Astrogeometry:
Towards Mathematical Foundations

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Abstract

In image processing (e.g., in astronomy), the desired black-and-white image is, from the mathematical viewpoint, a set. Hence, to process images, we need to process sets. To define a generic set, we need infinitely many parameters; therefore, if we want to represent and process sets in the computer, we must restrict ourselves to finite-parametric families of sets that will be used to approximate the desired sets. The wrong choice of a family can lead to longer computations and worse approximation. Hence, it is desirable to find the family that it is the best in some reasonable sense. In this paper, we show how the problems of choosing the optimal family of sets can be formalized and solved.

As a result of the described general methodology, for astronomical images, we get exactly the geometric shapes that have been empirically used by astronomers and astrophysicists; thus, we have a theoretical explanation for these shapes.

1 Introduction to the problem

1.1 Sets are needed for image processing

In image processing, our goal is to restore the actual image. For black-and-white images, the image can be identified with a set of its black points, i.e., with a set in a 2-D or in a 3-D space. So, in order to process images, we must be able to process sets.

1.2 In the computer, we can only use finite-parametric families of sets

Images can, in principle, be arbitrarily complicated.

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The ideal description of a set $X \subseteq \mathbb{R}^k$ would include, for any point $x \in \mathbb{R}^k$, an information whether this point $x$ belongs to the given set $X$ or not. This information requires infinitely many \textit{bits} (binary digits) to store. However, inside any given computer, we can only store finitely many bits, and therefore, we can represent the information only about finitely many points $x \in \mathbb{R}^k$. In computer imaging, these points are usually called \textit{pixels}.

A pixel-by-pixel representation is necessary for some images (e.g., to store a high quality photo) and for computer games (to create the most realistic picture). However, such a representation requires a lot of computer memory and makes processing the corresponding data extremely slow. Therefore, if we want to speed up the processing of these sets, we must somehow approximate arbitrarily complicated sets by sets that can be characterized by a few real-valued parameters, i.e., by sets that belong to some \textit{finite-dimensional family of sets}.

Several families of this type have been efficiently used in image processing. This leads us to a problem:

\subsection{1.3 Main problem: which families of sets should we choose}

In principle, different families of sets can be used. It turns out that often, the use of different approximating families leads to different quality of the resulting approximation. Therefore, it is important to choose the right approximating family.

Currently, this choice is mainly made \textit{ad hoc}, at best, by testing a few possible families and choosing the one that performs the best on a few benchmarks. Since only a few families are analyzed, we are not sure that we did not miss the real good approximating family. (And since only a few benchmarks are used for comparison, we are not sure that the chosen family is indeed the best one.) It is, therefore, desirable to find the \textit{optimal} family of approximating sets.

\subsection{1.4 What we are planning to do}

In this paper, we will describe a general framework for finding the optimal family, and illustrate this general idea on the example of the astronomic imaging.

\section{What does “optimal” mean? Motivations for the following definitions}

\subsection{2.1 What is “optimality criterion”}

When we say “optimal”, we mean optimal \textit{w.r.t.} to some \textit{optimality criterion}. When we say that some \textit{optimality criterion} is given, we mean that, given two different families of approximating sets, we can decide whether the first one
is better, or that the second one is better, or that these families are of the same quality w.r.t. the given criterion. In mathematical terms, this means that we have a \textit{pre-ordering relation} $\preceq$ on the set of all possible finite-dimensional families of sets.

\section{2.2 \textbf{We want to solve an ambitious problem: enumerate all finite-dimensional families of sets that are optimal relative to some natural criteria}}

One way to approach the problem of choosing the “best” family of sets is to select one optimality criterion, and to find a family of sets that is the best with respect to this criterion. The main drawback of this approach is that there can be different optimality criteria, and they can lead to different optimal solutions. It is, therefore, desirable not only to describe a family of sets that is optimal relative to some criterion, but to describe all families of sets that can be optimal relative to different natural criteria. In this paper, we are planning to implement exactly this more ambitious task.

\section{2.3 \textbf{Examples of optimality criteria}}

\subsection{2.3.1 \textbf{Numerical optimality criteria}}

Pre-ordering is the general formulation of optimization problems in general, not only of the problem of choosing a family of sets. In general optimization theory, in which we are comparing arbitrary \textit{alternatives $A$, $B$, ..., from a given set $A$}, the most frequent case of such a pre-ordering is when a \textit{numerical criterion} is used, i.e., when a function $J: A \rightarrow R$ is given for which $A \preceq B$ iff $J(A) \leq J(B)$.

Several natural numerical criteria can be proposed for choosing the best family of sets: if we approximate the actual set of possible values $X$ by an element $\hat{X}$ from the chosen family, then we can measure the quality of the approximation by computing the Lebesgue measure of the difference between the two sets, or by computing the Hausdorff distance between these two sets. As an optimality criterion, we can, e.g., choose the \textit{average} value of this quality measure (average in the sense of some natural probability measure on the class of all problems).

Alternatively, we can fix a class of the problem, and take the \textit{largest} (worst-case) value of the quality measure for problems of this class as the desired (numerical) optimality criterion.

\subsection{2.3.2 \textbf{Non-numerical optimality criteria naturally appear}}

For “worst-case” optimality criteria, it often happens that there are several different alternatives that perform equally well in the worst case, but whose performance differ drastically in the average cases. In this case, it makes sense,
among all the alternatives with the optimal worst-case behavior, to choose the one for which the average behavior is the best possible. This very natural idea leads to the optimality criterion that is not described by a numerical optimality criterion $J(A)$; in this case, we need two functions: $J_1(A)$ describes the worst-case behavior, $J_2(A)$ describes the average-case behavior, and $A \preceq B$ if either $J_1(A) < J_2(B)$, or $J_1(A) = J_1(B)$ and $J_2(A) \leq J_2(B)$.

We could further specify the described optimality criterion and end up with a natural criterion. However, as we have already mentioned, the goal of this paper is not to find a family of sets that is optimal relative to some criterion, but to describe all families of sets that are optimal relative to some natural optimality criteria. In view of this goal, in the following text, we will not specify the criterion, but, vice versa, we will describe a very general class of natural optimality criteria.

So, let us formulate what “natural” means.

2.4 Which optimality criteria are natural

2.4.1 The criterion must be invariant

Problems related to geometric sets often have natural symmetries. For example, let us consider astronomical images. These images are sets in $R^3$ (or in $R^2$). For such sets, there are three natural symmetries:

- First, if we change the starting point of the coordinate system from the previous origin point $O = (0,0,0)$ to the new origin $O'$ whose coordinates were initially $a = (a_1,a_2,a_3)$, then each point $x$ with old coordinates $(x_1,x_2,x_3)$ gets new coordinates $x'_i = x_i - a_i$. As a result, in the new coordinates, each set $X \in A$ from a family of images $A$ will be described by a “shifted” set $T_a(X) = \{x - a | x \in X\}$, and the family turns into $T_a(A) = \{T_a(X) | X \in A\}$. It is reasonable to require that the relative quality of the two families of sets do not depend on the choice of the origin. In other words, we require that if $A$ is better than $B$, then the “shifted” $A$ (i.e., $T_a(A)$) should be better than the “shifted” $B$ (i.e., $T_a(B)$).

- Second, the choice of a rotated coordinate system is equivalent to rotating all the points $(x \to R(x))$, i.e., going from a set $X$ to a set $R(X) = \{R(x) | x \in X\}$, and from a family $A$ to a new family $R(A) = \{R(X) | X \in A\}$. It is natural to require that the optimality criterion is invariant w.r.t. rotations, i.e., if $A$ is better than $B$, then $R(A)$ is better than $R(B)$.

- Third, it is often difficult to find the exact distance to the observed object. Therefore, we are not sure whether the observed image belongs to a small nearby object, or to a larger but distant one. As a result of this uncertainty, the actual image is only known modulo homothety (similarity, dilation) $x \to \lambda \cdot x$ for some real number $\lambda > 0$. It is, therefore, natural to require that the desired optimality criterion be invariant w.r.t. homothety.
2.4.2 The criterion must be final

If the criterion does not select any family as an optimal one, i.e., if, according to this criterion, none of the families is better than the others, then this criterion is of no use in selection.

If the criterion considers several different families equally good, then we can always use some other criterion to help select between these “equally good” ones, thus designing a two-step criterion. If this new criterion still does not select a unique family, we can continue this process until we arrive at a combination multi-step criterion for which there is only one optimal family.

Therefore, we can always assume that our criterion is final in the sense that it selects one and only one optimal family.

3 Definitions and the main result

Our goal is to choose the best finite-parametric family of sets. To formulate this problem precisely, we must formalize what a finite-parametric family is and what it means for a family to be optimal. In accordance with our informal description, both formalizations will use natural symmetries. So, we will first formulate how symmetries can be defined for families of sets, then what it means for a family of sets to be finite-dimensional, and finally, how to describe an optimality criterion.

**Definition 1.** Let \( g : M \to M \) be a 1-1-transformation of a set \( M \), and let \( A \) be a family of subsets of \( M \). For each set \( X \in A \), we define the result \( g(X) \) of applying this transformation \( g \) to the set \( X \) as \( \{ g(x) | x \in X \} \), and we define the result \( g(A) \) of applying the transformation \( g \) to the family \( A \) as the family \( \{ g(X) | X \in A \} \).

**Definition 2.** Let \( M \) be a smooth manifold. A group \( G \) of transformations \( M \to M \) is called a Lie transformation group, if \( G \) is endowed with a structure of a smooth manifold for which the mapping \( g, a \to g(a) \) from \( G \times M \) to \( M \) is smooth.

We want to define \( r \)-parametric families sets in such a way that symmetries from \( G \) would be computable based on parameters. Formally:

**Definition 3.** Let \( M \) and \( N \) be smooth manifolds.

- By a multi-valued function \( F : M \to N \) we mean a function that maps each \( m \in M \) into a discrete set \( F(m) \subseteq N \).

- We say that a multi-valued function is smooth if for every point \( m_0 \in M \) and for every value \( f_0 \in F(m) \), there exists an open neighborhood \( U \) of \( m_0 \) and a smooth function \( f : U \to N \) for which \( f(m_0) = f_0 \) and for every \( m \in U \), \( f(m) \subseteq F(m) \).
Definition 4. Let $G$ be a Lie transformation group on a smooth manifold $M$.

- We say that a class $A$ of closed subsets of $M$ is $G$--invariant if for every set $X \in A$, and for every transformation $g \in G$, the set $g(X)$ also belongs to the class.

- If $A$ is a $G$--invariant class, then we say that $A$ is a finitely parametric family of sets if there exist:
  - a (finite-dimensional) smooth manifold $V$;
  - a mapping $s$ that maps each element $v \in V$ into a set $s(v) \subseteq M$; and
  - a smooth multi-valued function $\Pi : G \times V \rightarrow V$

such that:

- the class of all sets $s(v)$ that corresponds to different $v \in V$ coincides with $A$, and

- for every $v \in V$, for every transformation $g \in G$, and for every $\pi \in \Pi(g,v)$, the set $s(\pi)$ (that corresponds to $\pi$) is equal to the result $g(s(v))$ of applying the transformation $g$ to the set $s(v)$ (that corresponds to $v$).

- Let $r > 0$ be an integer. We say that a class of sets $B$ is a $r$--parametric class of sets if there exists a finite-dimensional family of sets $A$ defined by a triple $(V,s,\Pi)$ for which $B$ consists of all the sets $s(v)$ with $v$ from some $r$--dimensional sub-manifold $W \subseteq V$.

Definition 5. Let $A$ be a set, and let $G$ be a group of transformations defined on $A$.

- By an optimality criterion, we mean a pre-ordering (i.e., a transitive reflexive relation) $\preceq$ on the set $A$.

- An optimality criterion is called $G$-invariant if for all $g \in G$, and for all $A,B \in A$, $A \preceq B$ implies $g(A) \preceq g(B)$.

- An optimality criterion is called final if there exists one and only one element $A \in A$ that is preferable to all the others, i.e., for which $B \preceq A$ for all $B \neq A$.

- An optimality criterion is called natural if it is $G$--invariant and final.
Theorem. Let $M$ be a manifold, let $G$ be a $d$-dimensional Lie transformation group on $M$, and let $\mathcal{A}$ be a natural (i.e., $G$-invariant and final) optimality criterion on the class $\mathcal{A}$ of all $r$-parametric families of sets from $M$, $r < d$. Then:

- the optimal family $A_{\text{opt}}$ is $G$-invariant; and
- each set $X$ from the optimal family is a union of orbits of

\[ (d - r) \text{-dimensional subgroups of the group } G. \]

(For readers’ convenience, the proof is given in the last section.)

4 Astrogometry: physical application of the main result

Celestial bodies such as galaxies, stellar clusters, planetary systems, etc., have different geometric shapes (e.g., galaxies can be spiral or circular, etc.). Usually, complicated physical theories are used to explain these shapes; for example, several dozen different theories explain why many galaxies are of spiral shape; see, e.g., [8, 7, 10, 2]. Some rare shapes are still unexplained.

In this section, we show that to explain these “astroshapes”, we do not need to know the details of physical equations: practically all the shapes of celestial bodies can be explained by simple geometric invariance properties. This fact explains, e.g., why so many different physical theories lead to the same spiral galaxy shape.

4.1 The symmetry group that corresponds to astrogometry

In astrogometry (i.e., in analysis of geometric astronomical images), we are interested in images $X \subset R^3$. As have already mentioned, for astronomical images, the natural group of symmetries $G_a$ is generated by shifts, rotations, and dilations.

So, to apply our main result to astrogometry, we must describe all orbits of subgroups of $G_a$.

4.2 How to describe orbits of subgroups of $G_a$

A 1-D orbit is an orbit of a 1-D subgroup. This subgroup is uniquely determined by its “infinitesimal” element, i.e., by the corresponding element of the Lie algebra of the group $G$. This Lie algebra is easy to describe. For each of its elements, the corresponding differential equation (that describes the orbit) is reasonably easy to solve.
2-D forms are orbits of ≥ 2-D subgroups, so, they can be enumerated by combining two 1-D subgroups.

Comment. An alternative (slightly more geometric) way of describing 1-D orbits is to take into consideration that an orbit, just like any other curve in a 3-D space, is uniquely determined by its curvature $\kappa_1(s)$ and torsion $\kappa_2(s)$, where $s$ is the arc length measured from some fixed point. The fact that this curve is an orbit of a 1-D group means that for every two points $x$ and $x'$ on this curve, there exists a transformation $g \in G$ that maps $x$ into $x'$. Shifts and rotations do not change $\kappa_i$, they may only shift $s$ (to $s + s_0$); dilations also change $s$ to $s \to \lambda \cdot s$ and change the numerical values of $\kappa_i$. So, for every $s$, there exist $\lambda(s)$ and $s_0(s)$ such that the corresponding transformation turns a point corresponding to $s = 0$ into a point corresponding to $s$. As a result, we get functional equations that combine the two functions $\kappa_i(s)$ and these two functions $\lambda(s)$ and $s_0(s)$. Taking an infinitesimal value $s$ in these functional equations, we get differential equations, whose solution leads to the desired 1-D orbits.

4.3 As a result of applying our main idea, we get exactly all observable astroshapes

4.3.1 Possible orbits

The resulting description of 0−, 1−, and 2−dimensional orbits of connected subgroups $G_\alpha$ of the group $G$ is as follows:

0: The only 0−dimensional orbit is a point.

1: A generic 1−dimensional orbit is a conic spiral that is described (in cylindrical coordinates) by the equations $z = k\rho$ and $\rho = R_0 \exp(c\varphi)$. Its limit cases are:

- a logarithmic (Archimedean) spiral: a planar curve ($z = 0$) that is described (in polar coordinates) by the equation $\rho = R_0 \exp(c\varphi)$.
- a cylindrical spiral, that is described (in appropriate coordinates) by the equations $z = k\phi$, $\rho = R_0$.
- a circle ($z = 0$, $\rho = R_0$);
- a semi-line (ray);
- a straight line.
2: Possible 2-D orbits include:

- a plane;
- a semi-plane;
- a sphere;
- a circular cone;
- a circular cylinder, and
- a logarithmic cylinder, i.e., a cylinder based on a logarithmic spiral.

4.3.2 Possible orbits are exactly possible shapes

Comparing these orbits (and ellipsoids, the ultimate stable shapes) with astroshapes enumerated, e.g., in [10], we conclude that:

- First, our scheme describes all observed connected shapes.
- Second, all above orbits, except the logarithmic cylinder, have actually been observed as shapes of celestial bodies.

For example, according to Chapter III of [10], galaxies consist of components of the following geometric shapes:

- bars (cylinders);
- disks (parts of the plane);
- rings (circles);
- arcs (parts of circles and lines);
- radial rays;
- logarithmic spirals;
- spheres, and
- ellipsoids.

It is easy to explain why logarithmic cylinder was never observed: from whatever point we view it, the logarithmic cylinder blocks all the sky, so it does not lead to any visible shape in the sky at all. With this explanation, we can conclude that we have a perfect explanation of all observed astroshapes.
4.3.3 Comment: we can also explain difficult-to-explain disconnected shapes

In the above description, we only considered connected continuous subgroups $G_0 \subseteq G$. Connected continuous subgroups explain connected shapes.

It is natural to consider disconnected (in particular, discrete) subgroups as well; the orbits of these subgroups lead to disconnected shapes. Thus, we can explain these shapes, most of which modern astrophysics finds pathological and difficult to explain (see, e.g., [10], Section I.3). For example, an orbit $O$ of a discrete subgroup $G_0'$ of the 1-D group $G_0$ (whose orbit is a logarithmic spiral) consists of points whose distances $r_n$ to the center forms a geometric progression: $r_n = r_0 \cdot k^n$. Such dependence (called Titius-Bode law) has indeed been observed (as early as the 18th century) for planets of the Solar system and for the satellites of the planets (this law actually led to the prediction and discovery of what is now called asteroids). Thus, we get a purely geometric explanation of the Titius-Bode law.

Less known examples of disconnected shapes that can be explained in this manner include:

- several parallel equidistant lines ([10], Section I.3);
- several circles located on the same cone, whose distances from the cone’s vertex form a geometric progression ([10], Section III.9);
- equidistant points on a straight line ([10], Sections VII.3 and IX.3);
- “piecewise circles”: equidistant points on a circle; an example is MCG 0-9-15 ([10], Section VII.3);
- “piecewise spirals”: points on a logarithmic spiral whose distances from a center form a geometric progression; some galaxies of Sc type are like that [10].

Comment. V.I. Arnold has shown (see, e.g., [9, 1]) that dynamical systems theory explains why the observed shape should be topological homeomorphic to a spiral. We have explained even more: not only that this shape is homeomorphic to the spiral, but that geometrically, this shape is exactly a logarithmic spiral.

4.3.4 This idea also explains: evolution of geometric shapes, their relative frequency, directions of rotation and of magnetic field

Our main idea can be used to explain not only the shapes themselves, but also how they evolve, which are more frequent, etc. (For details, see, e.g., [6, 5, 4, 3].) This explanation is, however, still on the physical level, so we still need to describe it in precise mathematical terms.
5 Proof of the Theorem

Since the criterion \( \preceq \) is final, there exists one and only one optimal family of sets. Let us denote this family by \( A_{\text{opt}} \).

1. Let us first show that this family \( A_{\text{opt}} \) is indeed \( G \)-invariant, i.e., that \( g(A_{\text{opt}}) = A_{\text{opt}} \) for every transformation \( g \in G \).

Indeed, let \( g \in G \). From the optimality of \( A_{\text{opt}} \), we conclude that for every \( B \in \mathcal{A} \), \( g^{-1}(B) \preceq A_{\text{opt}} \). From the \( G \)-invariance of the optimality criterion, we can now conclude that \( B \preceq g(A_{\text{opt}}) \). This is true for all \( B \in \mathcal{A} \) and therefore, the family \( g(A_{\text{opt}}) \) is optimal. But since the criterion is final, there is only one optimal family; hence, \( g(A_{\text{opt}}) = A_{\text{opt}} \). So, \( A_{\text{opt}} \) is indeed invariant.

2. Let us now show an arbitrary set \( X_0 \) from the optimal family \( A_{\text{opt}} \) consists of orbits of \( (d - r) \)-dimensional subgroups of the group \( G \).

Indeed, the fact that \( A_{\text{opt}} \) is \( G \)-invariant means, in particular, that for every \( g \in G \), the set \( g(X_0) \) also belongs to \( A_{\text{opt}} \). Thus, we have a (smooth) mapping \( g \to g(X_0) \) from the \( d \)-dimensional manifold \( G \) into the \( (d - r) \)-dimensional set \( G(X_0) = \{ g(X_0) \mid g \in G \} \subseteq A_{\text{opt}} \). In the following, we will denote this mapping by \( g_0 \).

Since \( r < d \), this mapping cannot be 1-1, i.e., for some sets \( X = g'(X_0) \in G(X_0) \), the pre-image \( g^{-1}_0(X) = \{ g \mid g(X_0) = g'(X_0) \} \) consists of one than one point. By definition of \( g(X) \), we can conclude that \( g(X_0) = g'(X_0) \) iff \( (g')^{-1}(g(X_0)) = X_0 \). Thus, this pre-image is equal to \( \{ g \mid (g')^{-1}(g(X_0)) = X_0 \} \).

If we denote \( (g')^{-1}g \) by \( \tilde{g} \), we conclude that \( g = g'	ilde{g} \) and that the pre-image \( g^{-1}_0(X) = g^{-1}(g(X_0)) \) is equal to \( \{ g' \tilde{g} \mid g(X_0) = X_0 \} \), i.e., to the result of applying \( g' \) to \( \{ \tilde{g} \mid g(X_0) = X_0 \} \) = \( g^{-1}_0(X) \). Thus, each pre-image \( g^{-1}_0(X) = g^{-1}_0(g(X_0)) \) can be obtained from one of these pre-images (namely, from \( g^{-1}_0(X_0) \)) by a smooth invertible transformation \( g' \). Thus, all pre-images have the same dimension \( D \).

We thus have a stratification (fiber bundle) of a \( d \)-dimensional manifold \( G \) into \( D \)-dimensional strata, with the dimension \( D_f \) of the factor-space being \( \leq r \). Thus, \( d = D + D_f \), and from \( D_f \leq r \), we conclude that \( D = d - D_f \geq n - r \).

So, for every set \( X_0 \in A_{\text{opt}} \), we have a \( D \geq (n - r) \)-dimensional subset \( G_0 \subseteq G \) that leaves \( X_0 \) invariant (i.e., for which \( g(X_0) = X_0 \) for all \( g \in G_0 \)). It is easy to check that if \( g, g' \in G_0 \), then \( gg' \in G_0 \) and \( g^{-1} \in G_0 \), i.e., that \( G_0 \) is a subgroup of the group \( G \). From the definition of \( G_0 \) as \( \{ g \mid g(X_0) = X_0 \} \) and the fact that \( g(X_0) \) is defined by a smooth transformation, we conclude that \( G_0 \) is a smooth sub-manifold of \( G \), i.e., a \( (n - r) \)-dimensional subgroup of \( G \).

To complete our proof, we must show that the set \( X_0 \) is a union of orbits of the group \( G_0 \). Indeed, the fact that \( g(X_0) = X_0 \) means that for every \( x \in X_0 \), and for every \( g \in G_0 \), the element \( g(x) \) also belongs to \( X_0 \). Thus, for every element \( x \) of the set \( X_0 \), its entire orbit \( \{ g(x) \mid g \in G_0 \} \) is contained in \( X_0 \). Thus, \( X_0 \) is indeed the union of orbits of \( G_0 \). Q.E.D.
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