SPACE-TIME IS "SQUARE TIMES" MORE DIFFICULT TO APPROXIMATE THAN EUCLIDEAN SPACE

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Abstract. How many points do we need to approximate a given metric space S (e.g., a ball in the Euclidean space) with a given accuracy $\varepsilon > 0$? To be more precise, how many points do we need to reproduce the metric $\rho(X, Y)$ on S with an accuracy ε? This problem is known to be equivalent to the following geombinatoric problem: *find the smallest number of balls of given radius ε that cover a given set S.*

A similar approximation problem is also important for *space-times.* In this case, instead of a regular metric $\rho(X, Y)$ that describes *distance* between points X and Y, we have a *kinematic metric* $\tau(X, Y)$ that describes the *proper time* between events X and Y. It turns out, rather surprisingly, that this space-time analogue of the above geombinatoric problem require much more points to approximate: e.g., to approximate a compact set in a 4-D Euclidean space, we need $\approx \varepsilon^{-4}$ points, while to approximate a similar compact in a 4-D space-time, we need $\approx \varepsilon^{-8}$ points, approximately the square of the previous number.

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1. INFORMAL DESCRIPTION OF THE PROBLEM: HOW MANY POINTS DO WE NEED TO DESCRIBE SPACE-TIME WITH A GIVEN ACCURACY?

According to the models used in modern physics, space-time consists of infinitely many points (*events*). If we fix a coordinate system, then each event can be characterized by a pair (x_0, x), where x_0 is a real number (that describes the time of the event), and x is an element of the 3-D space (that describes the spatial location of the event). This subdivision into time and space essentially depends on our choice of a coordinate systems, and thus, does not have any direct physical meaning.

To get a physically meaningful quantity, we must consider *two* different events $X = (x_0, x)$ and $Y = (y_0, y)$. Some pairs of events are related by the *causality* relation $X \leq Y$ (meaning that X can influence Y). If X can influence Y, then we can define *proper time* $\tau(X, Y)$ as a time that a (geodesic) particle will "feel" when moving from X to Y (according to relativity theory, all processes like decay of a particle, aging of a human astronaut, etc., depend on proper time). This function τ is only defined on causally related pairs. In geometry of space-time, this function is often extended to arbitrary pairs so that $\tau(X, Y) = 0$ if $X \not\leq Y$ (see, e.g., (Busemann 1967), (Pimenov 1970)). This extended function is called the *kinematic metric.*

In particular, when gravity is negligible (and it is negligible for the majority of Earth measurements of space-time), we can use the formulas of special relativity, in which

$$X \leq Y \leftrightarrow y_0 - x_0 \geq \rho(x, y),$$

where $\rho(x, y)$ is the standard Euclidean metric on R^3, and

$$\tau_R(X, Y) = \sqrt{(y_0 - x_0)^2 - \rho^2(x, y)}.$$

In this paper, we will also consider two generalizations of this formula:

- The above formulas are consistent with all known measurement results. However, this does not mean that formulas of special relativity are necessarily absolutely precise: Indeed, measurements are never 100% precise; therefore, formulas that lead to close expression for $\tau(X,Y)$ are also consistent with all measurement results. In particular, Busemann (Busemann 1967) proposed the following formula

  $$\tau_\alpha(X,Y) = ((y_0 - x_0)^\alpha - \rho^\alpha(x,y))^{1/\alpha}$$

  with a parameter α that can take any value from 1 to ∞. For $\alpha \approx 2$, the "Finsler-type" space described by this formula is consistent with the results of all known experiments of special relativity. In this paper, we will consider generalizations that correspond to different value of α.
- Traditionally, physicists considered 4-dimensional space-time (one temporal dimension + 3 spatial ones). However, in modern physics, space-time is often assumed to be of different dimension (see, e.g., (Brink *et al.* 1988) and (Siegel 1988)). In view of this possibility, in the following text, we will consider space-time models of arbitrary dimension d.

Since there are infinitely many events, an ideal (exhaustive) description of space-time would consist of describing these *infinitely* many events. At any given moment of time, we can only have records about the finite number of events. The more events we record, the better is our knowledge of the space-time; in particular, the better is our approximation of the kinematic metric τ (that describes the space-time). It is, therefore, natural to ask the following question:

> *How many events do we need to record in order to know the space-time (i.e., to be precise, the kinematic metric) with a given accuracy $\varepsilon > 0$?*

2. FORMAL DESCRIPTION OF THE PROBLEM

Definition 1. *Let S be a set, and let $\tau : S \times S \to R_0^+$ be a function that assigns a non-negative number to every pair (X, Y) of elements of S. Let $\varepsilon > 0$ be a real number. We say that a finite set $\{X_1, \ldots, X_n\}$ is an $\varepsilon-$approximation to S iff there exists a function $\pi : S \to \{X_1, \ldots, X_n\}$ such that for every $X, Y \in S$, the following three inequalities hold:*

- $|\tau(X, \pi(Y)) - \tau(X, Y)| \le \varepsilon.$
- $|\tau(\pi(X), Y) - \tau(X, Y)| \le \varepsilon.$
- $|\tau(\pi(X), \pi(Y)) - \tau(X, Y)| \le 2\varepsilon.$

In these terms, the problem is:

> *For a given area of space-time, to find the smallest possible number of point that form an $\varepsilon-$approximation to this area.*

In this paper, we will solve this problem for the space-time of special relativity and for its generalization proposed by Busemann. These results partially appeared in (Kreinovich 1979).

3. ANALOGY WITH METRIC SPACES REVEALS THE GEOMBINATORIC NATURE OF THIS PROBLEM

Many method of *space-time* geometry first appeared as a natural generalization of the *traditional* geometry (that is intended to describe only space). It is, therefore, natural, before we start solving a space-time problem, to try to solve a similar problem for a normal metric space, i.e., for the case in which $\tau(X, Y)$ is a normal metric $\rho(X, Y)$ (with triangle inequality). In this case, Definition 1 is reduced to the well-known notion of a $\varepsilon-$net (see, e.g., (Lorentz 1966)):

Definition 2. *A set $\{X_1, \ldots, X_n\}$ of points from a metric space S with a metric ρ is an $\varepsilon-$net for S iff for every $X \in S$, there exists an i for which $\rho(X, X_i) \le \varepsilon$.*

PROPOSITION 1. *For a metric space, a set is an $\varepsilon-$approximation iff it is a $\varepsilon-$net.*

(For reader's convenience, all the proofs are delayed until the special Proofs section.)

For metric spaces, the definition of an $\varepsilon-$net can be reformulated in purely geometric terms: *a set* $\{X_1, \ldots, X_n\}$ *is an* $\varepsilon-$*net for a metric space* S *iff* S *can be covered by* n *balls of radius* $\varepsilon > 0$ *with centers in* X_i. Thus, for metric spaces, the problem of finding the $\varepsilon-$approximating set with the smallest possible number of elements can be reformulated as a geombinatoric problem:

> *For a given metric space* S *and a given* $\varepsilon > 0$, *how many balls of radius* ε *do we need to cover* S?

Comments.

1. Since for metric spaces, approximability can be reformulated as a geombinatoric problem, our main problem (approximability of space-time) can be thus viewed as a natural space-time analogue of this geombinatoric problem.

2. For sets in the Euclidean space R^d, at least the asymptotics of the smallest number N_ε of such balls is known: e.g., for a compact set S with a non-zero interior, we have $N_\varepsilon(S) \sim C/\varepsilon^d$ (Lorentz 1966). In fact, this formula is so well known, that in *fractal theory* it is often used as a *definition* of the dimension d (see, e.g., (Mandelbrot 1982)): if for some set S, we have the above asymptotics for some real number d, then we say that this set is of dimension d. To determine d from the experimentally determined values of $N_\varepsilon(S)$, we take logarithms of both sides and determine d as $H_\varepsilon(S)/\log_2 \varepsilon$, where $H_\varepsilon(S) = \log_2 N_\varepsilon(S)$ is called an $\varepsilon-$*entropy* of the set S. (For sets in Euclidean space, $H_\varepsilon(S) \sim d \cdot \log_2 \varepsilon$.) To use the analogy with metric spaces, we will use a similar logarithmic measure for space-times:

Definition 3. (Kreinovich 1974) *For a given set* S *with a metric* τ, *by a kinematic* $\varepsilon-$*entropy* $\tilde{H}_\varepsilon(S)$, *we mean a binary logarithm of the smallest possible number of elements in a* $\varepsilon-$*approximation to* S.

4. MAIN RESULT

Comment. To get a finite approximation to a metric space S, we must restrict ourselves to *compact* sets S only. For Euclidean space, a natural way to get a compact set is to restrict ourselves to a ball. For the space-time of special relativity, a natural way to restrict ourselves to a compact set is to consider *intervals* $[X, Y]$ defined as $[X, Y] = \{Z \mid X \leq Z \leq Y\}$. One more denotation is needed to describe our result:

Denotation. *We say that* $f(x) \asymp g(x)$ *iff there exists a constant* C *such that* $|f(x) - g(x)| \leq C$ *for all* x.

THEOREM. *If* $[A, B]$ *is an interval with an interior point in a* $d-$*dimensional* ($d \geq 2$) *space-time with a kinematic metric* τ_α*, then*

$$\tilde{H}_\varepsilon([A, B]) \asymp \alpha d \cdot \log_2 \varepsilon.$$

Comments.

1. So, for $\alpha = 2$, instead of $N_\varepsilon \approx \varepsilon^{-n}$ points that we need to approximate a (standard) metric with accuracy ε, we need $\tilde{N}_\varepsilon \approx \varepsilon^{-2n}$ points (approximately the *square* of the previous amount) to approximate the kinematic metric. We can thus say informally that *it is "square times" more difficult to approximate a 4-D space time than the 4-D Euclidean space.*

2. For 1-dimensional space-time, $\tilde{H}_\varepsilon([A, B]) \asymp \log_2 \varepsilon$, and so, there is no asymptotic difference between Euclidean and pseudo-Euclidean cases.

5. PROOFS

Proof of Proposition 1. If a finite set $F = \{X_1, \ldots, X_n\}$ is an $\varepsilon-$approximation to S, then, for an arbitrary X, from the first inequality for $X = Y$, we conclude that $\rho(X, \pi(X)) \leq \varepsilon$. Hence, F is an $\varepsilon-$net, with $\pi(X) \in F$.

Vice versa, if F is a $\varepsilon-$net, then we can take, for every X, the $\varepsilon-$close element from F as $\pi(X)$. Then, all three desired inequalities follow from the triangle inequality for the metric ρ. Q.E.D.

Proof of the Theorem. To prove our result, we will show two things:

1) We will first show that for some $C > 0$, for $\varepsilon' = C\varepsilon^\alpha$, if F is an $\varepsilon'-$net in the sense of the Euclidean-type metric

$$d(X,Y) = \sqrt{(x_0 - y_0)^2 + \rho^2(x,y)}, \quad (1)$$

then F is an $\varepsilon-$approximation for the interval $[A,B]$.

2) Vice versa, for some other constant $C' > 0$, we will show that if F is an $\varepsilon-$approximation for the interval $[A,B]$, then F is a $C'\varepsilon^\alpha-$net (in the sense of the above Euclidean-type metric) for a certain compact set with an interior point located inside the interval $[A,B]$.

Thus, the number of points in an $\varepsilon-$approximation set for $[A,B]$ can be approximated by the number of points in a $\varepsilon'-$net for the Euclidean metric, where $\varepsilon' \approx C\varepsilon^\alpha$; we already know that this number is asymptotically $\sim \varepsilon'^{-d}$, which is exactly $\sim \varepsilon^{-\alpha d}$.

Let us now prove the above two statements,

1) Let F be a $\varepsilon'-$net in the sense of the metric d. This means that for $X' = \pi(X)$, $d(X,X') \le \varepsilon'$ for all X. Hence, by definition of the metric d, $|x_0 - x_0'| \le d(X,X') \le \varepsilon'$ and similarly, $\rho(x,x') \le \varepsilon'$. From the triangle inequality, we conclude that for every $Y = (y_0, y)$,

$$|\rho(x,y) - \rho(x',y)| \le \rho(x,x') \le \varepsilon'$$

and, similarly, that

$$|\,|x_0 - y_0| - |x_0' - y_0|\,| \le \varepsilon'.$$

For $\alpha \ge 1$, the function $\tau^\alpha = t^\alpha - \rho^\alpha$ (defined on a compact set) is a Lipschitz function of its two variables t and ρ; therefore, for some constant $C_1 > 0$, we have

$$|\tau^\alpha(X,Y) - \tau^\alpha(\pi(X),Y)| \le C_1\varepsilon' = C_1 C\varepsilon^\alpha.$$

So, if we choose $C \le 1/C_1$ (so that $C_1 C \le 1$), we can conclude that $\tau^\alpha(X,Y) \le \tau^\alpha(\pi(X),Y) + \varepsilon^\alpha$, and that $\tau(X,Y) \le$

$(\tau^\alpha(\pi(X),Y)+\varepsilon^\alpha)^{1/\alpha}$. From the well-known (Minkowski) inequality $(a^\alpha+b^\alpha)^{1/\alpha} \le a+b$, we conclude that $\tau(X,Y) \le \tau(\pi(X),Y)+\varepsilon$.

Similarly, we can prove that $\tau(\pi(X),Y) \le \tau(X,Y) + \varepsilon$, and hence, that the first inequality in the definition of an $\varepsilon-$approximation holds. Two other inequalities are proven in a likewise manner.

2) Let us now show that if F is an $\varepsilon-$approximation set, then F is a $\varepsilon'-$net (in the sense of the metric d) for some compact set N. First, let us describe this set N.

We have assumed that the interval $[A,B]$ has an interior point; let us denote this point by $I = (i_0, i)$. Since this is an interior point, there exists a number $\delta > 0$ such that the $\delta-$neighborhood $[i_0 - \delta, i_0 + \delta] \times B_\delta(i)$ of I is contained in the interval $[A,B]$ (here, $B_\delta(i)$ denotes the ball of radius δ with a center in i). Let us take $r = \delta/2$, and take a "half-size" neighborhood $N = [i_0-r, i_0+r]\times D_r(i)$. Let us show that if $X \in N$, then $d(X,\pi(X)) \le \varepsilon'$, where $\varepsilon = C'\varepsilon^\alpha$. From this implication, it will follow that F is a $\varepsilon'-$net for N (in the sense of the metric d).

To prove that the distance $d(X,\pi(X))$ between X and $X' = \pi(X)$ is indeed $\le C'\varepsilon^\alpha$, we will prove the following two inequalities:

- that $|x_0 - x_0'| \le C_2\varepsilon^\alpha$; and
- that $\rho(x,x') \le C_3\varepsilon^\alpha$

for some appropriate constants C_i. Then, we will have $d(X,X') = \sqrt{|x_0 - x_0'|^2 + \rho^2(x,x')} \le \sqrt{C_2^2 + C_3^2} \cdot \varepsilon^\alpha$, i.e., the desired inequality for $C' = \sqrt{C_2^2 + C_3^2}$.

To prove the first inequality, let us assume (without loss of generality) that $x_0 \ge x_0'$. We will use the fact that

$$|\tau(X,Y) - \tau(X',Y)| \le \varepsilon \tag{2}$$

for all $Y \in [A,B]$. Actually, we will use this inequality for one specifically designed point Y:

- To get the *spatial* part y of this point, let us do the following:
  - take a geodesic line γ (i.e., for our case, a straight line) that connects x and x';
  - extend this line past x', and
  - take a point y that is $r/4$ past x' on this extension.

  Since N was a half of the neighborhood that belongs to $[A, B]$, we get a point y that belongs to $B_\delta(i)$, and for which $\rho(x', y) = r/4$ and $\rho(x, x'') = \rho(x, x') + r/4$.

- Let us now choose the *temporal* coordinate y_0 of this point Y in such a way that $\tau(X', Y) = 2\varepsilon$. From the formula for τ_α, we can easily conclude that for that, we must take $y_0 = x'_0 + ((r/4)^\alpha + (2\varepsilon)^\alpha)^{1/\alpha}$.

From the inequality (2), it now follows that $\tau(X, Y) \geq \tau(X', Y) - \varepsilon = 2\varepsilon - \varepsilon = \varepsilon > 0$. Hence, $y_0 - x_0 \geq \rho(y, x)$. Substituting the known values for y_0 and $\rho(x, y)$ into this inequality, we conclude that

$$x'_0 + ((\frac{r}{4})^\alpha + (2\varepsilon)^\alpha)^{1/\alpha} - x_0 \geq \rho(x, x') + \frac{r}{4}.$$

Since $x_0 \geq x'_0$, we have $x'_0 - x_0 = -|x_0 - x'_0|$. By moving this term to the right-hand side of the inequality, and $r/4$ to the left-hand side, we conclude that $\rho(x, x') + |x_0 - x'_0| \leq D$, where by D, we denoted the difference

$$D = ((\frac{r}{4})^\alpha + (2\varepsilon)^\alpha)^{1/\alpha} - \frac{r}{4}.$$

The function $((r/4)^\alpha + z)^{1/\alpha}$ is smooth for small z, and its value for $z = 0$ is $r/4$; so, we have $((r/4)^\alpha + z)^{1/\alpha} \leq r/4 + C_4|z|$ for some constant C_4. Hence, $D \leq C_4(2\varepsilon)^\alpha = C_2\varepsilon^\alpha$ (where we denoted $C_2 = C_4 \cdot 2^\alpha$). Hence, from the fact that the sum $\rho(x, x') + |x_0 - x'_0|$ of two non-negative numbers does not exceed $D \leq C_2\varepsilon^\alpha$, we can conclude that each of these numbers does not exceed $C_2\varepsilon^\alpha$, i.e., that $\rho(x, x') \leq C_2\varepsilon^\alpha$ and $|x_0 - x'_0| \leq C_2\varepsilon^\alpha$. This is exactly what we wanted to prove.

The theorem is proven.

Proof of the Comment. For 1-D space-time, basically, $\varepsilon-$approximation is the same as an $\varepsilon-$net.

6. OTHER POSSIBLE RESULTS AND AN OPEN PROBLEM

6.1. Our main result: in brief

In this paper, we described the approximability estimates for the space-time of special relativity and for some natural generalizations of this space-time model.

6.2. Other possible results

Some further generalizations of our result can be made in a reasonably easy manner: e.g., our proof is applicable not only to Euclidean proper space, but it also proves the following result:

Definition 4. *We say that a metric space M with a metric ρ is locally geodesically connected, if for every $i \in M$, there exists a $r > 0$ for which every two points x, x' from the ball $B_r(i)$ can be connected by a geodesic line that can be extended for at least $r/4$ past each of these points x and x'.*

PROPOSITION 2. *Let M be a locally geodesically connected metric space, and let a metric τ_α be defined on the set $R \times M$. Then, for an arbitrary interval $[A, B]$ with an interior point, there exist numbers c_i, real line intervals Δ_i, and balls B_i for which for every ε:*

$$H_{c_1\varepsilon^\alpha}(\Delta_1 \times B_1) \le \tilde{H}_\varepsilon([A, B]) \le H_{c_2\varepsilon^\alpha}(\Delta_2 \times B_2),$$

where $H_{\varepsilon'}$ denotes $\varepsilon'-$entropy in the sense of the metric (1).

6.3. An important open problem

It is desirable to find similar estimates for more complicated (and *more realistic*) *models of space-time*, e.g., for pseudo-Riemann spaces, and for spaces with singularities (see, e.g., (Misner *et al.* 1973).

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