FUZZY IMPLICATION REVISITED: A NEW TYPE OF FUZZY IMPLICATION EXPLAINS YAGER'S IMPLICATION OPERATION

I. B. Türkşen[1] and V. Kreinovich[2]

[1]Department of Industrial Engineering
University of Toronto
Toronto, Ontario
M5S 1A4 Canada
email turksen@ie.utoronto.ca

[2]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
email vladik@cs.utep.edu

1. Formulation of the problem

1.1. It is necessary to represent implication in expert systems

Our knowledge of complex systems is often incomplete, and therefore, we have to rely on the expert's statements, These statements are usually formulated not in mathematical terms, but in words of natural languages. Fuzzy logic is a methodology for formalizing such statements. Since a large part of expert knowledge consists of if–then statements, it is therefore important to formalize implication (i.e., if–then statements) in fuzzy logic.

1.2. How implication is represented now : two general ideas

There has been a lot of research on representing &, $\vee$, and $\neg$ in fuzzy logic. So, let us assume that we already have operations $f_\&, f_\vee : [0,1] \times [0,1] \to [0,1]$ and $f_\neg : [0,1] \to [0,1]$ that correspond to &, $\vee$, and $\neg$. In this situation, there are two basic ways to describe an implication operation $f_\to$ in fuzzy logic (see, e.g., [3,8], and reference therein; for other references, see, e.g., [2,6]):

- Descriptions based on *explicit* representations of $\to$ in terms of &, $\vee$, and $\neg$: e.g., $A \to B = B \vee \neg A$ leads to $f_\to(a,b) = f_\vee(b, f_\neg(a))$. The resulting functions $f_\to : [0,1] \times [0,1] \to [0,1]$ are called *$S-$operations.*
- Descriptions based on *implicit* representations of $\to$ in terms of &, $\vee$, and $\neg$: E.g., $A \to B$ is the weakest statement C with the property that $C\&A$ implies B. This representation leads to $f_\to(a,b) = \inf\{c | f_\&(c,b) \geq a\}$. The resulting functions $f_\to : [0,1] \times [0,1] \to [0,1]$ are called *$R-$operations.*

1.3. Some reasonable implication operations cannot be represented in this form

There are, however, some reasonable implication operations (e.g., Yager's b^a [11]) that cannot be easily described in this manner.

1.4. There is a way to represent these implication operations, but the resulting representation is rather artificial

In [4,5,3], it was shown that such operations can be described if we allow *non-commutative* $\&-$operations $f_\&$.

This is a rather artificial representation because for such operations, the resulting degree of belief $f_\&(a,b)$ in a statement $A\&B$ may be different from the degree of belief $f_\&(b,a)$ in a seemingly equivalent statement $B\&A$.

1.5. Formulation of the problem

It would be nice to find a natural way to represent the new implication operations, without using artificial non-commutative "and"-operations.

2. Main result

2.1. The main idea

In the present paper, we show that Yager's operation naturally appears for the simplest $\&-$operation ($f_{\&}(a,b) = a \cdot b$) if in addition to $S-$ and $R-$operations, we allow the new, *third* type of implication operations: we call them $A-$*operations*, because they are uniquely determined by some reasonable *a*xioms. Let us describe these axioms:

2.2. The new axioms and their motivation

(I0) For $a, b \in \{0,1\}$, $\rightarrow$ should be consistent with the classical implication, i.e., $a \rightarrow b = 1$ unless $a = 1$, $b = 0$, in which case, $a \rightarrow b = 0$.

(I1) $a \rightarrow (b\&c) \equiv (a \rightarrow b)\&(a \rightarrow c)$.

Motivation. If A implies $B\&C$, this means that A implies B, and that A implies C.

Definition. We will say that an operation $f_{\rightarrow}$ *satisfies* the axiom (I1) if $f_{\rightarrow}(a, f_{\&}(b,c)) = f_{\&}(f_{\rightarrow}(a,b), f_{\rightarrow}(a,c))$ for all a, b, and c. A similar definition can be repeated for other axioms.

(I2) $a \rightarrow (b \rightarrow c) \equiv (a\&b) \rightarrow c$.

Motivation. If A implies that "B implies C", this means that whenever we have both A and B, we can deduce C. Vice versa, if $A\&B$ implies C, this means that if we have A, then from B, we can deduce C.

(I3) $(0.5 \rightarrow b)\&(0.5 \rightarrow b) \equiv b$.

Motivation. If we have a statement A about which we know nothing (so it is safe to assume that the degree of belief in both A and $\neg A$ is equal to 0.5), and if we know that B can be deduced from both A and $\neg A$, then B must be true. Vice versa, if B is true, then B can be deduced from both A and $\neg A$.

(I4) $a \rightarrow b \equiv \neg b \rightarrow \neg a$.

Motivation. This is a known property of the implication.

2.3. The result itself

MAIN RESULT. *Let $f_{\&}(a,b) = a \cdot b$ and $f_{\neg}(a) = 1 - a$. Let us assume that an implication operation $f_{\rightarrow}(a,b)$ is continuous for $a, b > 0$. Then:*

- *If $f_{\rightarrow}$ satisfies* (I0), (I1), *and* (I2), *then $f_{\rightarrow}(a,b) = b^{a^r}$ for some $r > 0$.*
- *If $f_{\rightarrow}$ satisfies* (I0), (I1), (I2), *and* (I3), *then $f_{\rightarrow}(a,b) = b^a$.*
- *If $f_{\rightarrow}$ satisfies* (I0), (I1), *and* (I4), *then $f_{\rightarrow}(a,b) = \exp(k \ln(a) \cdot \ln(1-b))$ for some $k > 0$.*

2.4. Conclusions and comments

1. If we use axioms (I0)–(I3), then we get the desired explanation of Yager's implication. In two other cases, we get new operations that are worth trying.

2. $f_{\&}(a,b) = a \cdot b$ is a particular case of a *strict* $\&-$operation ($t-$norm). A generic case is $f^*_{\&}(a,b) = \varphi^{-1}(\varphi(a) \cdot \varphi(b))$ for some monotonic function $\varphi : [0,1] \rightarrow [0,1]$. For this case, the

above-described sets of axioms lead to $f^*_\to(a,b) = \varphi^{-1}(f_\to(\varphi(a),\varphi(b)))$, where $f_\to$ is an implication operation described in the Main Result.

3. A similar approach can be used to describe hedges of the type "very", "slightly" as functions $f : [0,1] \to [0,1]$ with the property $f(a \cdot b) = f(a) \cdot f(b)$ that corresponds to the condition that *very* $(A\&B)$ means the same as *very* A and *very* B. This condition leads to $f(a) = a^r$ for some $r > 0$ [7]. We can combine these two results by considering implications with the condition $h_1A_1\&...\&h_nA_n$ for some hedges h_i. For $f_\&(a,b) = a \cdot b$, the resulting degree of belief in this condition is $a_1^{r_1} \cdot \ldots \cdot a_n^{r_n}$, where a_i is the degree of belief in A_i, and r_i corresponds to h_i. This formula was proposed (for a different reason) by P. Werbos in [10] to make fuzzy logic more "elastic" (in [9], this idea is used for a medical expert system).

3. Proof

For every $a \in [0,1]$, let us denote $f_\to(a,b)$ by $f_a(b)$. Then, due to (I1), the function f_a satisfies the property $f_a(b \cdot c) = f_a(b) \cdot f_a(c)$. Since $f_\to$ is continuous, the function f_a is also continuous, and therefore, the solution of this functional equation is $f_a(b) = b^{p(a)}$ for some p depending on a (see, e.g., [1]). So, $f_\to(a,b) = b^{p(a)}$. Since $f_\to$ is continuous, the function $p(a)$ is also continuous.

From (I2), we can now conclude that $(c^{p(b)})^{p(a)} = c^{p(a)\cdot p(b)} = c^{p(a\cdot b)}$ for all c, hence, $p(a \cdot b) = p(a) \cdot p(b)$. This is the same functional equation as before, so, we know that $p(a) = a^r$ for some r. From (I0), we conclude that $r > 0$.

From (I3), we can now conclude that $b^{0.5^r} \cdot b^{0.5^r} = b^{2\cdot 0.5^r} = b$ for all b, so, $2 \cdot 0.5^r = 1$, $0.5^r = 0.5$, and $r = 1$.

For $f_\to(a,b) = b^{p(a)}$, condition (I4) takes the form $b^{p(a)} = (1-a)^{p(1-b)}$. Taking logarithms of both sides, we get $p(a) \cdot \ln(b) = p(1-b) \cdot \ln(1-a)$. If we divide both sides by both logarithms, we conclude that $p(a)/\ln(1-a) = p(b)/\ln(1-b)$. Since this is true for all a and b, we conclude that the fraction $p(a)/\ln(1-a)$ is a constant independent on a. If we denote this constant by k, we get $p(a) = k \cdot \ln(1-a)$, and hence, the desired formula for $f_\to$. Q.E.D.

Acknowledgments

This work was partially supported by a NSF Grants No. CDA-9015006 and No. EEC-9322370, and by NASA Research Grant No. 9-757. The authors are thankful to Hung T. Nguyen for valuable comments, and to Vladimir Dimitrov for the encouragement.

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