Multi-Criteria Optimization – an Important Foundation of Fuzzy System Design

Hung T. Nguyen[1] and Vladik Kreinovich[2]

[1]*Department of Mathematical Sciences, New Mexico State University*
Las Cruces, NM 88003, *email* `hunguyen@nmsu.edu`

[2]*Department of Computer Science, University of Texas at El Paso*
El Paso, TX 79968, *email* `vladik@cs.utep.edu`

Abstract. *In many real-life design situations, there are several different criteria that we want to optimize, and these criteria are often in conflict with each other. Traditionally, such* multi-criteria *optimization situations are handled in an* ad hoc *manner, when different conflicting criteria are artificially combined into a single combination objective that is then optimized. The use of unnatural* ad hoc *tools is clearly not the best way of describing a very natural aspect of human reasoning. Fuzzy logic describes a much more natural way of handling multi-criterion optimization problems: when we cannot maximize each of the original conflicting criteria 100%, we optimize each to a certain extent.*

Several methods have been proposed in fuzzy logic to handle multi-criteria optimization. These methods, however, still use some ad hoc *ideas. In this paper, we show that some approaches to multi-objective optimization can be justified based on the fuzzy logic only and do not require any extra* ad hoc *tools.*

Keywords. Multi-criteria optimization, fuzzy system design, fuzzy sets, fuzzy logic.

1 Introduction

In some real-life situations, when we design a complicated system, we know exactly what we want to to optimize. For example, when we design a race car, our goal is to maximize its speed. In such situations, the problem of finding the best design becomes a clearly defined mathematical problem. Let X denote the set of all possible designs. Then, the problem can be formulated as follows:

GIVEN:

- a (crisp) *objective* function $f : X \to R$, and
- a (crisp) set $C \subseteq X$ (of all designs that satisfy certain *a priori* criteria)

TO FIND $x \in X$ for which

$$f(x) \to \max_{x \in C}.$$

There are several methods of formalizing and solving the maximization problem for more realistic cases in which the conditions on x are formulated in uncertain terms, and are, therefore, described by a *fuzzy* set $\mathbf{C} \subseteq X$ (see, e.g., survey [3] and references therein).

In the majority of real-life situations, however, the objectives of the designed system are not easy to formulate in precise terms. Usually, there are many different criteria $f_1(x), \ldots, f_n(x)$ that we want to optimize, and these criteria are often in conflict with each other. For example, the optimal design for an atomic power station must be both maximally safe and maximally money-saving. If we simply formulate these two maximalities in crisp terms, we will get an inconsistent criterion, because the design that is 100% safe will make the station hundreds of time more expensive, and the cheapest design is clearly not safe. Such situations are called "multi-criteria optimization".

Traditionally, such situations are handled in a somewhat *ad hoc* manner, when different conflicting criteria $f_1(x), \ldots, f_n(x)$ are (rather artificially) combined into a single combination objective $f(x)$ that is then optimized. This combination is usually performed by using an aggregation function $h(y_1, \ldots, y_n)$: $f(x) = h(f_1(x), \ldots, f_n(x))$. The simplest (and most frequently used) aggregation function is a linear function $h(y_1, \ldots, y_n) = w_1 \cdot y_1 + \ldots + w_n \cdot y_n$.

The use of (not very natural) *ad hoc* tools is clearly not the best way of describing a very natural aspect of human reasoning. Fortunately, fuzzy logic describes a much more natural way of handling multi-criterion optimization problems: when we cannot maximize each of the original conflicting criteria 100%, we optimize each *to a certain extent*.

Several methods have been proposed in fuzzy logic to handle multi-criteria optimization; see, e.g., Hwang and Yoon [7], Chen and Hwang [4], Klir and Yuan [8], and references therein. These methods are much more natural than the crisp ones, because they are based on the laws of fuzzy logic that reflect the properties of human reasoning; however, the descriptions of all existing methods use, in additional to fuzzy logic, some ad hoc assumptions and formulas: most of these methods use an aggregation function to combine different criteria $f_i(x)$. In this paper, we show that some of these approaches can be justifies based on the fuzzy logic only and, thus, do not require any extra *ad hoc* tools.

This result builds on the theorems proved in our 1996 paper on fuzzy optimization [3].

2 Fuzzy multi-criteria optimization problems are difficult to formalize

Informally, the multi-criterion optimization problem can be described as follows:

GIVEN:

- a positive integer n;
- n (crisp) functions $f_1, \ldots, f_n : X \to R$, and
- a (fuzzy) set $\mathbf{C} \subseteq X$.

TO FIND $x \in X$ for which

$$f_i(x) \to \max_{x \in \mathbf{C}}$$

for all $i = 1, \ldots, n$.

What is given can be easily formalized:

Definition 1. *By a multi-criterion maximization problem under fuzzy constraints, we mean a tuple* $(f_1, \ldots, f_n, \mathbf{C})$*, where* $f_1, \ldots, f_n : X \to R$ *are (crisp) functions from a set* X *into the set* R *of all real numbers, and* $\mathbf{C} \subseteq X$ *is a fuzzy subset of* X*.*

However, *what we want* is not immediately clear. Since the condition on the desired element x is formulated in fuzzy terms, this problem is difficult to formalize even for a single objective function (see discussion in [3]). For several objective functions, it is even more difficult to formulate. To overcome this double difficulty, it is reasonable to try to handle the two difficulties one by one; in other words:

- first, we will try to formulate the multi-objective optimization problem for the case of crisp constraints (i.e., for the case when $\mathbf{C}$ is a *crisp* set);
- and then, we will try to use general fuzzy techniques to extend this formulation to the general case of fuzzy constraints (i.e., to the case when $\mathbf{C}$ is a fuzzy set).

Let us start with the first part. There are two main ways of representing crisp knowledge for a computer (i.e., in a computer-accessible form):

- as a more mathematics-oriented, *non-procedural* knowledge; usually, in terms of first order logic (or one of its modifications), and
- as a more computer-oriented, *procedural* knowledge; usually, in terms of if-then rules.

In the following sections, we will:

- formalize the crisp version of the multi-objective optimization problem in both languages;
- apply standard fuzzy extension methods (see, e.g., [5, 6]) to extend these descriptions to fuzzy case; and then
- analyze and compare the resulting definitions.

3 Multi-criteria optimization in terms of logic

Let us describe the statement "functions $f_1, \ldots, f_n$ attain their maxima on a set C at x" (denoted hereafter as $S(x)$) in terms of (classical) logic, and then translate it into fuzzy logic.

This statement means that:

- x belongs to C, and
- if y belongs to C, then $f_i(x) \geq f_i(y)$ for $i = 1, \ldots, n$.

Formally,

$$S(x) \leftrightarrow x \in C \,\&\, \forall y(y \in C \rightarrow \forall i(f_i(y) \leq f_i(x))).$$

We want to extend this expression to fuzzy logic. How to do that? Let's start with atomic formulas $x \in C$ and $f_i(y) \leq f_i(x)$:

- For a fuzzy set $\mathbf{C}$, the formula $x \in \mathbf{C}$ is described by a membership function $\mu_C(x)$.
- The system of inequalities $\forall i(f_i(y) \leq f_i(x))$ is a crisp statement, so it can be represented by its truth value $t[\forall i(f_i(y) \leq f_i(x))]$ ($t[A] = 1$ if A is true, and $t[A] = 0$ if A is false).

To combine these atomic formulas, we must choose fuzzy operations $f_\&$, $f_\forall$, and $f_\rightarrow$ that correspond to &, $\forall$, and $\rightarrow$. Then, as a result, we will get the desired membership function for S:

$$\mu_S(x) = f_\&(\mu_C(x), f_\forall(f_\rightarrow(\mu_C(x), t[\forall i(f_i(y) \leq f_i(x))]))).$$

How do we choose these fuzzy operations? We consider & and $\forall$ together, because $\forall$ is nothing else but many "and"s. If we have a finite set X with elements $x_1, \ldots, x_n$, then $\forall x A(x)$ means $A(x_1) \,\&\, A(x_2) \,\&\, \ldots \,\&\, A(x_n)$. If we have an infinite set $X = \{x_1, x_2, \ldots, x_n, \ldots\}$, then we can consider $\forall x A(x)$ as an infinite "and" $A(x_1) \,\&\, A(x_2) \,\&\, \ldots \,\&\, A(x_n) \,\&\, \ldots$, and interpret it as a limit (in some reasonable sense) of finitely many "and"s.

So, it is sufficient to choose a fuzzy analogue of "and"; then, a fuzzy analogue of $\forall$ will be automatically known.

It turns out that the fact that we need to apply & infinitely many times, and still get a meaningful number, drastically restricts the choice of an &−operation. Namely, we must combine the values $f_\rightarrow(\mu_C(x), t[\forall i(f_i(y) \leq f_i(x))])$ that correspond to all possible y's. If we take $y_1, y_2, \ldots, y_n, \ldots$ all close to each other, then the aggregated degrees of certainty will also be close. The closer y_i to each other, the closer the aggregated values to each other. In the limit, we get the following problem: to combine infinitely many identical values a.

If we take $f_\& = \min$, then we get

$$f_\&(a, \ldots, a, \ldots) = \lim_{n \to \infty} f_\&(a, \ldots, a) \text{ (n times)} = \min(a, \ldots, a) = a.$$

For $f_\&(a, b) = a \cdot b$, we get

$$f_\&(a, \ldots, a, \ldots) = \lim_{n \to \infty} f_\&(a, \ldots, a) \text{ (n times)} =$$

$$\lim_{n \to \infty} a \cdot a \cdot \ldots \cdot a \text{ (n times)} = \lim a^n = 0$$

for all $a < 1$. So, for $f_\&(a,b) = a \cdot b$, we get a meaningless result $\mu_S(x) = 0$ for all x. It turns out that we get the same meaningless result for all $\&-$operations different from min. Let's formulate this result in precise terms.

Definition 2. [15, 8, 13] *An $\&-$operation ($t-$norm) is a continuous, symmetric, associative, monotonic operation $f_\& : [0,1]\times[0,1] \to [0,1]$ for which $f_\&(1,x) = x$.*

Usually, three types of $\&-$operations are used: min, strict operations, and Archimedean operations:

Definition 3. *An $\&-$operation is called Archimedean if $f_\&(x,x) < x$ for all $x \in (0,1)$, and strict if it is strictly increasing, as the function of each of the variables.*

The following result is known:

PROPOSITION 1. [3] *If $f_\&$ is an Archimedean or a strict operation, then for every $a \in (0,1)$,*

$$\lim_{n\to\infty} f_\&(a,\ldots,a) \ (n \text{ times}) = 0.$$

Due to this result, it is reasonable to choose $\& = \min$ and, correspondingly, $\forall = \inf$. Hence, we arrive at the following definition:

Definition 4. *Let $(f_1,\ldots,f_n,\mathbf{C})$ be a multi-criteria maximization problem under fuzzy constraints, and let $f_\to : [0.1]\times[0,1] \to [0,1]$ be a function. We will call $f_\to$ an implication operation. By a solution corresponding to $f_\to$, we mean*

$$\mu_S(x) = f_\&(\mu_C(x), \inf_y (f_\to(\mu_C(x), t[\forall i(f_i(y) \le f_i(x))])).$$

How can we choose $f_\to$? There exist many fuzzy analogues of $\to$ (see, e.g., [12, 14, 8, 13]. For our purposes, however, the choice is not so big, because in our formula, we only have crisp conclusions. Let's analyze how different implication operations behave in this case. We will consider the simplest implication operations first, and then we will discuss the general case.

3.1 Kleene-Dienes operation

This operation (see, e.g., [2]) is based on the well-known expression from classical logic: $(a \to b) \leftrightarrow (\neg a \vee b)$. To use this formula, we must know $\neg$ and $\vee$.

Definition 5. *By a $\neg-$operation, we mean a strictly decreasing continuous function $f_\neg : [0,1] \to [0,1]$ such that $f_\neg(0) = 1$ and $f_\neg(f_\neg(a)) = a$.*

Definition 6. [15, 8, 13] *An $\vee-$operation ($t-$conorm) is a continuous, symmetric, associative, monotonic operation $f_\vee : [0,1]\times[0,1] \to [0,1]$ for which $f_\vee(0,x) = x$.*

Definition 7. *Assume that $f_\vee$ and $f_\neg$ are $\vee-$ and $\neg-$operations. The function $f_\to(a,b) = f_\vee(f_\neg(a), b)$ will be called a Kleene-Dienes implication.*

PROPOSITION 2. *Let $(f_1, \ldots, f_n, \mathbf{C})$ be a multi-criteria maximization problem with fuzzy constraints. Then, the solution corresponding to Kleene-Dienes implication has the form*

$$\mu_{KD}(x) = \min(\mu_C(x), f_\neg(\sup_{y:\, \exists i(f_i(y) > f_i(x))} \mu_C(y))).$$

Proof. Since $b \in \{0, 1\}$, we can eliminate $f_\vee$: indeed, $f_\vee(0, x) = x$, and $f_\vee(x, 1) = 1$ for an arbitrary $\vee-$operation. Q.E.D.

Comment. In particular, for $f_\neg(z) = 1 - z$, we get the expression

$$\mu^*_{KD}(x) = \min(\mu_C(x), 1 - \sup_{y:\, \exists i(f_i(y) > f_i(x))} \mu_C(y)).$$

3.2 Zadeh's operator

This implication operator is based on another formula from classical logic: $(a \rightarrow b) \leftrightarrow (\neg a \vee (a \& b))$. Since we already know that $\& = \min$, we arrive at the following definition:

Definition 8. *Assume that $f_\vee$ and $f_\neg$ are $\vee-$ and $\neg-$operations. The function $f_\rightarrow(a, b) = f_\vee(f_\neg(a), \min(a, b))$ will be called a Zadeh implication.*

PROPOSITION 3. *Let $(f_1, \ldots, f_n, \mathbf{C})$ be a multi-criteria maximization problem with fuzzy constraints. Then, the solution corresponding to Zadeh's implication has the form*

$$\mu_Z(x) =$$
$$\min(\mu_C(x), f_\neg(\sup_{y:\, \exists i(f_i(y) > f_i(x))} \mu_C(y)), \sup_{y:\, \forall i(f_i(y) \leq f_i(x))} f_\vee(\mu_C(y), f_\neg(\mu_C(y))).$$

Proof easily follows from considering the cases $b = 0$ and $b = 1$. Q.E.D.

3.3 Other implication operations

It is easy to check that for crisp b, all other known implication operations either turn into one of these two, or lead to a crisp formula. For example, let's consider the most frequently used operations listed in [14]:

- Lukasiewicz's $\min(1, 1 - a + b)$ turns into $1 - a$ if $b = 1$ and 1 if $b = 0$ (same as Kleene-Dienes).
- Gödel's [11] 1 if $a \leq b$, b else, gets only crisp values if b is crisp.
- Gaines's [11] 1 if $a \leq b$ and b/a else leads to 1 if $b = 1$ and to 0 if $b = 0$, i.e., also, only to crisp values.
- Kleene-Dienes-Lukasiewicz [2] $1 - a + a \cdot b$ for crisp b coincides with Kleene-Dienes's.
- Willmott's $\min(\max(1 - a, b), \max(a, 1 - a), \max(b, 1 - b))$ reduces for crisp b to Zadeh's formula [16].

This "lack of choice" can be partially explained (see also [3]) by the fact that usually, two methods of describing an $\rightarrow -$operation are used:

- We can describe $\rightarrow$ directly in terms of &, $\vee$, and $\neg$. We have already considered these methods.
- We can also describe $a \rightarrow b$ indirectly: as a statement that, being added to a, implies b (i.e., as a kind of a "solution" of the equation $f_{\&}(a, a \rightarrow b) = b$). If this equation has several solutions, we can choose, e.g., the largest one, or more generally, the largest c for which $f_{\&}(a, c) \leq b$. Since b is crisp, we get a degenerate solution:
  - If $b = 1$, then $f_{\&}(a, c) \leq 1$ is always true, so $c = 1$.
  - If $b = 0$, then $f_{\&}(a, c) = 0$ is usually only true for $c = 0$.

  So, for crisp b, this definition leads to an operation with crisp values only.

4 Multi-criteria optimization in terms of if-then rules

Let's describe the (crisp) conditional multi-criteria optimization problem in terms of if-then rules. Computational algorithms that compute the maximum are usually iterative, so it is difficult to find if-then rules that would directly select the desired solution. However, it is very easy to describe rules that will delete everything but the desired solution:

- If x does not satisfy the condition, then x is not the desired solution.
- If for some x and for some i, there exists another element y that satisfies the constraint C and for which $f_i(y) > f_i(x)$, then x is not the desired solution.

In logical terms, these rules take the following form:

$$\neg C(x) \rightarrow \neg S(x),$$

$$C(y) \,\&\, (f_i(y) > f_i(x)) \rightarrow \neg S(y).$$

To generalize these rules to the case when the constraint set $\mathbf{C}$ is fuzzy, we will use the standard (Mamdani's) methodology from fuzzy control (see, e.g., [10, 9, 3]). According to this methodology, if we have a set of rules, then for a certain conclusion to be true, it is necessary and sufficient that for one of the rules that lead to this conclusion, all the conditions are satisfied. In logical terms,

$$\neg S(x) \leftrightarrow \neg C(x) \vee (C(y_1) \,\&\, (f_1(y_1) > f_1(x))) \vee \ldots \vee (C(y_1) \,\&\, (f_n(y_1) > f_n(x))) \vee$$

$$(C(y_2) \,\&\, (f_1(y_2) > f_1(x))) \vee \ldots \vee (C(y_2) \,\&\, (f_n(y_2) > f_n(x))) \vee \ldots$$

Here, $\vee$ is applied to statements that correspond to all possible values of y.

Next, in fuzzy control, we substitute degrees of membership instead of atomic statements, and use $\&-$, $\vee-$, and $\neg-$operations instead of $\&$, $\vee$, and $\neg$. As a result, we get the following formula:

$$\mu_{\neg S}(x) = f_\vee[\mu_{\neg C}(x),$$
$$f_\&(\mu_C(y_1), t[f_1(y_1) > f_1(x)]), \ldots, f_\&(\mu_C(y_1), t[f_n(y_1) > f_n(x)]),$$
$$f_\&(\mu_C(y_2), t[f_1(y_2) > f_1(x)]), \ldots, f_\&(\mu_C(y_2), t[f_n(y_2) > f_n(x)]), \ldots].$$

Here, the $\vee-$operation combines infinitely many terms, so (similarly to what we have shown in Section 3), we can conclude that the only way to avoid the meaningless situation in which $\mu_{\neg S}(x) = 1$ for all x is to use $f_\vee = \max$. So, we arrive at the following definition.

Definition 9. *Let $(f_1, \ldots, f_n, \mathbf{C})$ be a maximization problem under fuzzy constraints, and let $f_\neg$ be a $\neg-$operation. A fuzzy set $\mathbf{S}_R$ will be called a rule-based solution to this problem if its membership function $\mu_R(x)$ satisfies the following equality:*

$$f_\neg(\mu_R(x)) = \max[f_\neg(\mu_C(x)), \sup_{y,i} f_\&(\mu_C(y), t[f_i(y) > f_i(x)])].$$

It turns out that this solution coincides with the one described in the Section 3.1:

PROPOSITION 4. *For every multi-criteria maximization problem under fuzzy constraints, the rule-based solution $\mu_R(x)$ coincides with the solution $\mu_{KD}(x)$ corresponding to Kleene-Dienes implication.*

Proof. The predicate $f_i(y) > f_i(x)$ is crisp, so its truth value is either 0, or 1. By definition of an $\&-$operation, $f_\&(a, 0) = 0$, and $f_\&(a, 1) = a$. Therefore:

- If $f_i(y) \le f_i(x)$, we have $t[f_i(y) > f_i(x)] = 0$, and $f_\&(\mu_C(y), 0) = 0$.
- If $f_i(y) > f_i(x)$, then $t[f_i(y) > f_i(x)] = 1$, and $f_\&(\mu_C(y), 1) = \mu_C(y)$.

When computing sup of a set of non-negative numbers, we can neglect 0's, and thus consider only y for which $\exists i(f_i(y) > f_i(x))$. So, we get

$$f_\neg(\mu_R(x)) = \max[f_\neg(\mu_C(x)), \sup_{y:\, \exists i(f_i(y) > f_i(x))} \mu_C(y)].$$

Now, because of the properties of an $\neg-$operation, we have $\mu_R(x) = f_\neg(f_\neg(\mu_R(x)))$, so

$$\mu_R(x) = f_\neg(\max[f_\neg(\mu_C(x)), \sup_{y:\, \exists i(f_i(y) > f_i(x))} \mu_C(y)]).$$

Since $f_\neg$ is decreasing, $f_\neg(\max(a, b)) = \min(f_\neg(a), f_\neg(b))$, so

$$\mu_{DR}(x) = \min[\mu_C(x), f_\neg(\sup_{y:\, \exists i(f_i(y) > f_i(x))} \mu_C(y))].$$

This is exactly the expression μ_{KD} for the Kleene-Dienes implication. Q.E.D.

5 Scale-invariance: an important property of the proposed solution

We will see that the proposed solution has the following important property: the resulting solution $\mu_S(x)$ does not depend on the choice of units in which we measure the values of the objective functions $f_i(x)$, or on any other re-scaling of these functions.

This "scale-invariance" is an important requirement: e.g., if we want to design the largest spherical oil reservoir, which is possible under the current technology, then we can formulate this problem as $f_1(x) = R \to \max$, where R is the radius of the reservoir x, or as $f_1'(x) = V \to \max$, where $V = (4/3)\,\pi \cdot R^3$ is the volume of this reservoir. From the user's viewpoint, maximizing the radius is exactly the same problem as maximizing the volume, so it is desirable that the solutions corresponding to f_1 and f_1' be the same.

- This, unfortunately, is not always the case for traditional *ad hoc* multi-criteria optimization methods: for example, if we maximize a linear combination of different criteria, then maximizing the combination $f(x) = w_1 \cdot f_1(x) + w_2 \cdot f_2(x) + \ldots$ can lead to different results than optimizing the combination $f'(x) = w_1 \cdot f_1'(x) + w_2 \cdot f_2(x) + \ldots$

- Let us show that this requirement of scale-invariance is satisfied for the proposed solution:

Definition 10. *Let a set X be fixed.*

- *By a re-scaling, we mean a strictly increasing function $g : R \to R$ from real numbers to real numbers.*

- *We say that a function $f_i' : X \to R$ is a re-scaling of the function $f_i : X \to R$ if $f_i'(x) = g_i(f_i(x))$ for some re-scaling function $g_i(y)$.*

- *We say that a multi-criteria maximization problem $\mathcal{P}' = (f_1', \ldots, f_n', \mathbf{C})$ under fuzzy constraints is a re-scaling of the maximization problem $\mathcal{P} = (f_1, \ldots, f_n, \mathbf{C})$ if each function f_i' is a re-scaling of the corresponding function f_i.*

PROPOSITION 5. *Let $\mathcal{P}$ be a multi-criteria optimization problem under fuzzy constraints, and let $\mathcal{P}'$ be a re-scaling of the problem $\mathcal{P}$. Then, for a given implication function, a function $\mu_S(x)$ is the solution to the problem $\mathcal{P}$ if and only if it is the solution to the re-scaled problem $\mathcal{P}'$.*

Proof. The proof easily follows from the fact that the definition of the solution (Definition 4) does not use the actual values of f_i, it only uses the relation $f_i(y) \le f_i(x)$ which is preserved under any strictly increasing transformation $f_i(x) \to g_i(f_i(x))$. Therefore, the solutions that correspond to $\mathcal{P}$ and to $\mathcal{P}'$ indeed coincide. Q.E.D.

6 From fuzzy solution to crisp solution

In the above sections, we described the *fuzzy solution* $\mu_S(x)$ to the multi-criteria optimization problem. This fuzzy solution, in effect, supplies the user with a list of possible designs x, together with the degrees $\mu_S(x)$ to which each design x satisfies all the required criteria.

In some real-life situations, this is all the user wants, so that she can make the final decision herself. In some cases, however, the user would prefer a computer to choose the design for him. In these cases, it is natural to choose a design x^* with the largest degree of "requirements satisfaction" $\mu_S(x)$. The experience of fuzzy optimization shows that this is indeed a reasonable choice (see, e.g., Zimmermann [18]).

Definition 11. *Let $(f_1, \ldots, f_n, \mathbf{C})$ be a multi-criteria optimization problem under fuzzy constraints, and let $\mu_S(x)$ be the solution to this problem. We say that an element $x^* \in X$ is a crisp solution to this problem if*

$$\mu_S(x^*) = \max_{x \in X} \mu_S(x).$$

Comments.

- It is worth mentioning that in some cases, better results can be obtained by using more complicated defuzzification procedures [1, 17].
- From the fact the (fuzzy) solution $\mu_S(x)$ is invariant under an arbitrary re-scaling of the objective functions, we can now conclude that the crisp solution (defined as the maximum of $\mu_S(x)$) is also scale-invariant:

PROPOSITION 6. *Let $\mathcal{P}$ be a multi-criteria optimization problem under fuzzy constraints, and let $\mathcal{P}'$ be a re-scaling of the problem $\mathcal{P}$. Then, for a given implication function, an element $x^* \in X$ is a crisp solution to the problem $\mathcal{P}$ if and only if it is a crisp solution to the re-scaled problem $\mathcal{P}'$.*

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