

Fuzzy Implication can be Arbitrarily Complicated: A Theorem

Francisco G. Fernandez and Vladik Kreinovich

Department of Computer Science,
University of Texas at El Paso,
El Paso, TX 79968,
emails {ffernand,vladik}@cs.utep.edu.

Abstract

In fuzzy logic, there are several methods of representing implication in terms of $\&$, \vee , and \neg ; in particular, *explicit* representations define a class of *S* implications, *implicit* representations define a class of *R* implications. Some reasonable implication operations have been proposed, such as Yager's a^b , that are difficult to represent as *S* or *R* implications. For such operations, a new class of representations has been recently proposed called *A* implications, for which the relationship between implications and the basic operations $\&$, \vee , and \neg is even more complicated.

A natural question is: is this complexity really necessary? In other words, is it true that *A* operations cannot be described as *S* or *R* ones, or they can, but we simply have not found these representations?

In this paper, we show that yes, the complexity is necessary, because there are operations that cannot be represented in a simpler form.

1 Formulation of the problem

1.1 Fuzzy (i.e., multi-valued) logics

For every problem, be it medical, engineering, etc., we would like to use the best experts. Best experts are few, and it is impossible to use them for all practical problems. So, it is desirable to design computer-based systems that will use the knowledge of the best experts to make decisions comparable in quality with the decisions of these experts. Designing such *expert systems* is, therefore, an extremely important area of computer science.

To develop an expert system, we must describe the expert knowledge in a way that a computer can use it. Expert knowledge, in addition to well-formulated

precise statements about which experts are 100% sure, usually contains statements that are not 100% verified, that are somehow more intuitive than precise. In other words, experts may have different degrees of belief in the statements that comprise their knowledge.

Hence, when we describe expert statements, then instead of simply classifying them as “true” or “false” (i.e., instead of using the traditional two-valued logic), we must describe the expert’s *degree of belief* in these statements; this degree of belief may range from “absolutely sure” (that correspond to “true”) to “absolutely wrong” (which corresponds to “false”), but in addition to these two extreme cases, there may be many intermediate degrees of belief that an expert expresses by words like “possible”, “most probable”, etc. In other words, instead of a *two-valued* logic, we have a *multi-valued* logic, in which instead of a truth value that can take only two possible values, we have degree of belief that has many possible values.

The most widely used multi-valued logic is *fuzzy logic* (see, e.g., [8, 11]), in which degrees of belief are described by numbers from the interval $[0, 1]$: 1 corresponds to “absolutely true”, 0 corresponds to “absolutely false”, and values in between 0 and 1 describe intermediate degrees of belief.

1.2 The basic operators: $\&$, \vee , \neg

As with traditional two-valued logic, we would like to use basic logical operations such as $\&$, \vee , and \neg to describe formulas in the system. Because we are no longer dealing with simple two-valued logic, these basic operators must be defined to have a meaning in the multi-valued system which is dependent on the particular application. For example, one set of interpretations for these operators could be $\& = \min(a, b)$, $\vee = \max(a, b)$, and $\neg = 1 - a$.

1.3 How to describe non-basic logical operations: S-operations

In addition to basic operators, we also need to describe other operations in the multi-valued system, such as fuzzy implication, in order to describe more interesting formulas. In traditional logic, such operations are representable in terms of $\&$, \vee , and \neg (e.g., using CNF, DNF). The natural idea is to use the same representation in multi-valued logics. One way to do this is to use descriptions based on *explicit* representations (see [3, 4, 7, 10, 14]) of implication in terms of $f_{\&}$, f_{\vee} , and f_{\neg} : e.g.,

$$f_{\rightarrow}(a, b) = f_{\vee}(b, f_{\neg}(a)). \quad (1)$$

The resulting functions $f_{\rightarrow} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ are called *S-operations*(see [3, 4, 7, 10, 14]).

1.4 How to describe non-basic logical operations: R-operations

Another possibility is to use descriptions based on *implicit* representations of implication (see [3, 4, 7, 10, 14]) in terms of $\&$, \vee , and \neg : e.g., $A \rightarrow B$ is the weakest statement C with the property that $C\&A$ implies B . This representation leads to

$$f_{\rightarrow}(a, b) = \inf\{c \mid f_{\&}(c, b) \geq a\}. \quad (2)$$

The resulting functions $f_{\rightarrow} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ are called *R-operations*.

1.5 Implication operations not easily describable as S- or R-operations: A-operations

There are still some implication operations, such as an operation b^a introduced by R. Yager in [15], that cannot be easily described in this manner. Attempts have been made to show that such operations can be described if we allow *non-commutative* $\&$ -operations $f_{\&}$ (see, e.g., [4, 5, 6]). Such representations, however, are somewhat artificial in the sense that in such operations, the resulting degree of belief $f_{\&}(a, b)$ in a statement $A\&B$ may be different from the degree of belief $f_{\&}(b, a)$ in a seemingly equivalent statement $B\&A$.

To improve on this artificial representation, the authors of [14] propose a new class of implication operations, which they call *A-operations*, because these operations are uniquely determined by some reasonable *axioms*. In particular, they consider the following axioms:

(I0) For $a, b \in \{0, 1\}$, \rightarrow should be consistent with the classical implication, i.e., $a \rightarrow b = 1$ unless $a = 1, b = 0$, in which case, $a \rightarrow b = 0$.

(I1) $a \rightarrow (b\&c) \equiv (a \rightarrow b)\&(a \rightarrow c)$.

Motivation. If A implies $B\&C$, this means that A implies B , and that A implies C .

Definition. We will say that an operation f_{\rightarrow} *satisfies* the axiom (I1) if $f_{\rightarrow}(a, f_{\&}(b, c)) = f_{\&}(f_{\rightarrow}(a, b), f_{\rightarrow}(a, c))$ for all a, b , and c . Similar definitions can be repeated for other axioms.

(I2) $a \rightarrow (b \rightarrow c) \equiv (a\&b) \rightarrow c$.

Motivation. If A implies that “ B implies C ”, this means that whenever we have both A and B , we can deduce C . Vice versa, if $A\&B$ implies C , this means that if we have A , then from B , we can deduce C .

(I3) $(0.5 \rightarrow b)\&(0.5 \rightarrow b) \equiv b$.

Motivation. If we have a statement A about which we know nothing (so it is safe to assume that the degree of belief in both A and $\neg A$ is equal to 0.5), and

if we know that B can be deduced from both A and $\neg A$, then B must be true. Vice versa, if B is true, then B can be deduced from both A and $\neg A$.

In [14], it is proven that $f_{\rightarrow}(a, b)$ is the only operation that satisfies these properties (I0)–(I3).

1.6 The problem

The natural question is: are these A operations really necessary? Or, we can, in principle, describe every A operation as a S or a R operation?

2 Definitions and the main result

2.1 The difference between S-, R-, and A-operations reformulated in logical terms

It turns out that the above descriptions of S -, R -, and A - operations can be reformulated in terms of their *logical complexity*.

S. S -operations are explicitly defined in terms of the basic operations, and thus, do not need any additional logical symbols such as logical connectives ($\&$, \vee , \neg) or quantifiers.

R. An R -operation $f_{\rightarrow}(a, b)$ is defined indirectly, as the largest number c for which a certain property holds. This indirect representation can be described in terms of first order logic, namely, the fact that z is the largest number that satisfies a certain property means that, first, z itself satisfies this property, and second, that if any other number z satisfies the same property, then $c \leq z$. In other words, we can reformulate the definition of an R operation as follows:

$$f_{\rightarrow}(a, b) = z \leftrightarrow \\ f_{\&}(z, a) \geq b \& \forall c (f_{\&}(c, a) \geq b \rightarrow c \leq z).$$

We have reformulated an R operation as a formula with quantifiers over real numbers. Such formulas are called *first order formulas*.

A. Finally, the definition of an A operation can be reformulated in logical terms as follows: since $f_{\rightarrow}(a, b) = a^b$ is the only operation that satisfies the properties (I)–(I3), the equation $a^b = z$ is satisfied if and only if $f(a, b) = z$ for all functions f that satisfy the conditions (I)–(I3). In other words:

$$f_{\rightarrow}(a, b) = z \leftrightarrow \forall f [\\ [f(0, 0) = 1 \& f(0, 1) = 1 \& f(1, 0) = 0 \& f(1, 1) = 1 \& \\ \forall a, b, c (f(a, f_{\&}(b, c)) = f_{\&}(a, b), f(a, c)) \& \\ \forall a, b, c (f(a, f(b, c)) = f(f_{\&}(a, b), c)) \& \\ \forall b (f_{\&}(f(0.5, b), f(0.5, b)) = b)] \rightarrow \\ f(a, b) = z].$$

2.2 Three classes of operations: S^* , R^* , A^*

In order to prove our main result, we must give a formal description of the classes of operations that correspond to S , R , and A . We will denote the corresponding formally defined classes by, correspondingly, S^* , R^* , and A^* . The comment given above prompts the following natural definition of these classes:

Definition. Let operations $f_{\&}$, f_{\vee} , and f_{\neg} be fixed.

S^* We say that a binary operation $f(a, b)$ (or any other operation $f(a, \dots, b)$) is an S^* operation if this function f can be represented as a composition of basic functions $f_{\&}$, f_{\vee} , and f_{\neg} (i.e., if f can be explicitly defined in terms of basic operations).

R^* We say that an operation $f(a, b)$ is an R^* operation if the relation $f(a, b) = z$ can be represented by a first order formula in the language with basic functions $f_{\&}$, f_{\vee} , f_{\neg} , basic operations $=$, $<$, \leq , and with first order quantifiers, i.e., quantifiers $\forall x$, $\exists x$ that run over real numbers.

A^* We say that an operation $f(a, b)$ is an A^* operation if the relation $f(a, b) = z$ can be represented by a second order formula in the language with basic functions $f_{\&}$, f_{\vee} , f_{\neg} , basic operations $=$, $<$, \leq , with first order quantifiers $\forall x$, $\exists x$, and with second order quantifiers $\forall f$, $\exists f$ that run over all possible functions from real numbers to real numbers.

In particular, S implications are examples of S^* operations, R implications are R^* operations, and A implications are examples of A^* operations.

The definitions of the three classes correspond to increasing logical complexity. Since the definitions of each class allows more operations than the previous one, each class includes the previous one: $S^* \subseteq R^* \subseteq A^*$. From this definition, however, it is not automatically clear whether these three classes are indeed all different, i.e., whether, e.g., fuzzy implication can indeed be arbitrarily complicated. It could, in principle, happen that every second-order operation can be expressed in terms of first order logic, and thus, no fuzzy implication operations would be so complicated that they require the second order logic.

We will show that the three classes are indeed different, and thus, operations can be arbitrarily complicated.

2.3 Main result: all three classes are different (i.e., fuzzy implication can be arbitrarily complicated)

As the basic functions, let us choose $f_{\&}(a, b) = a \cdot b$, $f_{\vee}(a, b) = a + b - a \cdot b$, and $f_{\neg}(a) = 1 - a$. Then, the following two results are true:

Theorem 1. *There exists an R^* operation that is not S^* .*

Theorem 2. *There exists an A^* -operation that is not R^* .*

For both cases, we will have the explicit examples of operations that cannot be represented in the simpler form. These operations will be implication operations.

2.4 Proofs

2.4.1 Proof of Theorem 1

As an R^* operation that is not a S^* operation, let us take an implication operation that formalizes the phrase “if A is true, then B is lightly (to some extent) true”. In fuzzy logic, the degree of belief in “slightly” is usually represented as the square root of the degree of belief in A itself (see, e.g., [8]). Therefore, if we interpret “if A then slightly B ” as “slightly B or not A ”, we get the following operation: $f_{\rightarrow}(a, b) = f_{\vee}(\sqrt{b}, f_{\neg}(a))$. For our choice of $f_{\&}$, f_{\vee} , and f_{\neg} , we get $f_{\rightarrow}(a, b) = \sqrt{b} + (1 - a) - \sqrt{b} \cdot (1 - a)$.

All three basic operations are polynomials, therefore, any function that can be obtained as a composition of these operations, is also a polynomial. Thus, every S^* operation is a polynomial. The above operation is not a polynomial, and hence, it is not S^* .

Let us now show that this operation is R^* . Indeed, the above function can be reformulated as follows:

$$f_{\rightarrow}(a, b) = z \leftrightarrow \exists c(c = \sqrt{b} \& z = f_{\vee}(c, f_{\neg}(a))).$$

The relation $\sqrt{b} = c$, in its turn, is equivalent to $c^2 = b$, i.e., to $f_{\&}(c, c) = b$. Thus, the above function can be expressed by the following first order formula:

$$f_{\rightarrow}(a, b) = z \leftrightarrow \exists c(f_{\&}(c, c) = b \& z = f_{\vee}(c, f_{\neg}(a))).$$

Thus, this is an R^* -operation that is not S^* . Q.E.D.

2.4.2 Proof of Theorem 2

We will prove that Yager’s implication $f_{\rightarrow}(a, b) = a^b$ is an A^* -operation that is not R^* . We have already shown that it is a A^* operation. So, to complete our proof, it is sufficient to show that it is not R^* .

Indeed, by definition, every R^* operation is defined by a first order formula, with the basic operations being $f_{\&}$, f_{\vee} , and f_{\neg} . In our case, these three operations are polynomials, i.e., compositions of additions and multiplications, and therefore, for each R^* operation f , the formula $f(a, b) = z$ can be expressed as a first order formula in which the elementary operations are addition and multiplication. There is a theorem, proved initially by A. Tarski [13] (see also [12, 1, 2]), that every function $f(a, b)$ that is defined by such a first-order formula is *algebraic*, i.e., it is a solution of some polynomial equation (i.e., equation of the type $P(f, a, b) = 0$ for some polynomial P).

However, it is known that the function a^b is not algebraic [9]. Hence, Yager's operation is not R^* . Q.E.D.

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