Classical-Logic Analogue of a Fuzzy "Paradox"

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Abstract

One of the main reasons why classical logic is not always the most adequate tool for describing human knowledge is that in many real-life situations, we have some arguments in favor of a certain statement A and some arguments in favor of its negation $\neg A$. As a result, we want to consider both A and $\neg A$ to be (to some extent) true. Classical logic does not allow us to do that, while in fuzzy logic, if the degree of belief in A is different from 0 and 1, then we have a positive degree of belief in a statement $A\&\neg A$.

In classical logic, $A\&\neg A$ is always false, so, when we get both A and $\neg A$ in classical logic (a paradox), this means that something was wrong. In view of that, the fact that $A\&\neg A$ is possible in fuzzy logic, is viewed by some logicians as a "paradox" of fuzzy logic.

In this paper, we show that in classical logic, although we cannot directly have $A\&\neg A$, we can, in some sense, have confidence in both A and $\neg A$. In other words, we show that what classical logicians consider a fuzzy paradox has a direct analogue in classical logic.

1. Introduction

1.1. "Paradoxes" as One of the Reasons Why Classical Logic is Not Always Adequate for Representing Knowledge

In many real-life situations, we have some arguments in favor of a certain statement A and some arguments in favor of its negation $\neg A$. As a result, we want to consider both A and $\neg A$ to be (to some extent) true.

Classical logic does not allow us to do that, because in classical logic, if we have A and $\neg A$ at the same time, we have a *paradox*, a *contradiction* which means that something was wrong.

1.2. "Paradoxes" are Easily Captured by Fuzzy Logic

The inadequacy described above was one of the main reasons for introducing a new logic for describing human knowledge: *fuzzy logic* (to be more precise, fuzzy logic was first introduced in the disguise of *fuzzy set theory* [5], and only later, in logical terms).

In fuzzy logic, if the degree of belief in A is different from 0 (="absolutely false") and 1 (="absolutely true"), and if we use min or algebraic product for &, then we have a positive degree of belief in a statement $A\&\neg A$.

1.3. The Fact That Fuzzy Logic is Able to Capture "Paradoxes" Alienates Some Classical Logicians

In classical logic, $A\&\neg A$ is always false, so, when we get both A and $\neg A$ in classical logic (a *paradox*), this means that something was wrong. In view of that, the fact that $A\&\neg A$ is possible in fuzzy logic, is viewed by some logicians as a "paradox" of fuzzy logic.

As a result, logicians sometimes treat fuzzy logic as in some sense *inferior* to the classical logic.

1.4. What We Are Planning to Do

In this paper, we show that in classical logic, although we cannot directly have $A\&\neg A$, we can, in some sense, have confidence in both A and $\neg A$. In other words, we show that what classical logicians consider a fuzzy paradox has a direct analogue in classical logic.

The preliminary version of some of the results appeared in a technical report [3].

2. Our Main Idea: Informal Description

2.1. A Working Theory

The goal of a (working) theory is to describe some mathematical or physical phenomena. For example, *formal arithmetic* describes integers, *geometry* describes geometric objects, *classical mechanics* describes the movements and locations of solid bodies, etc.

2.2. Axioms

To make a logical theory work, we must postulate some *axioms*, from which we will be able to deduce properties of the analyzed objects.

Some of these statements taken as *axioms* are formulated directly in terms of these objects; these statements have, therefore, a direct meaning, and we can have some intuition about them.

2.3. It Is Often Helpful to Have Additional Axioms Formulated in Terms of More General Concepts

It is well known that sometimes, it is easier to prove results about, say, integers, by using more complicated concepts such as arbitrary sets. Therefore, in addition to the axioms that are formulated in terms of the basic objects of study, we may need some axioms formulated in terms of more complicated concepts.

If we add these new concepts, we will get a new theory T, of which the original working theory is a sub-theory.

2.4. For These New Axioms, We Often Have No Intuition About Whether They Or Their Negations Should be Chosen

The problems with these extra axioms is that, unlike the basic objects, for which we have some intuition, we often have much fewer intuition about these more complicated objects, and therefore, for some statements S that we would like to add, we have no idea whether to add the statement S itself or its negation $\neg S$.

This question, of course, arises only when this statement S is not equivalent to any statement from our original theory: otherwise, we would have some intuition about it that would help us make this decision.

2.5. Our Result: Informal Description

At first glance, it may seem that we have two alternatives:

- We can add S and thus prove *some* statements from the basic (working) theory.
- We can also add $\neg S$ and prove some other statements from the basic theory.

We cannot, of course, add *both* S and $\neg S$, because that would be a contradiction.

However, as we will show in this paper, we can consistently add the *consequences* of both axioms. In this sense, from the viewpoint of the working theory, *both* statements S and $\neg S$ are correct in the sense that all their consequences that can be formulated in terms of a working theory are true.

In this sense, we can have *both* S and $\neg S$ true. This situation is, therefore, a direct classical-logic analogue of the fuzzy "paradox" mentioned above.

3. Definitions and the Main Result

3.1. A Very Brief Introduction to Classical First Order Logic

To make this paper easily available to a general reader, let us first recall how a theory is defined in classical logic (for an introduction to classical logic, see, e.g., [4, 2, 1]). Readers who already know formal logic can skip this subsection.

Definition 1. *Let a set $\mathcal{S}$ be fixed. Its elements will be called symbols.*

- *By a language L, we mean a tuple $(P, F, ar, \mathcal{V})$, where:*
  - *P and F are non-intersecting finite subsets of $\mathcal{S}$;*
    - *elements of P are called predicate symbols, and*
    - *elements of the set F are called function symbols.*
  - *ar is a function from $P \cup F$ to the set N of all non-negative integers;*
    - *the value $ar(p)$ (resp., $ar(f)$) is called an arity of a predicate p (resp., of a function f);*
    - *if $ar(f) = n$, we say that f is $n-ary$.*
  - *$\mathcal{V}$ is a set; its elements will be called variables.*
- *A language $L' = (P', F', ar', \mathcal{V})$ is called a sub-language of the language $L = (P, F, ar, \mathcal{V})$ if $P' \subseteq P$, $F' \subseteq F$, and ar' is a restriction of ar to P' and F'.*

- *For every language L, a term is defined by the following recursive definition:*
  - *every variable is a term;*
  - *if $t_1, \ldots, t_n$ are terms, and f is an $n-$ary predicate, then $f(t_1, \ldots, t_n)$ is a term.*
- *For a language L, an elementary formula is defined as $p(t_1, \ldots, t_n)$, where p is an $n-$ary predicate symbol, and t_i are terms.*
- *A formula is defined as follows:*
  - *Every elementary formula is a formula.*
  - *If A and B are formulas, and v is a variable, then $A\&B$, $A \vee B$, $A \to B$, $A \leftrightarrow B$, $\neg A$, $\forall v\, A$ and $\exists v\, A$ are formulas.*

Comment. The next step is to define a *closed formula* (also called *statement*), i.e., a formula in which every variable is covered by a quantifier (e.g., $\forall x \exists y\, (x > y)$). This notion is intuitively evident but somewhat technically complicated, so we skip a formal definition here. Informally, a closed formula is a one that can either be true or false *by itself*, as opposed to formulas with free variables (like "$x = 3$") that can be true or false depending on the value of the variables.

Definition 2. *By a first order theory (or simply a theory, for short), we understand a pair (L, A), where:*

L *is a (first order) language (in the sense of Definition 1); and*

A *is a set of closed formulas in the language L; these formulas are called axioms of the theory.*

Comment. In logic, we then describe a *model* of a theory, i.e., a mapping that maps every function symbol to a function, and every predicate symbol to a predicate in such a way that all axioms are true.

Definition 3.

- *A set of statements is called consistent with the theory T (or simply consistent, for short) if there exists a model of T in which all these statements are true.*
- *We say that a formula A follows from the theory T if A is true in every model of T.*
- *We say that formulas A and B are equivalent w.r.t. T if the formula $A \leftrightarrow B$ follows from T.*
- *We say that a formula A follows from a formula B (or, that B implies A) if A is true in every model of T in which B is true.*

Comment. It is well known (and reasonably easy to prove) that A follows from B if and only if the implication $B \to A$ follows from the theory T. It is also well known and easy to prove that if A follows from B, and B follows from C, then A follows from C.

3.2. Formulation of Our Main Result

THEOREM. *Let:*

$T = (L, A)$ *be a theory;*

L' *be a sub-language of the language L, and*

S *be a statement in L that is not equivalent to any statement in L'.*

Then, if we take:

- *all statements in L' that follow from S, and*
- *all statements in L' that follow from $\neg S$,*

then the resulting (united) set of statements is consistent with the theory T.

Comment. In our interpretation, L' is a language of *working theory* (e.g., arithmetic), and L is a language that includes more complicated objects (e.g., sets). In these terms, the theorem states that if we have a statement S that is formulated in terms of more complicated objects and that cannot be reformulated in terms of working theory, then we can consider *both* statements from working theory that follow from S and statements that follow from $\neg S$ and still do not get any contradiction.

This result shows that in some reasonable sense (namely, in the sense of taking only statements from the working theory into consideration), it *is* possible to assume both that S is true and that $\neg S$ is true. This result is thus a classical-logic analogue of the fuzzy "paradox" mentioned above.

4. Proof

We will prove our theorem by reduction to a contradiction. Let us assume that the union $D(S) \cup D(\neg S)$ is inconsistent, where $D(S)$ denotes the set of statements in the language L' that follow from S, and $D(\neg S)$ denotes the set of statements in the language L' that follow from $\neg S$.

According to a well-known result from mathematical logic (sometimes called *compactness theorem*) [4, 2, 1], if a set of formulas is inconsistent, then it has an inconsistent finite subset. In our case, this means that there exist a finite set of statements $A_1, \ldots, A_n$

that follow from S and a finite set of statements $B_1, \ldots, B_m$ that follow from $\neg S$ such that the union $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ is inconsistent.

The fact that all A_i follow from the statement S means that each of the statements A_i is true in every model of T in which S is true. Therefore, the conjunction $A_1 \& \ldots \& A_n$ (we will denote it by A) is also true in every such model, and thus, also follows from S.

Similarly, the conjunction $B_1 \& \ldots \& B_m$ (we will denote it by B) follows from $\neg S$.

The inconsistency of the set

$$\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$$

means that in no model of T, all A_i and all B_j are true. This, in its turn, means that if all A_i is true (i.e., if A is true), then $B = B_1 \& \ldots \& B_m$ cannot be true. This means that A implies $\neg B$.

So, in a theory T, S implies A, $\neg S$ implies B, and A implies $\neg B$. From the fact that $\neg S$ implies B, we can conclude that $\neg B$ implies S. So:

- on one hand, S *implies* A;
- on the other hand, A implies $\neg B$, and $\neg B$ implies S; therefore, A *implies* S.

Hence, the statement S is equivalent to the statement A in the language A'. This equivalence contradicts to our assumption that S is not equivalent to any statement in L'. This contradiction show that our initial assumption was false, and the set $D(S) \cup D(\neg S)$ is indeed consistent. Q.E.D.

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