A NEW CLASS OF FUZZY IMPLICATIONS (AXIOMS OF FUZZY IMPLICATION REVISITED)

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Abstract. *Many different fuzzy implication operators have been proposed; most of them fit into one of the two classes: implication operations that are based on an explicit representation of implication $A \to B$ in terms of &, $\vee$, and $\neg$ (e.g., S-implications that are based on the formula $B \vee \neg A$), and R-implications that are based on an implicit representation of implication $A \to B$ as the weakest C for which $C\&B$ implies A. However, some fuzzy implication operations (such as b^a) cannot be naturally represented in this form. To describe such operations, we propose a new (third) class of implication operations called A-implications whose relation to &, $\vee$, and $\neg$ is described by (implicit) axioms.*

Keywords: Basic theory: fuzzy logic, operators

1. Formulation of the problem

1.1. It is necessary to represent implication in expert systems

Our knowledge of complex systems is often incomplete, and therefore, we have to rely on the expert's statements, These statements are usually formulated not in mathematical terms, but in words of natural languages. Fuzzy logic is a methodology for formalizing such statements. Since a large part of expert knowledge consists of if–then statements, it is therefore important to formalize implication (i.e., if–then statements) in fuzzy logic.

1.2. How implication is represented now : two general ideas

There has been a lot of research on representing &, $\vee$, and $\neg$ in fuzzy logic. So, let us assume that we already have operations $f_\&, f_\vee : [0,1] \times [0,1] \to [0,1]$ and $f_\neg : [0,1] \to [0,1]$ that correspond to &, $\vee$, and $\neg$. In this situation, there are two basic ways to describe an implication operation $f_\to$ in fuzzy logic (see, e.g., [3,9,11], and reference therein; for other references, see, e.g., [2,6]):

- Descriptions based on *explicit* representations of $\to$ in terms of &, $\vee$, and $\neg$, such as:
  - the representation $A \to B = B \vee \neg A$ leads to $f_\to(a,b) = f_\vee(b, f_\neg(a))$; the resulting functions $f_\to : [0,1] \times [0,1] \to [0,1]$ are called *S-implications*;
  - the representation $A \to B = B \vee (A \& \neg B)$ leads to $f_\to(a,b) = f_\vee(n, f_\&(a, f_\neg(b)))$; the resulting functions $f_\to : [0,1] \times [0,1] \to [0,1]$ are called *Q-implications* or *QL-implications*.
- Descriptions based on *implicit* representations of $\to$ in terms of &, $\vee$, and $\neg$: E.g., $A \to B$ is the weakest statement C with the property that $C\&A$ implies B. This representation leads to $f_\to(a,b) = \sup\{c | f_\&(c,a) \le b\}$. The resulting functions $f_\to : [0,1] \times [0,1] \to [0,1]$ are called *R-implications*.

1.3. Some reasonable implication operations cannot be represented in this form

There are, however, some reasonable implication operations that cannot be easily described in this manner: e.g., the implication operation $f_\to(a,b) = b^a$ (for $a > 0$ or $b > 0$) and $f_\to(0,0) = 1$, introduced by one of the authors [15].

That this function is indeed a reasonable implication operation can be seen, e.g., from the fact that it satisfies five of nine possible properties of fuzzy implication (and more generally, multi-valued logic implication) listed by Klir and Yuan in [7]; for comparison, the original Zadeh's implication also satisfies five out of nine properties, and several other frequently used implication operations satisfy even less than five properties (some four and some even two).

1.4. There is a way to represent these implication operations, but the resulting representation is rather artificial

In [4,5,3], it was shown that such implication operations can be described if we allow *non-commutative* &−operations $f_\&$.

This is a rather artificial representation because for such &−operations, the resulting degree of belief $f_\&(a,b)$ in a statement $A\&B$ may be different from the degree of belief $f_\&(b,a)$ in a seemingly equivalent statement $B\&A$.

1.5. Formulation of the problem

It would be nice to find a natural way to represent the new implication operations, without using artificial non-commutative "and"-operations.

2. Main result

2.1. The main idea

In the present paper, we show that the implication operation $f_{\rightarrow}(a,b) = b^a$ (and $f_{\rightarrow}(0,0) = 1$) from [15] naturally appears for the simplest &−operation ($f_{\&}(a,b) = a \cdot b$) if in addition to S- and R-implications, we allow the new, *third* type of implication operations: we call them A-*implications*, because they are uniquely determined by some reasonable *a*xioms. Let us describe these axioms:

2.2. The new axioms and their motivation

Remark. In this section, we will assume that two functions are given: $f_{\&} : [0,1] \times [0,1] \rightarrow [0,1]$ and $f_{\neg} : [0,1] \rightarrow [0,1]$.

(I0) For $a, b \in \{0,1\}$, $\rightarrow$ should be consistent with the classical implication, i.e., $a \rightarrow b = 1$ unless $a = 1$, $b = 0$, in which case, $a \rightarrow b = 0$.

Definition 0. *We say that a function* $f_{\rightarrow} : [0,1] \times [0,1] \rightarrow [0,1]$ *satisfies the axiom* (I0) *if* $f_{\rightarrow}(0,0) = f_{\rightarrow}(0,1) = f_{\rightarrow}(1,1) = 1$ *and* $f_{\rightarrow}(1,0) = 0$.

(I1) $a \rightarrow (b \& c) \equiv (a \rightarrow b) \& (a \rightarrow c)$.

Motivation. If A implies $B \& C$, this means that A implies B, and that A implies C.

Definition 1. *We say that a function* $f_{\rightarrow} : [0,1] \times [0,1] \rightarrow [0,1]$ *satisfies the axiom* (I1) *if* $f_{\rightarrow}(a, f_{\&}(b,c)) = f_{\&}(f_{\rightarrow}(a,b), f_{\rightarrow}(a,c))$ *for all* a, b, *and* c.

(I2) $a \rightarrow (b \rightarrow c) \equiv (a \& b) \rightarrow c$.

Motivation. If A implies that "B implies C", this means that whenever we have both A and B, we can deduce C. Vice versa, if $A \& B$ implies C, this means that if we have A, then from B, we can deduce C.

Definition 2. *We say that a function* $f_{\rightarrow} : [0,1] \times [0,1] \rightarrow [0,1]$ *satisfies the axiom* (I2) *if* $f_{\rightarrow}(a, f_{\rightarrow}(b,c)) = f_{\rightarrow}(f_{\&}(a,b), c)$ *for all* a, b, *and* c.

(I3) $(0.5 \rightarrow b) \& (0.5 \rightarrow b) \equiv b$.

Motivation. If we have a statement A about which we know nothing (so it is safe to assume that the degree of belief in both A and $\neg A$ is equal to 0.5), and if we know that B can be deduced from both A and $\neg A$, then B must be true. Vice versa, if B is true, then B can be deduced from both A and $\neg A$.

Definition 3. *We say that a function* $f_{\rightarrow} : [0,1] \times [0,1] \rightarrow [0,1]$ *satisfies the axiom* (I3) *if* $f_{\&}(f_{\rightarrow}(0.5,b), f_{\rightarrow}(0.5,b)) = b$ *for all* b.

(I3′) $(a \to b) \& (\neg a \to b) \equiv b$.

Motivation. If B is true, then B follows from both A and $\neg A$. Vice versa, if we can prove that both in case A is true and in case A is false (i.e., $\neg A$ is true), the statement B is true, then B is true irrespective on whether A is true or not (i.e., B is simply true).

Definition 3′. *We say that a function* $f_\to : [0,1] \times [0,1] \to [0,1]$ *satisfies the axiom* (I3′) *if* $f_\&(f_\to(a,b), f_\to(f_\neg(a),b)) = b$ *for all* a *and* b.

(I4) $a \to b \equiv \neg b \to \neg a$.

Motivation. This is a known property of the implication.

Definition 4. *We say that a function* $f_\to : [0,1] \times [0,1] \to [0,1]$ *satisfies the axiom* (I4) *if* $f_\to(a,b) = f_\to(f_\neg(b), f_\neg(a))$ *for all* a *and* b.

2.3. The result itself

MAIN RESULT. *Let* $f_\&(a,b) = a \cdot b$ *and* $f_\neg(a) = 1 - a$. *Let us assume that a function* $f_\to(a,b)$ *is continuous in all points* (a,b) *except maybe the points* $(0,0)$ *and* $(1,1)$. *Then:*

(i) *If* $f_\to$ *satisfies* (I0), (I1), *and* (I2), *then for some* $r \geq 0$, *the function* $f_\to$ *has the following form:*
- $f_\to(a,b) = b^{a^r}$ *if* $a > 0$ *or* $b > 0$, *and*
- $f_\to(0,0) = 1$.

(ii) *If* $f_\to$ *satisfies* (I0), (I1), (I2), *and* (I3), *then* $f_\to$ *has the following form:*
- $f_\to(a,b) = b^a$ *if* $a > 0$ *or* $b > 0$, *and*
- $f_\to(0,0) = 1$.

(iii) *if* $f_\to$ *satisfies* (I0), (I2), *and* (I3′), *then* $f_\to$ *has the following form:*
- $f_\to(a,b) = b^a$ *if* $a > 0$ *or* $b > 0$, *and*
- $f_\to(0,0) = 1$.

(iv) *If* $f_\to$ *satisfies* (I0), (I1), *and* (I4), *then, for some* $k > 0$, $f_\to$ *has the following form:*
- $f_\to(a,b) = \exp(-k \ln(1-a) \cdot \ln(b))$ *if* $a < 1$ *and* $b > 0$;
- $f_\to(1,b) = 0$ *for* $b < 1$;
- $f_\to(a,0) = 0$ *for* $a > 0$;
- $f_\to(0,0) = f_\to(1,1) = 1$.

Comment. This result was partly published in [13].

2.4. Conclusions and comments

1. If we use axioms (I0)–(I3) (or (I0), (I2), and (I3′)), then we get the desired explanation (at least for $a, b > 0$) of the implication operation $f_\to(a,b) = b^a$. In two other cases, we get new operations that are worth trying.

2. $f_{\&}(a,b) = a \cdot b$ is a particular case of a *continuous strictly Archimedean* $\&-$operation ($t-$norm) (see, e.g., [7,10]). A generic case is $f^*_{\&}(a,b) = \varphi^{-1}(\varphi(a) \cdot \varphi(b))$ for some continuous strictly monotonic function $\varphi : [0,1] \to [0,1]$. For this case, the above-described sets of axioms lead to $f^*_{\to}(a,b) = \varphi^{-1}(f_{\to}(\varphi(a), \varphi(b)))$, where $f_{\to}$ is an implication operation described in the Main Result.

3. A similar approach can be used to describe hedges of the type "very", "slightly" as functions $f : [0,1] \to [0,1]$ with the property $f(a \cdot b) = f(a) \cdot f(b)$ that corresponds to the condition that "*very* $(A \& B)$" means the same as "*very* A and *very* B". This condition leads to $f(a) = a^r$ for some $r > 0$ [8]. We can combine these two results by considering implications with the condition $h_1 A_1 \& ... \& h_n A_n$ for some hedges h_i. For $f_{\&}(a,b) = a \cdot b$, the resulting degree of belief in this condition is $a_1^{r_1} \cdot ... \cdot a_n^{r_n}$, where a_i is the degree of belief in A_i, and r_i corresponds to h_i. This formula was proposed (for a different reason) by P. Werbos (see, e.g., [14]) to make fuzzy logic more "elastic" (in [12], this idea is used for a medical expert system).

3. Proof

3.1. (I0) and (I1)

Let us first see what we can conclude from (I0) and (I1). For every $a \in (0,1)$, let us denote $f_{\to}(a,b)$ by $f_a(b)$. Then, due to (I1), the function f_a satisfies the property $f_a(b \cdot c) = f_a(b) \cdot f_a(c)$. Since $f_{\to}$ is continuous, the function f_a is also continuous, and therefore, the solution of this functional equation is (see, e.g., [1]):

- either $f_a(b) = 0$,
- or $f_a(b) = b^{p(a)}$ for some p depending on a.

Let us show that the case $f_a(b) = f_{\to}(a,b) = 0$ for all $a, b \in (0,1)$ is impossible.

Indeed, in this case, from the assumed continuity of $f_{\to}$, we would get

$$f_{\to}(0,1) = \lim_{\varepsilon \to 0} f_{\to}(\varepsilon, 1-\varepsilon) = \lim 0 = 0,$$

which contradicts to (I0).

So, $f_{\to}(a,b) = b^{p(a)}$ for some $p(a)$.

From $b < 1$ and $f_{\to}(a,b) \le 1$, we conclude that $p(a) \ge 0$. Since $f_{\to}$ is continuous, the function $p(a)$ is also continuous.

So, we arrive at the following conclusion:

If a function $f_{\to}$ satisfies (I0) and (I1), it has the form $f_{\to}(a,b) = b^{p(a)}$ for all $a, b \in (0,1)$.

3.2. (I0), (I1), and (I2)

If, in addition to (I0) and (I1), we also assume (I2), then, from (I2), we can now conclude that $(c^{p(b)})^{p(a)} = c^{p(a) \cdot p(b)} = c^{p(a \cdot b)}$ for all c. Therefore, $p(a \cdot b) = p(a) \cdot p(b)$. This is the same functional equation as before, so, we know that:

- either $p(a) = 0$ for all a,
- or $p(a) = a^r$ for some r.

Let us show that the case $p(a) = 0$ for all a is impossible. Indeed, in this case, $f_{\rightarrow}(a, b) = f_a(b) = b^{p(a)} = b^0 = 1$ for all $a, b \in (0, 1)$. From the continuity of $f_{\rightarrow}$, we would now be able to conclude that $f_{\rightarrow}(1, 0) = 1$, which is inconsistent with (I0). So, this case is indeed impossible, and $p(a) = a^r$ for some r.

Let us show that $r \geq 0$. Indeed, if $r < 0$, then for every $b \in (0, 1)$, from $f_{\rightarrow}(a, b) = b^{a^r}$ and continuity of $f_{\rightarrow}$, we will be able to conclude, for $a \to 0$, that

$$f_{\rightarrow}(0, b) = \lim_{a \to 0} b^{a^r} = \lim_{q \to \infty} b^q = 0.$$

In the limit $b \to 1$, we conclude that $f_{\rightarrow}(0, 1) = 0$, which contradicts to (I0).

So, the cases $r < 0$ is impossible, $r \geq 0$, and thus, $f_{\rightarrow}(a, b) = b^{a^r}$ for all $a, b \in (0, 1)$.

The desired equality of the two expressions $f_{\rightarrow}(a, b)$ and b^{a^r} for *all* pairs (a, b) that are not equal to $(0, 0)$ and $(1, 1)$ (i.e., for $(a, b) \neq (0, 0)$ and $(a, b) \neq (1, 1)$), follows from the equality of these expressions for $a \in (0, 1)$ and $b \in (0, 1)$ and from the fact that both functions $f_{\rightarrow}(a, b)$ and b^{a^r} are continuous for $(a, b) \neq (0, 0)$ and $(a, b) \neq (1, 1)$.

The values of $f_{\rightarrow}(0, 0)$ and $f_{\rightarrow}(1, 1)$ follow from the axiom (I0). This completes the proof of part (i) of our result.

3.3. (I0), (I1), (I2), and (I3)

Let us now assume (I0), (I1), (I2), and (I3). We already know the general form of the operations that satisfy (I0), (I1), and (I2). From (I3), we can now conclude that $b^{0.5^r} \cdot b^{0.5^r} = b^{2 \cdot 0.5^r} = b$ for all b, so, $2 \cdot 0.5^r = 1$, $0.5^r = 0.5$, and $r = 1$. This proves part (ii) of our result.

3.4. (I0), (I1), and (I4)

Let us now prove part (iv).

Let us first prove the desired formula for $a \in (0, 1)$ and $b \in (0, 1)$. We already know that for such a and b, the conditions (I0) and (I1) imply that $f_{\rightarrow}(a, b) = b^{p(a)}$ for some positive continuous function $p(a)$. For this expression $f_{\rightarrow}(a, b) = b^{p(a)}$, condition (I4) takes the form

$$b^{p(a)} = (1 - a)^{p(1-b)}.$$

Taking logarithms of both sides, we get $p(a) \cdot \ln(b) = p(1 - b) \cdot \ln(1 - a)$. If we divide both sides by both logarithms, we conclude that $p(a)/\ln(1 - a) = p(b)/\ln(1 - b)$.

Since this equality is true for all a and b, we conclude that the fraction $p(a)/\ln(1 - a)$ is a constant that is independent on a. Since $p(a) > 0$ and $\ln(1 - a) < 0$, this constant is negative. If we denote this constant by $-k$, we get the formula $p(a) = -k \cdot \ln(1-a)$, and hence, the desired formula for $f_{\rightarrow}$. For $a \in (0,1)$ and $b \in (0,1)$, the formula from part (iv) is proven.

Similarly to the proof of part (i), we can now use continuity and the axiom (I0) to show that the desired formula is true for all pairs (a,b).

3.5. (I0), (I2), and (I3′)

To complete the proof, we must now prove part (iii).

Let us first prove that for every $a \in (0,1)$ and for every $x \in (0,1)$, we have $f_{\rightarrow}(a,x) = x^a$.

In order to prove this equality, we will first prove a weaker statement: that this equality holds for all numbers a of the type $k/2^n$, where n is an arbitrary positive integer, and k is a positive integer that is $< 2^n$. In other words, we want to prove that for for every $x \in (0,1)$, for every positive integer $n > 0$, and for every positive integer $k < 2^n$, we have

$$f_{\rightarrow}(k/2^n, x) = x^{k/2^n}. \qquad (n\,.\,k)$$

If we prove this equality $(n\,.\,k)$ for all n and k, then, from the continuity of $f_{\rightarrow}$ and from the condition (I0), we will be able to conclude that the desired equality is true for arbitrary real numbers a and x.

We will prove $(n\,.\,k)$ by induction over n.

3.5.1. Induction base

For $n = 1$, the only positive integer k with the property $k < 2^n = 2^1 = 2$ is $k = 1$. So, the desired equation $(n\,.\,k)$ takes the form

$$f_{\rightarrow}(1/2, x) = x^{1/2}. \qquad (1.1)$$

To prove (1.1), we notice that from (I3′) for $a = 1/2$, it follows that for every x,

$$f_{\rightarrow}(1/2, x) \cdot f_{\rightarrow}(1/2, x) = x,$$

and hence, $f_{\rightarrow}(1/2, x) = \sqrt{x} = x^{1/2}$.

3.5.2. Induction step

Let us assume that for some n, the equality $(n\,.\,k)$ is proven for all $k < 2^n$. Let us prove that for every $k < 2^{n+1}$, we have

$$f_{\rightarrow}(k/2^{n+1}, x) = x^{k/2^{n+1}}. \qquad (n+1\,.\,k)$$

To prove this equality, we will consider three possible cases: $k = 2^n$, $k < 2^n$, and $k > 2^n$.

If $k = 2^n$, then $k/2^n = 1/2$, and the desired equality coincides with the already proven equality (1.1).

If $k < 2^n$, then, by induction assumption, we have $f_{\rightarrow}(k/2^n, x) = x^{k/2^n}$. Hence, from (I2) for $a = 1/2$ and $b = k/2^n$, we conclude that

$$f_{\rightarrow}(1/2, f_{\rightarrow}(k/2^n, x)) = f_{\rightarrow}(1/2 \cdot k/2^n, x).$$

The right-hand side of this equality is equal to $f_{\rightarrow}(k/2^{n+1}, x)$. The left-hand side, due to the induction assumption and to the already proven formula (1.1), is equal to $(x^{k/2^n})^{1/2} = x^{k/2^{n+1}}$. Hence, for these k, the desired inequality is proven.

Finally, let $k > 2^n$. Then, if we define the difference $2^{n+1} - k$ by k', we can conclude that $k' < 2^n$. Therefore, as we have just proven,

$$f_{\rightarrow}(k'/2^{n+1}, x) = x^{k'/2^{n+1}}. \qquad (n+1\,.\,k')$$

Since $k' + k = 2^{n+1}$, we have $k/2^{n+1} + k'/2^{n+1} = 1$, hence, from (I2), it follows that

$$f_{\rightarrow}(k/2^{n+1}, x) \cdot f_{\rightarrow}(k'/2^{n+1}, x) = x.$$

Therefore, from the equation $(n+1\,.\,k')$, we conclude that

$$f_{\rightarrow}(k/2^{n+1}, x) = \frac{x}{f_{\rightarrow}(k'/2^{n+1}, x)} = \frac{x}{x^{k'/2^{n+1}}} = x^{k/2^{n+1}}.$$

The induction step is proven for all k, and thus, part (iii) is proven. Q.E.D.

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