Hilbert Problems (Almost) 100 Years Later (From the Viewpoint of Interval Computations)

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Hilbert problems. In 1900, the world-wide mathematical community asked David Hilbert, the leading mathematician of his time, to prepare the list of challenging mathematical problems for the coming 20th century. After careful analysis, Hilbert selected 23 problems that he considered to be the most important and the most promising. These problems were delivered in his famous 1900 lecture [12].

From the mathematical viewpoint, Hilbert's selection was extremely successful. The short list of problems indeed inspired breakthroughs and developments that formed the core of the 20th century mathematics. Some of these problems even mentioned the notion of what we would now call an algorithm, the notion that was still in infancy in 1900, but which now shapes the world of science and the humanity in general.

Hilbert problems: a view from the year (almost) 2000. In the course of these developments, practically all Hilbert's problems were solved [6]. Solved – in the sense in which Hilbert himself would like them to be solved: if he asked whether a certain mathematical object existed, he meant to obtain a *proof* that such an object existed, not necessarily an explicit construction. Of course, a construction is always desirable but back then in 1900, before computers were invented, even the *existence* of a construction did not necessarily mean that we could actually construct: it could be that this construction required thousands of steps, which was, at that time, next to impossible.

Nowadays, computers routinely make millions and billions of steps, so a theoretical ability to construct most frequently means that we can actually construct (if not now, then in the near future). With this fascinating computational ability in mind, it is desirable to re-visit year-1900 Hilbert problems from this year-2000 computational viewpoint.

What exactly is the question? Some Hilbert problems are already formulated in computational terms: e.g., the 10th problem asks to design a process (we would now call it an *algorithm*) to check whether a given Diophantine equation has a solution (Matiyasevich proved [21] that there is no such algorithm). However, most problems are formulated in non-algorithmic terms. In most cases, even algorithmic re-formulation is not very straightforward:

- If the question is about the *existence* of a natural or a rational number, then it is natural to reformulate it as a question about the *algorithm* for producing such numbers.
- But what if the question is about the existence of a *real* number? Or a continuous *function* from real numbers to real numbers? What do we then mean by computing a number or a function?

These questions form the basis of the so-called *constructive mathematics* (see, e.g., [1, 2, 3, 4, 5, 19]) In particular, to *compute* a real number x means to be able, for any given accuracy k, to produce an *interval* of width 2^{-k} that is guaranteed to contain this number.

In terms of these formalizations, the question is: we know that an object exists; can we have an algorithm for computing this object?

For different Hilbert problems, this question has different answers.

First example: algorithm expected, algorithm found. The first Hilbert problem for which this question was successfully solved was 17th: *To show that if a rational function* $r(x_1,\ldots,x_n)=p(x_1,\ldots,x_n)/q(x_1,\ldots,x_n)$ *of several real variables* $x_1,\ldots,x_n$ *is always non-negative, then it can be represented as a sum of squares of rational functions:* $r(x_1,\ldots,x_n)=r_1^2(x_1,\ldots,x_n)+\ldots+r_m^2(x_1,\ldots,x_n)$. This problem was solved (by E. Artin) in 1927, but Artin's proof did not provide any algorithm for finding $r_1,\ldots,r_m$. The general belief was that such an algorithm must exist, and many researchers tried to find it, but it was actually found only in 1957, by G. Kreisel [15, 16] (see also [8, 9, 20]).

In this problem, researchers expected an algorithm, and an algorithm was actually found.

Second example: no algorithm expected, no algorithm possible. Let us give an example of a problem where an algorithm was not expected: 3rd, about the axiomatization of volume in elementary geometry. Traditional description of a volume requires not only additivity but also the so-called *method of exhaustion.*

- In the plane, every two polygons of equal area are equi-decomposable, and therefore, any additive function on polygons is either an area or a function of an area.
- In 3D case, in 1900, it was not known whether any two polytopes of equal volume are equi-decomposable.

Dehn has proved that some are not, and moreover, he proved the existence of an additive function that is neither a volume nor a function of a volume. This function was strongly non-constructive (used axiom of choice), and it was widely believed that no constructive function of this type is actually possible. This was proven in [14].

In this problem, researchers expected that no algorithm is possible, and this impossibility was indeed proven.

Comment. The non-existence of an algorithm may sound negative, but it can be re-formulated in positive terms: The only *constructive* additive function on the set of all polytopes is either a volume or a function of a volume.

Moreover, an even more positive result is possible [14]:

- To determine the area of a planar polygon, it is not sufficient to only use additivity. We must also use a limit process: e.g., in general, to determine the area of a square with an irrational side, we must approximate this square by squares with rational sides, and then tend to the limit. In this case, we use quadrangles (4-gons) to approximate a quadrangle (4-gon).
- Traditional way to describe 3D volume was to use the so-called "method of exhaustion" (also called, for polytopes, "devil's staircase"), a limit process in which, to determine the volume of a tetrahedron (4-tope), we approximate it with polytopes whose number of vertices tends to ∞.
- In [14], it is shown that such approximation with unlimited number of vertices is not necessary: it is sufficient to approximate a polytope by a sequence of polytopes with the same number of edges.

After two examples that confirm expectations, two more which went contrary to expectations. In the above examples, the algorithmic result simply *confirmed* what people already believed. Let us give two more examples in which the answer turned out to be the *opposite* to what was expected.

Third example: no algorithm expected, but still algorithm found. 13th problem asked to prove that some continuous functions of several variables cannot be represented as compositions of functions of one and two variables. At first glance, this seems to be a natural hypothesis, and, e.g., for *differentiable* functions, the corresponding impossibility results were indeed proven. However, for *continuous* function, Kolmogorov actually proved [13] that an arbitrary function of a bounded closed domain *can* be represented as a composition of functions of one variable and addition.

The original Kolmogorov's proof was highly non-constructive, and it was believed that no algorithm is possible that would, given a computable continuous function, find the desired representation. Furthremore, it was believed that Kolmogorov's result is of theoretical interest only, with no algorithmic consequences.

Contrary to this belief, Kolmogorov's theorem turned out to be computationally efficient and useful [10, 11, 17, 18, 24]. After this computational success, it was no longer surprising that a constructive version of Kolmogorov's theorem was proven [22, 23].

Comment. The original constructive version of Kolmogorov's theorem provides, crudely speaking, interval approximations for a *real-valued* continuous function. A natural next question is whether a similar approximation can be obtained for an *interval-valued* continuous function. This question was analyzed in [25, 28].

From intervals, a natural next step is to *nested* intervals (i.e., *fuzzy sets*) [26]. The corresponding generalizations of constructive Kolmogorov's theorem also turned out to be quite useful in knowledge representation [27, 28, 29].

Fourth example: algorithm expected, but no algorithm possible. This example is the above-mentioned 10th problem: Hilbert expected an algorithm, but Matiyasevich proved that no such algorithm exists.

Comment. Similarly to our second example (of the 3rd problem), this *negative* result (non-existence of an algorithm) is proven via first proving an important *positive* result: that every recursively enumerable set can be represented as a set of non-negative solution for an appropriate Diophantine equation [7].

Table. These four examples can be described by the following table:

	Algorithm expected	No algorithm expected
Algorithm found	17rd	13th
No algorithm possible	10th	3rd

Challenge. A reader may expect that after these four examples, we will classify all 23 Hilbert problems into one of these four groups. Alas, to the best of our knowledge, few other problems have been actively analyzed from the computational viewpoint. Since Hilbert problems are indeed important to mathematics, to extend the interval-based constructive analysis to other Hilbert problems is, thus, an important challenge.

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