

Pure Quantum States Are Fundamental,  
Mixtures (Composite States) Are Mathematical  
Constructions:  
An Argument Using Algorithmic Information  
Theory

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## Abstract

*From the philosophical viewpoint, two interpretations of the quantum measurement process are possible: According to the first interpretation, when we measure an observable, the measured system moves into one of the eigenstates of this observable (“the wave function collapses”); in other words, the Universe “branches” by itself, due to the very measurement procedure, even if we do not use the result of the measurement. According to the second interpretation, the system simply moves into a mixture of eigenstates, and the actual “branching” occurs only when an observer reads the measurement results. According to the first interpretation, a mixture is a purely mathematical construction, and in the real physical world, a mixture actually means that the system is in one of the “component” states.*

*In this paper, we analyze this difference from the viewpoint of the algorithmic information theory; as a result of this analysis, we argue that only pure quantum states are fundamental, while mixtures are simply useful mathematical constructions.*

# 1 Are Mixtures Real?

## Formulation of the Problem

### 1.1 For Whom Is This Paper Written?

This paper contains an application of algorithmic information theory to the foundations of quantum mechanics. In view of its interdisciplinary character, we wrote this paper with two audiences in mind:

- To make the paper understandable for specialists in algorithmic information theory, we included brief descriptions of the corresponding physical processes and problems; physicist readers can safely skip these parts of the text.
- To make the paper understandable for physicists, we included brief description of the main notions of algorithmic information theory; those who already know these notions can safely skip the corresponding parts of the text.

### 1.2 Measurement Process in Quantum Mechanics: A Brief Mathematical Description

In quantum mechanics (QM), states are described by unit vectors in Hilbert space  $H$ . A Hilbert space is, crudely speaking, an infinite-dimensional complex-valued generalization of an Euclidean space. In more mathematical terms, a Hilbert space is a linear space over complex numbers with a scalar product that satisfies several reasonable properties (similar to scalar product of vectors in Euclidean space).

The fact that  $H$  is a *linear space* means that the notion of linear combination is defined. The linear combination of two states is called their *superposition*. Superposition is a purely quantum notion that has no classical analogue and that is largely responsible for the “paradoxical” (from the viewpoint of macrophysical intuition) behavior of quantum objects.

As we have mentioned, for every two vectors  $\varphi$  and  $\psi$  from  $H$ , a *scalar (dot) product*  $(\varphi, \psi)$  is defined. A typical example of a Hilbert space (that corresponds to the simplest non-relativistic quantum systems) is the set of all square integrable complex-valued functions  $\psi(x_1, x_2, x_3)$  defined on a 3-D space. For this space, the scalar product is defined as  $(\varphi, \psi) = \int \varphi(x_1, x_2, x_3)\psi^*(x_1, x_2, x_3) dx_1 dx_2 dx_3$ , where  $\psi^*$  means complex conjugation.

The physical meaning of a scalar product is related to measuring measurable (observable) quantities (*observables*, for short). In this description, *observables* are linear operators from  $H$  to  $H$ . For example, a coordinate  $x_1$  is a linear operator that transforms a function  $\psi(x_1, x_2, x_3)$  into a function  $x_1 \cdot \psi(x_1, x_2, x_3)$ .

If we are in a state  $\psi$ , and we measure an observable  $A$  with eigenvectors  $\varphi_i$  and corresponding eigenvalues  $\lambda_i$ , then, with probability  $p_i = |(\varphi_i, \psi)|^2$ , the

result of this measurement is  $\lambda_i$ , and the measured object “jumps” into the state  $\varphi_i$ .

In particular, if a system was in the eigenstate  $\psi_i$  already, then with probability 100%, the result of measuring the observable  $A$  is  $\lambda_i$ . If a system was in a more general state, then measuring  $A$  can lead to different possible results, with different probabilities.

We start with a state that is, in general, a *superposition* of different eigenstates  $\varphi_i$ , and we end up in one these eigenstates. For example, if we measure coordinates  $\vec{x}$  and we start in a state  $\psi(\vec{x})$  for which  $\psi(\vec{x}) \neq 0$  for all  $\vec{x}$ , i.e., in which the particle is not located in any of the spatial points  $\vec{x}$ , then we end up in a state in which the particle is with 100% probability located in one of the spatial points (actually, we can only measure coordinates with a certain accuracy  $\delta$ , and therefore, rather than getting a state of a delta-function type, in which a particle is located in exactly one point, we get a state in which coordinates are located in a box of linear size  $\delta$ ).

If an observable has *degenerate* eigenvalues, i.e., if for some eigenvalues, we have an eigenspace  $L_i$  that may be more than 1-dimensional, then the state  $\psi$  “jumps” into an (orthogonal) *projection*  $P_i(\psi)$  onto one of eigenspaces  $L_i$ , and the probability of jumping into  $P_i(\psi)$  is equal to  $|P_i(\psi)|^2$ .

### 1.3 Two Possible Philosophical Interpretation of Quantum Measurement Process: Which Is Correct?

From the philosophical viewpoint, two interpretations of the quantum measurement process are possible:

1. According to the first interpretation, when we measure an observable  $A$  in a state  $\psi$ , the measured system moves into one of the eigenstates  $\psi_1 = P_1(\psi), \dots, \psi_i = P_i(\psi), \dots$  of this observable (“the wave function collapses”); in other words, the Universe “branches” by itself, due to the very measurement procedure, even if we do not use the result of the measurement.
2. According to the second interpretation, after the measurement, the system moves into a special new state  $m$  called a *mixture*. From the mathematical viewpoint, a mixture is a probability measure on the set of pure states. In case of measurement, we move into a mixture of eigenstates  $\psi_i = P_i(\psi)$ , in which each state  $\psi_i$  occurs with the probability  $p_i = |\psi_i|^2$ . We can denote this mixture  $m$  by  $p_1 \cdot \psi_1 \star \dots \star p_i \cdot \psi_i \star \dots$  (we use a special symbol  $\star$  to emphasize that this state  $m$  is *not* a linear combination of the states  $\psi_i$ ). In this interpretation, there is no “branching” related to the measurement process itself; the actual “branching” occurs only when an *observer* reads the measurement results.

*Which of the two interpretation is correct?*

The difference between these two interpretation can be reformulated in terms of mixtures:

1. According to the first interpretation, the *mixture* is a *purely mathematical construction*. This means that in reality, if a system is in a mixture of states  $\psi_1, \dots, \psi_i, \dots$ , then actually, the system is in one of these states  $\psi_i$ .
2. According to the second interpretation, the *mixture is a new physical state*, and it is possible that a system is in this mixture state without actually being in one of the component states.

#### 1.4 It Is Difficult to Distinguish These Two Interpretations Based on Measurement Results, Because These Two Interpretations Lead to the Same Probabilities of Different Measurement Results

The only meaningful way to distinguish between two physical situations is to propose a measurement that leads to different results in these situations. We already know what happens if we measure an observable  $A$  in a “normal” (*pure*) quantum state: as a result, we get some real numbers with different probabilities, i.e., we get a *probability measure* on the set of all real numbers. One measure (we will denote it by  $\mu_1$ ) corresponds to the state  $\psi_1$ , another probability measure (we will denote it by  $\mu_2$ ) corresponds to the state  $\psi_2$ , etc. So, if  $r$  is the result of measuring the observable  $A$  in the state  $\psi_i$ , then for every set  $S$ , the probability  $P(r \in S)$  that this result  $r$  belongs to the state  $S$  is equal to  $\mu_i(S)$ .

Let us now consider the situation in which we do not know what state we are in; we only know that with probability  $p_i$ , we are in the state  $\psi_i$ . These state  $\psi_i$  are incompatible in the sense that a system cannot be simultaneously in two different states  $\phi_i$  and  $\psi_j$ ,  $i \neq j$ . Therefore, applying standard formulas of probability theory, we can conclude that in this situation, the probability  $\mu(S)$  of the measurement result  $r$  to be in the set  $S$ , is equal to:

$$\mu(S) = p_1 \cdot \mu_1(S) + \dots + p_i \cdot \mu_i(S) + \dots$$

The probability of getting a result in a *mixed state*  $p_1 \cdot \psi_1 \star \dots \star p_i \cdot \psi_i \star \dots$  is defined as the probability combination of probabilities corresponding to the component states. To be more precise, the probability  $\mu(S)$  that the result of this measurement belongs to the set  $S$  is defined as the sum

$$p_1 \cdot \mu_1(S) + \dots + p_i \cdot \mu_i(S) + \dots$$

This is exactly the same expression as for the first interpretation.

Hence, these two interpretations lead to the same probabilities of different measurement results. Therefore, based on these probabilities, we cannot distinguish between these interpretations.

Can we distinguish them in any other way?

## 1.5 What We Are Planning to Do

In this paper, we will show, using a recent mathematical formalization of randomness called *algorithmic information theory*, that the original philosophical question can be reformulated in precise mathematical terms, and that the resulting mathematical question can be solved.

The resulting *answer* to the original philosophical question is as follows:

- From the observational viewpoint, we cannot distinguish these two cases.
- Therefore, using the well-known *Ockham razor principle* (*Entities should not be multiplied unnecessarily*), we can conclude that the special notion of a mixed state is not necessary.

In other words, if we use Ockham razor principle, then *only pure quantum states are fundamental, while mixtures are simply useful mathematical constructions*.

The notions and result from *algorithmic information theory* that are necessary for this conclusion will be described in the next section.

Our philosophical conclusion is based on a new mathematical result that is, we believe, of separate mathematical interest.

*Comment.* The main result of this paper was first announced in [3].

## 2 Our Approach to the Problem

### 2.1 Preliminary Comment: A Realistic Observable Can Have Only Finitely Many Eigenspaces

Before we describe our idea, let us make the following comment: From the purely mathematical viewpoint, an observable may have infinitely many eigenvalues. However, for real-life measuring devices, there usually is a bound  $\Delta$  on possible values of measured quantities, and there usually is a certain accuracy  $\varepsilon > 0$  with which we can measure. Therefore, as a result of this measurement, we can have only finitely many different values of the measured quantity (crudely speaking,  $-\Delta, -\Delta + \varepsilon, \dots, 0, \varepsilon, 2\varepsilon, \dots, \Delta$ ). Hence, we can assume that the observable  $A$  has only *finitely many* eigenvalues and eigenspaces.

Let us denote the total number of these eigenspaces by  $n$ .

## 2.2 Main Idea

The only way to say something about a state of a system is to undertake some measurement (or measurements) and to analyze the results of these measurements. Therefore, to tell whether mixtures are real or not, we will pick some sequence of measurements  $\mathcal{B} = B_1, \dots, B_m, \dots$ , and compare the predictions in two different situations:

- when we have one of the states  $\psi_i = P_i(\psi)$ ,  $1 \leq i \leq n$ ; and
- when we actually have a mixture.

The only thing that quantum mechanics actually predicts is probabilities, i.e., in mathematical terms, a *probability measure* on the set of possible measurement results (sequence of real numbers). Let us denote by  $\mu_{\mathcal{B}}^{\psi_i}$  the probability measure on the set of all possible measurement results that corresponds to the sequence of measurements  $\mathcal{B}$  and to the state  $\psi_i$ .

In these terms, our two interpretations lead to the following two probabilistic conclusions:

- According to the first interpretation, with probability  $p_i$ , the system is in a state  $\psi_i$ , and therefore, for some  $i = 1, \dots, n$ , the result of the measurement will be random with respect to (w.r.t.) a probability measure  $\mu_i = \mu_{\mathcal{B}}^{\psi_i}$ . In other words, according to the first interpretations, one of the following statements is true:
  - *The result of the measurement is random w.r.t. the probability measure  $\mu_1$ .*
  - ...
  - *The result of the measurement is random w.r.t. the probability measure  $\mu_i$ .*
  - ...
  - *The result of the measurement is random w.r.t. the probability measure  $\mu_n$ .*
- According to the second interpretation, the system is in the *mixture* (*composite*) state, and that therefore, the following statement is true:
  - *The result of measuring  $\mathcal{B}$  is random w.r.t. the composite measure*

$$\mu_{\mathcal{B}} = p_1 \cdot \mu_1 + \dots p_n \cdot \mu_n.$$

In order to compare the predictions caused by these two interpretations, we must specify what exactly is predicted by saying that “the result of measurements is random w.r.t. some measure  $\mu$ ”. This specification is provided by the *algorithmic information theory*.

## 2.3 Algorithmic Information Theory: A Brief Introduction

The formal definition of “random” was proposed by Kolmogorov’s student P. Martin-Löf [7]; for a current state of the art, see, e.g., [6]. (We will be using a version of this definition proposed by P. Benioff [1]; see also [4]).

To describe this definition, let us recall how physicists use the assumption that something is random. For example, what can we conclude if we know that the sequence of heads and tails obtained by tossing the coin is random? One thing we can conclude is that the fraction of heads in this sequence tends to  $1/2$  as the number of tosses tends to infinity. What is the traditional argument behind this conclusion? In mathematical statistics, there is a mathematical theorem saying that w.r.t. the natural probability measure on the set of all infinite sequences, for *almost all* sequences, the frequency of heads tends to  $1/2$ . In more mathematical terms, this means that the probability measure  $\mu(S)$  of the set  $S$  of all sequences for which the frequency does not tend to  $1/2$  is 0.

Because the property  $P$  holds for almost all sequences  $\omega$ , sequences that do not satisfy this property are (in some sense) *exceptional*. Because we have assumed that a given sequence  $\omega$  is *random*, it is, therefore, not exceptional, and hence, this sequence  $\omega$  must satisfy the property  $P$ .

The informal argument that justifies this conclusion goes something like that: if  $\omega$  does not satisfy the property  $P$ , this means that  $\omega$  possesses some property (not  $P$ ) that is very rare (is almost never true), and therefore,  $\omega$  is not truly random.

All other existing applications of statistics to physics follow the same pattern: we know that something is true for almost all elements, and we conclude that it is true for an element that is assumed to be random; in this manner, we estimate the fluctuations, apply random processes, etc. So, a random object is an object that satisfies all the properties that are true for almost all objects (almost all with respect to some reasonable probability measure). To give a definition of randomness, we must somewhat reformulate this statement.

To every property  $P$  that is true almost always, we can put into correspondence a set  $S_P$  of all objects that do not satisfy  $P$ ; this set has, therefore, measure 0. An object satisfies the property  $P$  if and only if it does not belong to the set  $S_P$ . Vice versa, if we have a definable set  $S$  of measure 0, then the property “not to belong to  $S$ ” is almost always true.

In terms of such sets, we can reformulate the above statement as follows: if an object is random, then it does not belong to any definable set of measure 0. So, if an object does not belong to any definable set of measure 0, we can thus conclude that it has all the properties that are normally deduced for random objects, and therefore, it can reasonably be called random.

Thus, we arrive at the following definition: *an object is random with respect to a probability measure  $\mu$  if and only if it does not belong to any definable set  $E$  of  $\mu$ -measure 0 ( $\mu(S) = 0$ ).* This, in effect, is the definition proposed by

Martin-Löf.

Now, we have an idea, and a means to formalize it. We are thus ready to formulate our result.

## 3 Main Result

### 3.1 Mathematical Result

**Definition.** Let a mathematical language  $L$  be fixed (e.g., language of set theory, or language of recursive objects).

- Sets defined by formulas from  $L$  will be called ( $L$ -)definable.
- Let  $\mu$  be a probability measure on a set  $X$ . An element  $x \in X$  is called random w.r.t.  $\mu$  if  $x$  does not belong to any  $L$ -definable set of  $\mu$ -measure 0.

*Comment.* For recursive  $L$ , we get Kolmogorov-Martin-Löf's definition of randomness.

**THEOREM.** Let  $\mu_1, \dots, \mu_n$  be measures on  $X$ , and let  $\alpha_i > 0$ ,  $\alpha_1 + \dots + \alpha_n = 1$ . Then, an element  $x \in X$  is random w.r.t. a composite measure  $\mu = \alpha_1 \cdot \mu_1 + \dots + \alpha_n \cdot \mu_n$  iff  $x$  is random w.r.t. one of the measures  $\mu_i$ .

*Comments.*

- This property was *postulated* when Levin defined tests of randomness for arbitrary measures [5] (see also the last section of [2]). In our definition, this property becomes a theorem.
- For reader's convenience, the proof of this theorem is placed in the last section.

### 3.2 Physical Consequence of Our Result

By applying the theorem to  $\alpha_i = p_i$  and  $\mu_i = \mu_{\mathcal{B}}^{\psi_i}$ , we can conclude that every sequence random w.r.t.  $\mu_{\mathcal{B}}$  is random w.r.t. one of the measures  $\mu_{\mathcal{B}}^{\psi_i}$ . We can reformulate it by saying that *if a particle is in a composite state, it actually is in one of the pure states.*

This conclusion is in accordance with the opinion of many physicists who view pure states as *real*, and composite states as a useful *mathematical* construction.

## 4 Proof of the Theorem

We must prove that an element  $x \in X$  is random w.r.t. a composite measure  $\mu = \alpha_1 \cdot \mu_1 + \dots + \alpha_n \cdot \mu_n \leftrightarrow x$  is random w.r.t. one of the measures  $\mu_i$ .

#### 4.1 →

Let us first prove that if an element  $x$  is random w.r.t. a composite measure  $\mu = \alpha_1 \cdot \mu_1 + \dots + \alpha_n \cdot \mu_n$ , then  $x$  is random w.r.t. one of the measures  $\mu_i$ .

We will prove this statement by reduction to a contradiction. Indeed, let us assume that  $x$  is random w.r.t.  $\mu$  and not random w.r.t. any of the measures  $\mu_1, \dots, \mu_n$ . By definition of a random element, the fact that  $x$  is not random w.r.t.  $\mu_i$  means that  $x$  is contained in some definable set  $E_i$  of  $\mu_i$ -measure 0. So, for every  $i$  from 1 to  $n$ , there exists a definable set  $E_i$  for which  $\mu_i(E_i) = 0$ .

Since the sets  $E_1, \dots, E_n$  are definable, their intersection  $E = E_1 \cap \dots \cap E_n$  is also definable. From  $x \in E_i$  for all  $i$ , we conclude that  $x \in E$ .

For every  $i$ , the set  $E$  is a subset of the set  $E_i$  which has  $\mu_i$ -measure 0. Therefore, this subset  $E$  also has  $\mu_i$ -measure 0:  $\mu_i(E) = 0$ . From  $\mu_i(E) = 0$  for all  $i$ , we can conclude that  $\mu(E) = \alpha_1 \cdot \mu_1(E) + \dots + \alpha_n \cdot \mu_n(E) = 0$ . So, an element  $x$  belongs to a definable set  $E$  of  $\mu$ -measure 0. This conclusion contradicts to our assumption that  $x$  is random w.r.t.  $\mu$ .

The resulting contradiction shows that it is impossible that  $x$  is not random w.r.t. all  $\mu_i$ . Hence,  $x$  is random w.r.t. at least one of  $n$  measures  $\mu_i$ . The → part is proven.

#### 4.2 ←

Let us now prove that if  $x$  is random w.r.t. one of the measures  $\mu_i$ , then  $x$  is random w.r.t. a composite measure  $\mu = \alpha_1 \cdot \mu_1 + \dots + \alpha_n \cdot \mu_n$ .

We will also prove this part by reduction to a contradiction. Let us assume that  $x$  is random w.r.t.  $\mu_i$  and  $x$  is not random w.r.t.  $\mu$ . The fact that  $x$  is not random w.r.t.  $\mu$  means that there exists a definable set  $E$  of  $\mu$ -measure 0 that contains  $x$ . For this set  $E$ , we have  $\mu(E) = \alpha_1 \cdot \mu_1(E) + \dots + \alpha_n \cdot \mu_n(E) = 0$ . All the terms in this sum are non-negative, therefore, the only way for this sum to be equal to 0 is when all the terms are 0; in particular,  $\alpha_i \cdot \mu_i(E) = 0$ . From  $\alpha_i > 0$ , we can now conclude that  $\mu_i(E) = 0$ . Since  $x \in E$  and  $E$  is a definable set, this conclusion contradicts to our assumption that  $x$  is random w.r.t.  $\mu_i$ . This contradiction concludes the proof of the ← part of the theorem.

The theorem is proven.

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