Intervals in space-time:
A. D. Alexandrov is 85

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On August 12, 1997, Alexander Danilovich Alexandrov, the well-known geometer, a member of the Russian Academy of Sciences and of several foreign academies, will turn 85.

A. D. Alexandrov's seminal research papers cover areas ranging from quantum mechanics to geometry to philosophy of science. In particular, his innovative approach to foundations of space-time geometry is directly related to intervals and interval computations. In this short text, we will briefly describe his interval-related results.

Before Alexandrov: 1-dimensional intervals that correspond to measuring time. Let us first describe a *non-relativistic* situation. In non-relativistic (Newtonian) physics, every event is characterized by a single number t (its moment of time). An event with a larger value of t occurs *after* the event with a smaller value of time. (In this sense, in Newtonian physics, temporal order is 1-dimensional.)

Measurement inaccuracies make this simple picture a little bit more complicated. Indeed, measurement is never 100% accurate, there is usually some uncertainty associated with it. As a result, if, e.g., we measure *time*, we do not get the *exact* actual value t of the time. Instead, we get an *approximate*

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measurement result $\widetilde{t}$, based on which, we can determine the *interval* $\mathbf{t} = [\underline{t}, \overline{t}]$ of possible values of t, where $\underline{t} = \widetilde{t} - \Delta$, $\overline{t} = \widetilde{t} + \Delta$, and Δ is an upper bound on the measurement error (this upper bound is usually given by a manufacturer of the time measuring instrument).

From the *physical* viewpoint, the interval means that the event, whose time we are measuring, occured *after* the moment $\underline{t}$ and *before* the moment $\overline{t}$. In other words, we have two events that bound the event in whose timing we are interested.

As a result, to describe the actual information about the events, we must use *intervals* $\mathbf{t}$ instead of numbers t.

Comments.

- In some cases, in addition to an interval, we also know the probabilities of different values on this interval, but often, the interval is all we know.

- The use of intervals to describe measurement of time was suggested by Norbert Wiener [10], who later became famous as the founder of cybernetics. Wiener's paper was one of the first uses of intervals in measurement theory.

- In 1-dimensional case, simply knowing which of the intervals are after and which are before is not enough. We need an additional structure, such as:

  - *addition* $t \to t + t_0$ that corresponds to the change in the starting point for measuring time;
  - *multiplication* $t \to \lambda \cdot t$, that corresponds to replacing the old time unit by a new unit that is λ times smaller, etc.

  Since these operations are usually defined on *moments* of time, there arrives a necessity to *extend* these operations to intervals, i.e., to use *interval computations.*

Relativistic case. To fully characterize an event, we need to describe not only its moment of time t, but also its *location* $\vec{x} = (x_1, x_2, x_3)$.

- In *Newtonian mechanics*, there is no limit on the communication speed, so, no matter how far away the two locations $\vec{x}$ and $\vec{x}'$ are, the event $e = (t, \vec{x})$ that occured before the event $e' = (t', \vec{x}')$ ($t < t'$) can, in principle, influence e'.

- In *special relativity*, the speed of every communication, every influence is bounded by the speed of light c. Thus, an event $e = (t, \vec{x})$ can influence the event $e' = (t', \vec{x}')$ (we will denote it by $e \preceq e'$) only if, in addition to $t < t'$, the distance $d(\vec{x}, \vec{x}') = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}$ does not exceed $c \cdot (t' - t)$.

Similarly to the Newtonian case, we can describe the uncertainty of our knowledge about an event e by describing *two* events: an event $\underline{e}$ that precedes e, and an event $\overline{e}$ that follows e. As a result, we only know that e belongs to a *(space-time) interval* $\mathbf{e} = [\underline{e}, \overline{e}] = \{e \mid \underline{e} \preceq e \preceq \overline{e}\}$.

Alexandrov's result. In 1950, Alexandrov has proven an unexpected result: that in relativistic case (in contrast to the Newtonian case) the ordering structure of intervals uniquely determines the linear structure on space-time. This fact (that a linear structure on space-time is uniquely determined by its ordering relation) can be formulated in the following precise terms: If f is a 1-1 mapping from R^4 onto R^4, and $e \in [\underline{e}, \overline{e}]$ if and only if $f(e) \in [f(\underline{e}), f(\overline{e})]$, then f is a linear mapping. Moreover, f is a composition of a 3-dimensional rotation, a 3-dimensional (spatial) shift, a 1-dimensional (temporal) shift, and a Lorentz transformation.

This theorem was first published in [2] (some preliminary results were announced in [1]). The theorem became widely known after E. C. Zeeman (later known as one of the authors of catastrophe theory) independently re-discovered this result in [11] and published it in a physics-oriented journal.

For a rather recent update, see, e.g., the 1992 monograph [7]. Of all results published after 1992, we would like to mention a paper [4], in which it is shown that Alexandrov's theorem holds even if take into consideration that the causality relation may also be only approximately known.

A follow-up of Alexandrov's theorem: interval-based foundation of space-time geometry. Alexandrov's result has shown that we can describe the foundations of special relativity in terms of space-time intervals. However, from a physical viewpoint, special relativity is only approximately true: in reality, the space-time is more complicated (e.g., it is curved). A natural next step, therefore, was to describe the foundations of *general* space-time geometry in similar interval terms. This task was undertaken, independently, by three groups of researchers [3, 5, 8].

These and other results formed the basis of a new branch of geometry, which Alexandrov called *chronogeometry*. Chronogeometry is an active field: There have been several conferences (the first organized in 1974 by Alexandrov himself), and lots of results.

In particular, I. Segal (see, e.g., [9]) used Alexandrov's results in the foundations of his *Chronometric Theory*. This theory follows up on Einstein's idea of geometrization of physics, and provides geometry-oriented alternative to traditional physical models in astrophysics, cosmology, elementary particles theory, and other areas of theoretical physics. At present, this theory is one of the few (if not the only) highly developed alternative to the more conventional (less geometric) approaches to theoretical physics. For a recent survey of the Chronometric Theory, see, e.g., [6].

Our best wishes. On behalf of all the editors and readers, we want to wish

A. D. Alexandrov further great results in all of his areas of interest, hopefully including interval-based approach to space-time.

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References

[1] A. D. Alexandrov, "On Lorentz transformations", *Uspekhi Math. Nauk*, 1950, Vol. 5, No. 3 (37), p. 187 (in Russian).

[2] A. D. Alexandrov and V. V. Ovchinnikova, "Remarks on the foundations of special relativity", *Leningrad University Vestnik*, 1953, No. 11, pp. 94–110 (in Russian).

[3] H. Busemann, *Timelike spaces*, PWN, Warszawa, 1967.

[4] V. Kreinovich. "Approximately measured causality implies the Lorentz group: Alexandrov-Zeeman result made more realistic", *International Journal of Theoretical Physics*, 1994, Vol. 33, No. 8, pp. 1733–1747.

[5] E. H. Kronheimer and R. Penrose, "On the structure of causal spaces", *Proc. Cambr. Phil. Soc.*, 1967, Vol. 63, No. 2, pp. 481–501.

[6] A. Levichev, "On mathematical foundations and physical applications of chronometry", in *Semigroups in Algebra, Geometry and Analysis*, Eds.: Hofmann/Lawson/Vinberg; Walter de Gruyter Co., Berlin – New York, 1995, pp. 77–103.

[7] G. L. Naber, *The geometry of Minkowski space-time*, Springer-Verlag, N.Y., 1992.

[8] R. I. Pimenov, *Kinematic spaces: Mathematical Theory of Space-Time*, N.Y., Consultants Bureau, 1970.

[9] I. E. Segal, *Mathematical Cosmology and Extragalactic Astronomy*, Academic Press, New York, 1976.

[10] N. Wiener, "A new theory of measurement: a study in the logic of mathematics", *Proceedings of the London Mathematical Society*, 1921, Vol. 19, pp. 181–205.

[11] E. C. Zeeman, "Causality implies the Lorentz group", *Journal of Mathematical Physics*, 1964, Vol. 5, pp. 490–493.