according to our definition, S is a symmetry.

Since rotations exist, the probability space (R^3, μ) has at least one symmetry and is, therefore, *non-trivial* in the sense of our definition.

The fact that $u = \vec{d} = \vec{0}$ can be now expressed as $S(u) = u$ for all rotations S.

6 The Main Result

Theorem. *An element u of a non-trivial probability space (U, μ) satisfies some law iff u is invariant w.r.t. some symmetry.*

Comments.

- In other words, every physical law can be reformulated in terms of some symmetries.
- Another corollary is that we can now reformulate the definition of randomness in terms of symmetries:

Corollary. *An element u of a non-trivial probability space U is random w.r.t. μ iff u is not invariant w.r.t. any symmetry.*

7 Proof of the Theorem

Let us first prove that if a given object u is invariant w.r.t. some symmetry S, then u is not random. Indeed, if u is invariant w.r.t. some symmetry S, then u

belongs to the set of invariant elements of the symmetry S; let us denote this set by $I(S)$. Since S is definable, this set $I(S)$ is also definable, and by definition of a symmetry, this set is of measure 0. Therefore, the element u is not random.

Let us now prove that if u is not random, then u is invariant w.r.t some symmetry. Indeed, by definition of randomness, the fact that u is not random means that u belongs to a definable set E of measure 0. Since the probability space (U, μ) is non-trivial, by definition, it has at least one symmetry S_0. Let us now define a symmetry $S(u)$ as follows:

- $S(v) = v$ for all v from the set $E \cup S_0(E) \cup S_0^2(E) \cup ... \cup S_0^{-1}(E) \cup ...$, and
- $S(v) = S_0(v)$ for all other v.

It is easy to check that $S(u) = u$. Let us show that S is a symmetry:

- By construction, the function S is 1-1.
- Outside the set $E \cup S_0(E) \cup S_0^2(E) \cup ... \cup S_0^{-1}(E) \cup ...$, the function S coincides with the measure-preserving transformation S_0. The set E has measure 0; since S_0 is a symmetry and hence, measure-preserving and 1-1, the sets $S_0(E)$, ..., all have measure 0. Thus, the union set is a union of countably many sets of measure 0 and hence, has a measure 0. So, S coincides with S_0 everywhere except a set of measure 0. Since S_0 is measure-preserving, we can conclude that S is measure-preserving.
- An element v is an invariant element of S iff either $v \in E$, or $v \in S_0(E)$, ..., or $S_0(v) = v$. Thus, $I(S) \subseteq E \cup S_0(E) \cup ... \cup I(S_0)$. We have already shown that the set $E \cup S_0(E) \cup S_0^2(E) \cup ... \cup S_0^{-1}(E) \cup ...$ has measure 0. The set $I(S_0)$ has a measure 0 since S_0 is a symmetry. Therefore, $I(S)$ is contained in the union of the sets of measure 0, and is, hence, itself a set of measure 0. So, for almost all v, we have $S(v) \neq v$.

So, S is a symmetry, and thus, u is invariant w.r.t. some symmetry. Q.E.D.

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