“Interval Rational = Algebraic” Revisited: A More Computer Realistic Result

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Abstract

In [1], it is shown that if we add “interval computations” operation to the list of arithmetic operations that define rational functions, then the resulting class of “interval-rational” functions practically coincides with the class of all algebraic functions. By “practically coincides”, we mean that first, every interval-rational function is algebraic, and that second, for every algebraic function $A(x)$, there exists an interval-rational function that coincides with $A(x)$ for almost all x. In [1], “almost all” was understood in terms of Lebesgue measure; this result is therefore, not very computer realistic, because all the numbers representable in a computer are rational and therefore, form a set of Lebesgue measure 0. In this article, we formulate a more computer-realistic version of the result that interval + rational = algebraic.

1 Reminder: Main Result From [1]

In [1], the following question is analyzed: what will happen if we add the “interval computations” operation to the list of basic arithmetic operations that define an algebraic function. Here, by an “interval computation” operation, we mean an operation that transforms the endpoints of the intervals $[x_i^-, x_i^+]$ into the endpoints of range $f([x_1^-, x_1^+], \ldots, [x_n^-, x_n^+])$ of a given function $f(x_1, \ldots, x_n)$.

In [1],,it has been proven that "interval + rational = algebraic" in the sense that every interval-rational function is algebraic, and every algebraic function can be locally represented by an interval-rational function. In this article, we show that the result from [1] can be made more realistic (more computer-oriented).

Let us recall the main definition:

Definition 1. [1] *By interval-rational functions of several real variables $x_1, \ldots, x_n$, we mean functions from the following class:*

- *Constants and variables x_i themselves are interval-rational functions.*
- *If f and g are interval-rational functions, then $f+g$, $f-g$, $f*g$, and f/g are interval-rational functions.*
- *If $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an interval-rational function, and $Y_1, \ldots, Y_m$ are intervals, then*

  $$g(x_1, \ldots, x_n) = \sup_{y_1 \in Y_1, \ldots, y_m \in Y_m} f(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

  is an interval-rational function.
- *If $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an interval-rational function, and $Y_1, \ldots, Y_m$ are intervals, then*

  $$g(x_1, \ldots, x_n) = \inf_{y_1 \in Y_1, \ldots, y_m \in Y_m} f(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

  is an interval-rational function.
- *Only functions that are obtained by these operations are interval-rational functions.*

Definition 3. [1] *Assume that $\mathcal{V}$ is an open domain in R^n. We say that an algebraic function $A : \mathcal{V} \to R$ can be locally represented by an interval-rational function if for almost every point $(y_1, \ldots, y_n) \in \mathcal{V}$ there exists a neighborhood $\mathcal{U}$ and an interval-rational function $f(x_1, \ldots, x_n)$ such that for all $\vec{x}$ from $\mathcal{U}$, $f(\vec{x}) = A(\vec{x})$.*

Comment. In [1], we understood *almost every* in the usual mathematical sense: a property is true for *almost every point* if the set of all points in which it is not true has Lebesgue measure 0.

THEOREM 1. [1]

(i) *Every interval-rational function is algebraic.*

(ii) *Every algebraic function can be locally represented by an interval-rational function.*

2 The Notion of "Almost All" Used In This Result Was Not Computer Realistic

In mathematics, "almost all" usually means "all points, except for points from a set of Lebesgue measure 0" (or, "except for points from a set of a small Lebesgue measure"). In the existing computers, however, only rational numbers are represented. The set of all rational numbers is countable and has, therefore, Lebesgue measure 0; so the standard mathematical notion of "almost all" is not very computer-realistic.

3 A New Formalization of "Almost All"

In real life, when we say that "an algorithm is applied to *real* numbers $x_1, \ldots, x_n$", we usually mean that this algorithm is applied to *rational* numbers $r_1, \ldots, r_n$ that are $\eta-$close to $x_1, \ldots, x_n$, where η is the computer precision. So, if we fix $\eta > 0$, we can say that an algorithm A works for n real numbers $x_1, \ldots, x_n$ if it works fine for all tuples of rational numbers $(r_1, \ldots, r_n)$ that are $\eta-$close to $(x_1, \ldots, x_n)$.

Now, we have *real-valued* inputs on which the algorithm works fine, and real-valued inputs on which it does not. For real-valued inputs, we can apply Lebesgue measure.[1] As a result, we arrive at the following definition (this definition is similar to the one given in [2]):

Definition 4.

- *Let $\eta > 0$ be a real number. We say that points $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^n$ are $\eta-$close if $|x_i - y_i| \le \eta$ for all i.*
- *Let $E \subset R^n$ be a bounded set, and let $\eta > 0$ be a real number. We say that a point $x \in R^n$ $\eta-$possibly belongs to E if there exists a point y that is $\eta-$close to x and that belongs to E.*
- *We say that a bounded set $E \subset R^n$ is $(\eta, \varepsilon)-$small if the set of all points that $\eta-$possibly belong to E has Lebesgue measure $\le \varepsilon$.*
- *We say that a bounded set E is* small *if for every ε, there exists a η for which the set E is $(\eta, \varepsilon)-$small.*
- *We say that a set $E \subseteq R^n$ (not necessarily bounded) is* small *if for every $\Delta > 0$, the intersection $E \cap [-\Delta, \Delta]^n$ is small.*
- *We say that a property $P(x)$ holds for* computer-realistically almost every *x if the set $\{x \mid \neg P(x)\}$ of all x for which P is false is small.*

[1] We are thankful to Luc Longprè, who helped us to come up with this idea.

Comments.

- If a property $P(x)$ is almost always true in this sense, this means, crudely speaking, that for any given ε, we can choose a computer precision η so that for all x except for a set of measure $\leq \varepsilon$, we can *guarantee* that $P(x)$ is true even when we only know components x_i with precision η.

- We can now reformulate the above Definition 3 and Theorem 1 in terms of this new computer-realistic definition of "almost all".

Definition 3′. *Assume that $\mathcal{V}$ is an open domain in R^n. We say that an algebraic function $A : \mathcal{V} \to R$ can be computer-realistically locally represented by an interval-rational function if for computer-realistically almost every point $(y_1, \ldots, y_n) \in \mathcal{V}$, there exists a neighborhood $\mathcal{U}$ and an interval-rational function $f(x_1, \ldots, x_n)$ such that for all $\vec{x}$ from $\mathcal{U}$, $f(\vec{x}) = A(\vec{x})$.*

THEOREM 1′.

(i) *Every interval-rational function is algebraic.*

(ii) *Every algebraic function can be computer-realistically locally represented by an interval-rational function.*

Comment. The proof is practically similar to the one presented in [1].

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References

[1] V. Kreinovich, "Interval rational = algebraic", *ACM SIGNUM Newsletter*, 1995, Vol. 30, No. 7, pp. 2–13.

[2] V. Kreinovich, A. Lakeyev, and J. Rohn, "Computational Complexity of Interval Algebraic Problems: Some Are Feasible And Some Are Computationally Intractable: A Survey", In: G. Alefeld and A. Frommer (eds.), *Scientific Computing and Validated Numerics*, Akademie-Verlag, Berlin, 1996 (to appear).