Fuzzy Numbers are the Only Fuzzy Sets That Keep Invertible Operations Invertible

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Abstract

In standard arithmetic, if we, e.g., accidentally add a wrong number y to the preliminary result x, we can undo this operation by subtracting y from the result $x+y$. In this paper, we prove the following two results:

- First, a similar possibility to invert (undo) addition holds for fuzzy numbers (although in case of fuzzy numbers, we cannot simply undo addition by subtracting y from the sum).
- Second, if we add a single fuzzy set that is not a fuzzy number, we lose invertibility.

Thus, invertibility requirement leads to a new characterization of the class of all fuzzy numbers.

Keywords: fuzzy sets, fuzzy numbers, invertible operations

1 Formulation of the problem

1.1 Why data processing?

In many real-life situations, we are interested in the value of some quantity Z that is impossible (or difficult) to measure or estimate directly. For example, it is difficult to estimate or directly measure the amount of oil in a given field. To get the desired estimate, we find some quantities $X_1, \ldots, X_n$ that are related to Z in a known way. Based on this known relation, we develop an algorithm f that transforms the values x_i of the quantities X_i into an estimate $z = f(x_1, \ldots, x_n)$ for Z.

Then, we will be able to estimate Z as follows:

- First, we measure (or otherwise estimate) the values $x_1 \ldots, x_n$ of the quantities $X_1, \ldots, X_n$.
- Second, starting with the estimates $x_1, \ldots, x_n$, we apply (step-by-step) the algorithm f and finally, get the desired estimate $z = f(x_1, \ldots, x_n)$.

The values $x_1, \ldots, x_n$ are called *input data*, and the computational process of computing $f(x_1, \ldots, x_n)$ is called *data processing*.

1.2 It is sometimes necessary to undo (invert) the results of arithmetic operations

- Computers are not 100% perfect. It may so happen that one of the operations was applied to a wrong number.
- Our algorithms are not perfect. We may have done a wrong step.

In both cases, it is desirable to be able to "undo" ("invert") what was done wrong and continue computations without having to start everything anew.

1.3 Arithmetic operations with real numbers are invertible

It is always easy to "undo" an (elementary) arithmetic operation with real numbers: e.g., if have we accidentally added to a value x a wrong number y, we can always reconstruct the original value x by simply subtracting y from the result of the addition: $(x + y) - y = x$.

Similarly, we can undo the results of practically all elementary arithmetic operations. The only case when it is impossible to undo the result of an arithmetic operation is when we have multiplied a number x by $y = 0$.

We can express this possibility to "invert" the result of an arithmetic operation by saying that the corresponding computer operation with real numbers (in particular, addition $x \to x + y$) is *invertible*.

1.4 Case of expert (fuzzy) knowledge

In many real life situations, we cannot directly measure the values $x_1, \ldots, x_n$ of the quantities $X_1, \ldots, X_n$. Instead, we only have the *expert's estimates* of these values. Instead of operations with real numbers, we thus have to perform operations with these estimates.

1.5 The main question

A natural question is: *are the arithmetic operations with expert estimates invertible?* And, if the resulting operations are not always invertible, under what conditions can we guarantee their invertibility?

In this paper, we will answer these two questions. The results of these paper partially appeared in [1].

2 Fuzzy sets, fuzzy numbers, and operations with fuzzy sets and numbers

2.1 Fuzzy sets as a natural description of expert knowledge

Expert estimates are usually formulated in terms of words of natural language (e.g., "x is approximately equal to 1"). In order to apply computer operations to such estimates, we must first describe them in computer-understandable (numerical) terms.

One of the most natural ways to describe the expert's uncertain ("fuzzy") knowledge about a quantity X is to describe, for each real number x, our degree of belief that x is a possible value of the quantity X. This degree of belief is usually denoted by $\mu_X(x)$, and the corresponding description is called a *fuzzy set* (see, e.g., [4, 11]).

If we know the fuzzy sets that correspond to different inputs $X_1, \ldots, X_n$, then we will be able, using the well-known *extension principle* [4, 11], to describe the fuzzy set Y, i.e., in other words, to describe, for each real number y, how possible it is that this number is the actual value of Y. This description is very informative, but in many real-life situations (like in the oil example) we are not so much interested in this "fine structure" of our beliefs as in making a simple decision of what values of Y are possible and what values are not.

To make such a "binary" ("yes-no") decision, we must select some threshold degree of belief $\alpha \in (0, 1]$ and separate all possible real numbers y into two groups:

- For some real numbers y, our degree of belief that y is a possible value of Y exceeds (or is equal to) the threshold α ($\mu_Y(y) \geq \alpha$). We assume that such values y are *possible* for Y.

- For some other values y, our degree of belief that y is a possible value of Y is smaller than the threshold α ($\mu_Y(y) < \alpha$). We assume that such values y are *not possible* for Y.

The set of all y selected as possible is called the $\alpha-$*cut* of the corresponding fuzzy set and denoted by ${}^{\alpha}Y$.

This necessity to make a decision leads to the following natural alternative representation of a fuzzy set: to describe a fuzzy set X, for every $\alpha \in (0, 1]$, we describe the set ${}^{\alpha}X$ of all the values that will be assumed possible if take α as a threshold. This family of sets $\{{}^{\alpha}X\}$ is monotonic (${}^{\alpha}X \subseteq {}^{\beta}X$ if $\alpha \geq \beta$), and completely describes the original fuzzy set.

2.2 Operations on fuzzy sets

The above representation of a fuzzy set as a family of its $\alpha-$cuts is a natural background for defining operations on fuzzy sets, in particular, operations that correspond to standard arithmetic operations.

In order to define the result $X \circ Y$ of applying an operation $\circ$ (e.g., addition, subtraction, multiplication, etc.) to fuzzy sets X and Y, let us fix a threshold α and find out what values of $X \circ Y$ are possible for this particular threshold. For this threshold, only values from ${}^{\alpha}X$ are possible values of X, and only values from ${}^{\alpha}Y$ are possible values of Y. By applying the operation $\circ$ to all possible pairs $x \in {}^{\alpha}X$ and $y \in {}^{\alpha}Y$, we get the set of all possible values of $X \circ Y$. In other words, the $\alpha-$cut ${}^{\alpha}(X \circ Y)$ of the desired fuzzy set $X \circ Y$ has the following form ([4], Section 4.4):

$$ {}^{\alpha}(X \circ Y) = \{x \circ y \,|\, x \in {}^{\alpha}X,\, y \in {}^{\alpha}Y\}. \tag{1} $$

In particular, for addition ($\circ = +$), we have

$$ {}^{\alpha}(X + Y) = \{x + y \,|\, x \in {}^{\alpha}X,\, y \in {}^{\alpha}Y\}. \tag{2} $$

Comment. Under certain reasonable conditions, this definition is equivalent to the more standard one, that stems from the extension principle [10, 2, 3].

2.3 Fuzzy sets used in data processing

In different situations, different fuzzy sets are possible; for example, we sometimes only know that the value of a certain quantity X is "large". The greater the value x, the greater our degree of belief that this x is large, so the corresponding $\alpha-$cuts are semi-infinite intervals.

Such knowledge is possible; however, for data processing, such vague information is practically useless. Since in this paper, we are only interested in data processing applications, we will therefore restrict ourselves only to the fuzzy sets in which for every α, the $\alpha-$cut is *bounded.*

There is one more property that is natural to assume: if the values $x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots$ are all possible, and the sequence $x^{(k)}$ converges to a certain number x, then no matter how accurately we compute x, we will always find a number $x^{(k)}$ that is indistinguishable from x and possible. Therefore, it is natural to assume that this limit value x is also possible. In other words, it is natural to assume that every $\alpha-$cut contains all its limit points, i.e., that it is a *closed* set.

Combining these two conditions, we arrive at the assumption that each $\alpha-$cut is bounded and closed. On the real line, bounded and closed sets are exactly compact sets, so, we will call the fuzzy sets with such $\alpha-$cuts *compact fuzzy sets*.

2.4 Fuzzy numbers

An important particular case of a compact fuzzy set is a *fuzzy number* in which each $\alpha-$cut is a (closed) interval.

In fuzzy data processing, mainly fuzzy numbers are used; however, more general fuzzy sets are also sometimes needed: For example, if for some quantity x, with some degree of belief α, we know that x^2 belongs to the interval $[1, 4]$, and we know nothing about the sign of x, then the corresponding $\alpha-$cut (set of possible values of x) is not an interval, but a union of two disjoint intervals $[-2, -1] \cup [1, 2]$.

2.5 The problem of invertibility is non-trivial for fuzzy numbers, because subtraction cannot reconstruct x from $x + y$ and y

For real numbers, the question of whether addition is invertible or not, is trivial: in order to reconstruct x from $x + y$ and y, we simply subtract y from $x + y$, and get $(x + y) - y = x$.

For fuzzy sets, both addition and subtraction can be defined, but, in general, the difference $(X + Y) - Y$ is *not* equal to X: e.g., even if we take *crisp* sets $X = Y = [0, 1]$, we have

$$X + Y = [0, 1] + [0, 1] = \{x + y \,|\, x \in [0, 1] \,\&\, y \in [0, 1]\} = [0, 2]$$

and

$$(X + Y) - Y = [0, 2] - [0, 1] =$$
$$\{z - y \,|\, z \in [0, 2] \,\&\, y \in [0, 1]\} = [-1, 2] \neq [0, 1] = X.$$

Thus, for fuzzy sets and fuzzy numbers, we cannot simply reconstruct x from $x + y$ and y by subtraction. Hence, for fuzzy sets and fuzzy numbers, the question of whether we can reconstruct x from $x + y$ and y becomes non-trivial.

2.6 Our answer to the main question

Our answer to the main question is:

- For fuzzy numbers, addition is invertible.
- For more general (compact) fuzzy sets, addition is not invertible.

3 Definitions and results

3.1 First result: for fuzzy numbers, addition is invertible, but if we add one extra fuzzy set, addition stops being invertible

Definition 1.

- *Let U be a set (called Universe of discourse). By a fuzzy subset X of the set U, we mean a mapping that maps each number $\alpha \in (0,1]$ into a set ${}^{\alpha}X \subseteq U$ in such a way that if $\alpha \geq \beta$, then ${}^{\alpha}X \subseteq {}^{\beta}X$. The set ${}^{\alpha}X$ is called an $\alpha-$cut of the fuzzy set X.*
- *A fuzzy set is called compact if all its $\alpha-$cuts are compact.*
- *A fuzzy subset of $U = \mathbb{R}$ is called a fuzzy number if all its $\alpha-$cuts are closed intervals.*

Definition 2. *Let $\mathcal{C}$ be a class of compact fuzzy subsets of the set $\mathbb{R}$ of all real numbers.*

- *For every two sets $X, Y \in \mathcal{C}$, we define their sum $X + Y$ by formula (2).*
- *We say that addition is invertible on $\mathcal{C}$ if for every three sets $X, X', Y \in \mathcal{C}$, $X + Y = X' + Y$ implies that $X = X'$.*

PROPOSITION 1.

- *Addition is invertible on the class of all fuzzy numbers.*
- *If a class $\mathcal{C}$ contains all fuzzy numbers and at least one compact fuzzy set C that is not a fuzzy number, then addition is not invertible on $\mathcal{C}$.*

Comment. For reader's convenience, all the proofs are placed in a special Proofs section.

3.2 Second result: if we require invertibility and start with arbitrary compact fuzzy sets, we end up with fuzzy numbers

Definition 3. *We say that compact fuzzy sets $X, X' \subset$ IRR are equivalent if there exists a compact fuzzy set Y for which $X + Y = X' + Y$.*

If we use arbitrary compact fuzzy sets, then we do not have invertibility. So, if we want invertibility, we must represent some sets X by their supersets $I \supset X$. In particular, if the sets X and X' are equivalent in the sense of the above definition, then we must represent the sets X and X' by one and the same superset.

PROPOSITION 2. *Two compact fuzzy sets X and X' are equivalent iff for every α, $\inf({}^\alpha X) = \inf({}^\alpha X')$ and $\sup({}^\alpha X) = \sup({}^\alpha X')$.*

Due to Proposition 2, if we want invertibility, then we must represent all equivalent fuzzy sets by a single set. The class of all equivalent sets is the class of all sets X with given infimum and supremum of each $\alpha-$cut. All these sets are contained in the fuzzy number with $\alpha-$cuts $[\inf({}^\alpha X), \sup({}^\alpha X)]$, which can thus serve as the desired representative. So, *if we want invertibility of addition, we get fuzzy numbers.*

3.3 Invertible operations beyond addition

We have already mentioned that there are other invertible operations besides addition: e.g., multiplication of positive numbers, exponentiation, etc. Like addition, these operations can also be extended from numbers to arbitrary fuzzy sets of numbers (see, e.g., [9]), in particular, to fuzzy numbers. For these operations, extensions to fuzzy numbers are also invertible.

Definition 4.

- *Let A and B be connected subsets of* IR. *We say that a continuous function $f : A \times B$ is invertible if $f(a, b) = f(a', b)$ implies $a = a'$, and $f(a, b) = f(a, b')$ implies $b = b'$.*
- *By an fuzzy extension of a function f, we mean a function that maps fuzzy sets X and Y into a fuzzy set $f(X, Y)$ defined by the following $\alpha-$cuts:*

$$ {}^\alpha f(X, Y) = \{f(x, y) \,|\, x \in {}^\alpha X \text{ and } y \in {}^\alpha Y\}. $$

- *We say that the fuzzy extension $f(X, Y)$ is invertible if for every fuzzy numbers X, X', Y, and Y', $f(X, Y) = f(X', Y)$ implies $X = X'$, and $f(X, Y) = f(X, Y')$ implies $Y = Y'$.*

PROPOSITION 3. *If a continuous function f is invertible, then its fuzzy extension is also invertible.*

Comment. Without continuity, this result is not necessarily true. For example, let $s(x)$ denote a function that swaps 0 and 1 and leaves all other values intact. Then, the function $f(a, b) = s(a) + b$ is invertible. However, for crisp intervals $X = [-0.5, 0.5]$, $X' = [-0.5, 1]$, and $Y = [0, 1]$, we have $s(X) = [-0.5, 0) \cup (0, 0.5] \cup \{1\}$, and $f(X, Y) = s(X) + Y = [-0.5, 1) \cup (0, 1.5] \cup [1, 2] = [-0.5, 2]$; also, $s(X') = X'$, and $f(X', Y) = X' + Y = [-0.5, 1] + [0, 1] = [-0.5, 2]$. So, here, $f(X, Y) = f(X', Y)$ but $X \neq X'$.

4 Invertibility of addition for fuzzy sets in multi-dimensional space

Definition 1$'$. *Let $d \geq 1$ be an integer, and let $\mathcal{C}$ be a class of compact fuzzy subsets of the set $\mathrm{I\!R}^d$. We say that addition is invertible on $\mathcal{C}$ if for every three sets $X, X', Y \in \mathcal{C}$, $X + Y = X' + Y$ implies that $X = X'$.*

To formulate the result, we must give a new definition:

Definition 5.

- *A fuzzy subset of the set $\mathrm{I\!R}^d$ is called convex if for every α, its $\alpha-$cut is convex.*
- *Let us define the convex hull $X' = \mathrm{conv}(X)$ of a fuzzy set X as a fuzzy set defined by $\alpha-$cuts ${}^\alpha(X') = \mathrm{conv}({}^\alpha X)$.*

PROPOSITION 1$'$.

- *Addition is invertible on the class of all compact convex fuzzy sets.*
- *If a class $\mathcal{C}$ contains all compact convex fuzzy sets and at least one non-convex compact fuzzy set C, then addition is not invertible on $\mathcal{C}$.*

Definition 2$'$. *Let $d \geq 1$ be an integer. We say that compact fuzzy sets $X, X' \subset \mathrm{I\!R}^d$ are equivalent if there exists a compact set $Y \subset \mathrm{I\!R}^d$ for which $X + Y = X' + Y$.*

PROPOSITION 2$'$. *Two compact fuzzy sets $X, X' \subset \mathrm{I\!R}^d$ are equivalent (in the sense of Definition 2$'$) iff $\mathrm{conv}(X) = \mathrm{conv}(X')$.*

So, if we want invertibility, we must restrict ourselves to convex fuzzy sets only.

5 Invertibility of addition for the case when & is represented not by min, but by an arbitrary strictly Archimedean t-norm

In the previous sections, we used a formula for the sum $X = X_1 + X_2$ of fuzzy sets X_1 and X_2 with membership functions $\mu_1(\vec{x})$ and $\mu_2(\vec{x})$ that was, in effect, equivalent to the extension principle

$$\mu(\vec{x}) = \sup_{\vec{y}}(T(\mu_1(\vec{y}), \mu_2(\vec{x} - \vec{y})) \tag{3}$$

for the case when we take $\min(a, b)$ as a t-norm $T(a, b)$.

It is reasonable to ask what happens to invertibility if, instead of minimum, we take algebraic product $T(a, b) = a \cdot b$, or, more generally, an arbitrary strictly Archimedean t-norm.

In order to formulate our answer to this question, let us recall the definitions:

Definition 6. (see, e.g., [4, 11]) *A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if it satisfies the following four conditions:*

- $T(1, a) = a$ *for all* a*;*
- $T(a, b) = T(b, a)$ *for all* a *and* b*;*
- $T(a, T(b, c)) = T(T(a, b), c))$ *for all* a*,* b*, and* c*;*
- *if* $a \le a'$ *and* $b \le b'$*, then* $T(a, b) \le T(a', b')$*.*

Definition 7. [4, 11]

- *A t-norm $T(a, b)$ is called Archimedean if it is continuous and $T(a, a) < a$ for all $a \in (0, 1)$.*
- *An Archimedean t-norm is called strictly Archimedean if it is strictly increasing for $a, b \in (0, 1)$.*

Proposition. [13, 6, 4, 11]

- *For every continuous strictly decreasing function $\varphi : [0, 1] \to [0, \infty]$, the function*
$$T(a, b) = \varphi^{-1}(\varphi(a) + \varphi(b)) \tag{4}$$
is a strictly Archimedean t-norm.
- *If $T(a, b)$ is a strictly Archimedean t-norm, then there exists a continuous strictly decreasing function $\varphi : [0, 1] \to [0, \infty]$ for which $T(a, b)$ can be represented by the formula (4).*

Example. In particular, the algebraic sum can be represented in this form for $\varphi(a) = -\log(a)$.

Comment. In this section, by a *fuzzy set* $X \subseteq \mathbb{R}^d$, we understand a continuous function $\mu : \mathbb{R}^d \to [0,1]$. (As one can see from the proof, the results are actually true for more general membership functions as well.)

Definition 8. *Let $T(a,b)$ be a strictly Archimedean t-norm defined by a formula (4). We say that a compact fuzzy set $X \subseteq \mathbb{R}^d$ with a membership function $\mu(\vec{x})$ is $T-$convex if the function $\vec{x} \to \varphi(\mu(\vec{x}))$ is a convex function from $\mathbb{R}^d$ to $[0,\infty]$.*

Comment. A fuzzy set is usually called *convex* if all its $\alpha-$cuts are convex. In this sense, every fuzzy number is convex. The notion of $T-$convexity is stronger: not only we require that each $\alpha-cut$ is convex, but we also require that the *function* $f(\vec{x}) = \varphi(\mu(\vec{x}))$ is convex, i.e., that for this function f,

$$f(\frac{\vec{x}+\vec{y}}{2}) \leq \frac{f(\vec{x})+f(\vec{y})}{2}$$

for all possible $\vec{x}$ and $\vec{y}$.

PROPOSITION 4. *Let addition of fuzzy sets be defined by the formula* (3) *with a strictly Archimedean t-norm $T(a,b)$ of type* (4). *Then:*

- *Addition is invertible on the class of all compact $T-$convex fuzzy sets.*
- *If a class $\mathcal{C}$ contains all compact $T-$convex fuzzy sets and at least one non–$T-$convex compact fuzzy set C, then addition* (3) *is not invertible on $\mathcal{C}$.*

Comments.

- For $T(a,b) = \min(a,b)$, it was sufficient to require that all fuzzy sets are fuzzy numbers, and invertibility followed. According to Proposition 4, for $T \neq \min$, being a fuzzy number is not sufficient for invertibility: it is necessary also to require that a certain function related to the membership function is convex.
- In particular, for algebraic product, invertibility is equivalent to the convexity of the function $-\log(\mu(a))$. If the membership function $\mu(x)$ is differentiable, then the required convexity is equivalent to non-negativeness $[-(\log(\mu(a))]'' \geq 0$ of the second derivative of $-(\log(\mu(a))$, i.e., to the inequality $\mu''(a) \leq (\mu'(a))^2/\mu(a)$.
- The invertibility problem for *multiplication* of positive real numbers and corresponding fuzzy sets can be reduced to invertibility for addition if we take into consideration that $x \cdot y = \exp(\log(x) + \log(y))$. Thus, the largest class of membership functions for which multiplication is invertible consists of exactly those fuzzy sets X for which addition is invertible for $\log(X)$. In other words, we must require that the mapping $x \to \varphi(\mu(\log(x)))$ is a convex function.

6 Proofs

For crisp sets, similar results were proved in [5]. In this paper, we extend the main ideas of [5] to fuzzy sets.

6.1 Proof of Proposition 1

If both X and Y are fuzzy numbers, i.e., if for every α, ${}^\alpha X = [{}^\alpha x^-, {}^\alpha x^+]$ and ${}^\alpha Y = [{}^\alpha y^-, {}^\alpha y^+]$, then addition is invertible: indeed, in this case, the sum $Z = X + Y$ is a fuzzy number for which for every α, the $\alpha-$cut is an interval $[{}^\alpha z^-, {}^\alpha z^+]$ with the bounds ${}^\alpha z^- = {}^\alpha x^- + {}^\alpha y^-$ and ${}^\alpha z^+ = {}^\alpha x^+ + {}^\alpha y^+$. Therefore, if we know Z and Y, we can reconstruct X (i.e., all $\alpha-$cuts of X) by using formulas ${}^\alpha x^- = {}^\alpha z^- - {}^\alpha y^-$ and ${}^\alpha x^+ = {}^\alpha z^+ - {}^\alpha y^+$.

Comment. This part of the proof follows the idea that was originally presented, for crisp sets, by S. Markov in [7, 8].

Let us now assume that $\mathcal{C}$ contains all fuzzy numbers, and also a compact fuzzy set C that is not a fuzzy number. Let us show that in this case, addition is not invertible. Indeed, since C is a compact fuzzy set, for every α, its $\alpha-$cut ${}^\alpha C$ is compact, and therefore, has finite infimum ${}^\alpha c^-$ and supremum ${}^\alpha c^+$. From the monotonicity of ${}^\alpha C$, it follows that the sequence of intervals $[{}^\alpha c^-, {}^\alpha c^+]$ is also monotonic. Thus, this family defines a fuzzy number. Let us denote this fuzzy number by Y, and let us take $X = C$ and $X' = Y$. Then, for every α, ${}^\alpha X' + {}^\alpha Y = [2 \cdot {}^\alpha c^-, 2 \cdot {}^\alpha c^+]$. Let us show that the sum $X + Y$ is also equal to the interval $[2 \cdot {}^\alpha c^-, 2 \cdot {}^\alpha c^+]$:

- First, due to our choice of ${}^\alpha c^+$ and ${}^\alpha c^-$, we have ${}^\alpha X = {}^\alpha C \subseteq [{}^\alpha c^-, {}^\alpha c^+] = {}^\alpha(X')$, hence,
$$ {}^\alpha X + {}^\alpha Y \subseteq {}^\alpha(X') + {}^\alpha Y = [2 \cdot {}^\alpha c^-, 2 \cdot {}^\alpha c^+]. $$
- Second, since $C = X$ is a compact fuzzy set, its $\alpha-$cuts are compact sets, and hence, we have $\{{}^\alpha c^-, {}^\alpha c^+\} \subseteq {}^\alpha X$, and therefore, for the sum $X + Y$ (as defined by Definition 2)
$$ {}^\alpha X + {}^\alpha Y \supseteq \{{}^\alpha c^-, {}^\alpha c^+\} + {}^\alpha Y = $$
$$ \{{}^\alpha c^-, {}^\alpha c^+\} + [{}^\alpha c^-, {}^\alpha c^+] = $$
$$ [2 \cdot {}^\alpha c^-, {}^\alpha c^- + {}^\alpha c^+] \cup [{}^\alpha c^- + {}^\alpha c^+, 2 \cdot {}^\alpha c^+] = $$
$$ [2 \cdot {}^\alpha c^-, 2 \cdot {}^\alpha c^+]. $$

So, ${}^\alpha X + {}^\alpha Y \subseteq [2 \cdot {}^\alpha c^-, 2 \cdot {}^\alpha c^+]$ and ${}^\alpha X + {}^\alpha Y \supseteq [2 \cdot {}^\alpha c^-, 2 \cdot {}^\alpha c^+]$; hence,
$$ {}^\alpha X + {}^\alpha Y = [2 \cdot {}^\alpha c^-, 2 \cdot {}^\alpha c^+] = {}^\alpha(X') + {}^\alpha Y $$
for all α. Hence, $X + Y = X' + Y$. However, $X = C$ is not a fuzzy number, and X' is; therefore, $X \neq X'$. Hence, this triple is the desired counterexample to invertibility. Q.E.D.

6.2 Proof of Proposition 2

Let us denote ${}^\alpha x^- = \inf({}^\alpha X)$, ${}^\alpha x'^- = \inf({}^\alpha X')$, ${}^\alpha y^- = \inf({}^\alpha Y)$, ${}^\alpha x^+ = \sup({}^\alpha X)$, ${}^\alpha x'^+ = \sup({}^\alpha X')$, and ${}^\alpha y^+ = \sup({}^\alpha Y)$.

The infimum of ${}^\alpha(X+Y) = {}^\alpha X + {}^\alpha Y$ is attained when x is the smallest, and y is the smallest, i.e.,

$$\inf({}^\alpha X + {}^\alpha Y) = {}^\alpha x^- + {}^\alpha y^-.$$

Similarly,

$$\inf({}^\alpha(X') + {}^\alpha Y) = {}^\alpha x'^- +^\alpha y^-.$$

Therefore, if $X+Y = X'+Y$, then for every α, $\inf^\alpha(X+Y) = \inf^\alpha(X'+Y)$, ${}^\alpha x^- + {}^\alpha y^- = {}^\alpha x'^- + {}^\alpha y^-$ and hence, ${}^\alpha x^- = {}^\alpha x'^-$. Likewise, we can prove that ${}^\alpha x^+ = {}^\alpha x'^+$.

Let us now show that if for all α, ${}^\alpha x^- = {}^\alpha x'^-$ and ${}^\alpha x^+ = {}^\alpha x'^+$, then ${}^\alpha X + {}^\alpha Y = {}^\alpha(X') + {}^\alpha Y$ for some Y. Indeed, as we have shown in the proof of Proposition 1, for the fuzzy number Y defined as ${}^\alpha Y = [{}^\alpha x^-, {}^\alpha x^+]$, we have ${}^\alpha(X+Y) = {}^\alpha(X'+Y) = [2\cdot {}^\alpha x^-, 2\cdot {}^\alpha x^+]$ and hence, $X+Y = X'+Y$. Q.E.D.

6.3 Proof of Proposition 3

Since f is invertible, for every a, the function $b \to f(a,b)$ is continuous and 1-to-1; therefore, this function is strictly monotonic (i.e., strictly increasing or strictly decreasing).

Let us show that either for every a the function $f_a(b)$ defined as $f_a(b) = f(a,b)$ is increasing, or for every a, this function is decreasing.

If the set B consists of only one point b, then this statement is trivially true. So, let us consider the case when the set B contains at least two different points $b' < b''$. In this case, if for some a, the function $f_a(b)$ is increasing, then $f(a,b') = f_a(b') < f_a(b'') = f(a,b'')$. Since f is continuous, we can conclude that $f(a',b') < f(a',b'')$ for all a' from some open neighborhood of a. Therefore, for these a', the function $f_{a'}(b)$ must also be strictly increasing. So, the set A_{in} of all $a \in A$ for which the function $f_a(b)$ is increasing is open. Similarly, the set A_{de} of all $a \in A$ for which the function $f_a(b)$ is decreasing is open. Therefore, the set A is represented as the union of two disjoint open sets: $A = A_{in} \cup A_{de}$. Since A is connected, one of these sets must be empty. Therefore, either the function $b \to f_a(b)$ is strictly increasing for all a, or the function $b \to f_a(b)$ is strictly decreasing for all a.

Similarly, either the function $a \to f(a,b)$ is strictly increasing for all b, or it is strictly decreasing for all b. Totally, we have four possible cases, depending on whether f is increasing or decreasing in a, and increasing or decreasing in b.

We are now ready to show that the extension of f to fuzzy numbers is invertible. We will show it for the case when f is strictly increasing in both a and b (the proof for the three other possible cases is similar). Indeed, in this case, due to monotonicity, $f([{}^\alpha x^-, {}^\alpha x^+], [{}^\alpha y^-, {}^\alpha y^+]) = [f({}^\alpha x^-, {}^\alpha y^-), f({}^\alpha x^+, {}^\alpha y^+)]$. If we know $f({}^\alpha x^-, {}^\alpha y^-)$ and ${}^\alpha y^-$, then, due to the fact that ${}^\alpha x \to f({}^\alpha x, {}^\alpha y^-)$ is strictly increasing, we can uniquely reconstruct ${}^\alpha x^-$. Similarly, we can uniquely reconstruct ${}^\alpha x^+$. So, the fuzzy extension of f is indeed invertible. Q.E.D.

6.4 Proof of Proposition 1′

The fact that addition is invertible on compact convex fuzzy sets easily follows from the known properties of convex sets: indeed, it is known (see, e.g., [12], Section 13) that a compact convex set C is uniquely determined by its *support function* $\delta^*(x\,|\,C) = \sup\{\langle x, x^*\rangle \,|\, x^* \in C\}$ (where $\langle a, b\rangle = \sum a_i \cdot b_i$ is a scalar (dot) product), and that the support function of the sum of two sets is equal to the sum of their support functions: $\delta^*(x\,|\,C_1 + C_2) = \delta^*(x\,|\,C_1) + \delta^*(x\,|\,C_2)$[1]. Hence, if we know the $\alpha-$cuts ${}^\alpha(X+Y) = {}^\alpha X + {}^\alpha Y$ and ${}^\alpha Y$, we can determine support functions for these sets, subtract these functions, get the support function for ${}^\alpha X$, and reconstruct ${}^\alpha X$.

Let us now show that if we add a non-convex compact fuzzy set C to the class of all convex compact sets, then we lose invertibility. Let us show that $X + Y = X' + Y$ for $X = C$, $X' = \mathrm{conv}(X)$ (a convex hull of X), and $Y = d \cdot \mathrm{conv}(X)$.

Namely, we will show that for each α,

$${}^\alpha X + {}^\alpha Y = {}^\alpha(X') + {}^\alpha Y = (d+1)\cdot \mathrm{conv}({}^\alpha X).$$

To simplify the exposition, in the remaining part of this proof, we will omit the index α and denote the set and its $\alpha-$cut by the same letter.

It is clear that $X' + Y = (d+1)\cdot\mathrm{conv}(X)$, and that $X + Y \subseteq X' + Y = (d+1)\cdot\mathrm{conv}(X)$. So, it is sufficient to prove that every point $z \in (d+1)\cdot\mathrm{conv}(X)$ belongs to the sum $X + Y$.

Indeed, from $z \in (d+1)\cdot\mathrm{conv}(X)$ it follows that $z/(d+1) \in \mathrm{conv}(X)$. Since X is a subset of a $d-$dimensional space, the point z can be represented as a convex combination of $d+1$ points from X: $z = \sum \alpha_k x_k$ for some $\alpha_k \ge 0$, $\sum \alpha_k = 1$, $x_k \in X$ (strictly speaking, we need at most $d+1$ points x_k; however, if less than $d+1$ points are sufficient, then we can add a few other points with $\alpha_k = 0$). Since the sum of $(d+1)$ values α_k is equal to 1, at least one of these values α_k is $\ge 1/(d+1)$. Let us denote one of such points by α_{k_0}. Let us take $\beta_k = \alpha_k$ for $k \ne k_0$ and $\beta_k = \alpha_{k_0} - 1/(d+1)$. Then,

$$\frac{z}{d+1} = \sum \alpha_k x_k = \frac{1}{d+1} x_{k_0} + \sum \beta_k x_k.$$

Here, $\beta_k \ge 0$, and

$$\sum \beta_k = 1 - \frac{1}{d+1} = \frac{d}{d+1}.$$

[1] *Caution*: This notion of the support function of a *convex (crisp) set* is standard in convex set theory (we have taken it from [12]). This notion is *different* from a notion of a support of a *fuzzy set* $\mu : X \to [0,1]$, that is usually defined, in fuzzy set theory, as $\{x \,|\, \mu(x) > 0\}$.

Hence, if we multiply both sides of the displayed equality by $d+1$, we can conclude that

$$z = x_{k_0} + d \cdot (\sum \gamma_k x_k),$$

where $\gamma_k = (d+1)\beta_k/d$ are non-negative numbers with the sum equal to 1. Here, $x_{k_0} \in X$, and $d \cdot (\sum \gamma_k x_k) \in d \cdot X' = Y$. Hence, $z \in X + Y$. Q.E.D.

6.5 Proof of Proposition 2′

In this proof, we will also omit the index α of the $\alpha-$cut.

If $X + Y = X' + Y$ for some X, X', and Y, then the support functions coincide:

$$\delta^*(x \mid X + Y) = \delta^*(x \mid X' + Y);$$

from $\delta^*(x \mid X+Y) = \delta^*(x \mid X) + \delta^*(x \mid Y) = \delta^*(x \mid X'+Y) = \delta^*(x \mid X') + \delta^*(x \mid Y)$, we conclude that $\delta^*(x \mid X) = \delta^*(x \mid X')$ and hence, that $\mathrm{conv}(X) = \mathrm{conv}(X')$.

Vice versa, if $\mathrm{conv}(X) = \mathrm{conv}(X')$, then, as we have proved in the proof of Proposition 1′, there exists a set Y (namely, $Y = d \cdot \mathrm{conv}(X)$) for which $X + Y = X' + Y$ $(= (d+1) \cdot \mathrm{conv}(X))$. Q.E.D.

6.6 Proof of Proposition 4

Formula (4) can be rewritten as $\varphi(T(a,b)) = \varphi(a) + \varphi(b)$. Thus, formula (3) can be rewritten as

$$M(\vec{x}) = \inf_{\vec{y}} (M_1(\vec{y}) + M_2(\vec{x} - \vec{y})), \tag{5}$$

where we denoted $M(x) = \varphi(\mu(x))$, $M_1(x) = \varphi(\mu_1(x))$, and $M_2(x) = \varphi(\mu_2(x))$. It is known (see, e.g., [12]) that if we take two sets $\Delta_1 = \{(\vec{x}, y) \mid y \geq M_1(\vec{x})\}$ and $\Delta_2 = \{(\vec{x}, y) \mid y \geq M_2(\vec{x})\}$ in the $(d+1)-$space $\mathbb{R}^{d+1}$, then the (Minkowski) sum $\Delta = \Delta_1 + \Delta_2 = \{\delta_1 + \delta_2 \mid \delta_1 \in \Delta_1 \,\&\, \delta_2 \in \Delta_2\}$ of these two sets can be represented as $\Delta = \{(\vec{x}, y) \mid y \geq M(\vec{x})\}$, where $M(\vec{x})$ is defined by the formula (5).

Thus, addition of fuzzy sets X_i (in the sense of formula (3)) is equivalent to addition of the corresponding crisp sets Δ_i (in the standard sense). Hence, arguing like in the proof of Proposition 1′, we can conclude that invertibility is attained when the sets Δ_i are convex sets. One can easily check (see also [12]) that convexity of a set $\Delta = \{(\vec{x}, y) \mid y \geq M(\vec{x})\}$ is equivalent to convexity of the corresponding function $M(x)$. Q.E.D.

Acknowledgments

This work was partially supported by NSF Grant No. EEC-9322370 and by NASA Grant No. NAG 9-757. It was partially done when one of the authors

(V.K.) was an Invited Professor at Laforia, University of Paris VI, Summer 1996.

The authors are thankful to José Herencia, Misha Koshelev, Svetoslav Markov, Radko Mesiar, Dan Ralescu, Sergei Ovchinnikov, I. C. Trelea, and to all the participants of the International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'96), Granada, Spain, July 1–5, 1996, for fruitful discussions, suggestions, and encouragement.

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