A New Universal Approximation Result For Fuzzy Systems, Which Reflects CNF–DNF Duality

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Abstract

There are two main fuzzy system methodologies for translating expert rules into a logical formula: In Mamdani's methodology, we get a DNF formula (disjunction of conjunctions), and in a methodology which uses logical implications, we get, in effect, a CNF formula (conjunction of disjunctions). For both methodologies, universal approximation results have been proven which produce, for each approximated function $f(x)$, two different approximating relations $R_{\rm DNF}(x,y)$ and $R_{\rm CNF}(x,y)$. Since in fuzzy logic, there is a known relation $F_{\rm CNF}(x) \le F_{\rm DNF}(x)$ between CNF and DNF forms of a propositional formula F, it is reasonable to expect that we would be able to prove the existence of approximations for which a similar relation $R_{\rm CNF}(x,y) \le R_{\rm DNF}(x,y)$ holds. Such existence is proved in our paper.

1 Introduction

1.1 Fuzzy control: in brief

Fuzzy control (see, e.g., [9]) is a methodology that translates the expert's if-then rules of the type

$$\text{if } A_i(x) \text{ then } B_i(y), \quad 1 \le i \le N, \tag{1}$$

or

$$\text{if } A_{i1}(x_1) \text{ and} \ldots \text{and } A_{in}(x_n) \text{ then } B_i(y), \tag{2}$$

in which the properties $A_i(x)$ and $B_j(x)$ are described by using words from natural languages (such as "x is small"), into a *control strategy*, i.e., into a function $f : X \to Y$ describing what exactly control we should apply for a given input $x \in X$. This methodology consists of three major steps:

- first, we *formalize* each "linguistic" property $A_i(x)$ or $B_i(y)$ as a *fuzzy set*, i.e., as a function $A_i : X \to [0, 1]$ which describes, for each object $x \in X$, to what extent this property holds for this x (e.g., to what extent x is small);
- then, we *combine* these fuzzy sets into a *fuzzy relation*, i.e. a function $R(x, y) : X \times Y \to [0, 1]$ which describes, for each input $x \in X$ and for each possible output $y \in Y$, to what extent this particular outputs satisfies the expert's rules;
- finally, we apply some *defuzzification procedure* to the *fuzzy relation* $R(x, y)$, and get the desired control strategy, as a function $\widetilde{f} : X \to Y$.

1.2 Mamdani's (DNF) approach

In most practical application of fuzzy control, *Mamdani's* approach is used in the combination (second) step. In this approach, the fuzzy relation $R(x, y)$ is represented by a logical formula

$$(A_1(x) \& B_1(y)) \vee \ldots \vee (A_N(x) \& B_N(y)), \tag{3}$$

or as

$$(A_{11}(x) \& \ldots \& A_{1n}(x_n) \& B_1(y)) \vee \ldots \vee (A_{N1}(x_1) \& \ldots \& A_{Nn}(x_n) \& B_N(y)) \tag{4}$$

where '$\&'$ and '$\vee'$ stand for connectives of conjunction and disjunction respectively. In logical terms, we have a disjunction of conjunctions $A_i(x) \& B_i(y)$, i.e., a formula in a Disjunctive Normal Form – DNF.

Then, we select an interpretation: a t-norm $f_\& : [0, 1] \times [0, 1] \to [0, 1]$ for conjunction and a t-conorm $f_\vee : [0, 1] \times [0, 1] \to [0, 1]$ for disjunction (see, e.g., [4, 10]), and use these operations in the corresponding formulas (3) and (4), resulting in:

$$R_{\rm DNF}(x,y)=f_\vee(f_\&(A_1(x),B_1(y)),\ldots,f_\&(A_N(x),B_N(y)),\qquad (5)$$

$$R_{\rm DNF}(x,y)=f_\vee[f_\&(A_{11}(x),\ldots,A_{1n}(x_n),B_1(y)),\ldots,$$
$$f_\&(A_{N1}(x_1),\ldots,A_{Nn}(x_n),B_N(y))].\qquad (6)$$

1.3 Logical implication (CNF) approach

From the logical viewpoint, it is somewhat more natural to represent the fuzzy relation $R(x,y)$ as a conjunction of implications:

$$(A_1(x)\to B_1(y))\&\ldots\&(A_N(x)\to B_N(y)),\qquad (7)$$

or

$$((A_{11}(x)\&\ldots\&A_{1n}(x_n))\to B_1(y))\&\ldots\&$$
$$((A_{N1}(x_1)\&\ldots\&A_{Nn}(x_n))\to B_N(y)).\qquad (8)$$

In this case, we select the interpretation: $f_\&:[0,1]\times[0,1]\to[0,1]$ for conjunction and $f_\to:[0,1]\times[0,1]\to[0,1]$, for implication and use these operations in the corresponding formulas (7) and (8), resulting in:

$$R_\to(x,y)=f_\&(f_\to(A_1(x),B_1(y)),\ldots,f_\to(A_N(x),B_N(y))),\qquad (9)$$

$$R_\to(x,y)=f_\&[f_\to(f_\&(A_{11}(x),\ldots,A_{1n}(x_n)),B_1(y)),\ldots,$$
$$f_\to(f_\&(A_{N1}(x_1),\ldots,A_{Nn}(x_n)),B_N(y))].\qquad (10)$$

In particular, since in classical logic $A\to B$ is equivalent to $\neg A\vee B$, and $(A_1(x)\&\ldots\&A_n)\to B$ to $\neg A_1\vee\ldots\neg A_n\vee B$, it makes sense to consider representations of formulas (7) and (8) in the following form

$$(\neg A_1(x)\vee B_1(y))\&\ldots\&(\neg A_N(x)\vee B_N(y)),\qquad (11)$$

or

$$(\neg A_{11}(x)\vee\ldots\vee\neg A_{1n}(x_n)\vee B_1(y))\&\ldots$$
$$\&(\neg A_{N1}(x_1)\vee\ldots\vee\neg A_{Nn}(x_n)\vee B_N(y)).\qquad (12)$$

In logical terms, we have a conjunction of disjunctions $A_i(x)\&B_i(y)$, i.e., a formula in a Conjunctive Normal Form – CNF.

In DNF, we have outside disjunction and inside conjunctions; in CNF, the roles of disjunction and conjunction are reversed: we have outside conjunction and inside disjunctions. In logic, conjunction and disjunction are often called *dual* logical operations; in view of this terminology, CNF and DNF are also often called *dual* forms.

Then, we select the interpretation: a t-norm $f_\& : [0,1] \times [0,1] \to [0,1]$ for conjunction, a t-conorm $f_\vee : [0,1] \times [0,1] \to [0,1]$ for disjunction and a *fuzzy negation* $f_\neg ; [0,1] \to [0,1]$, and use these operations in the corresponding formulas (11) and (12), resulting in:

$$R_{\rm CNF}(x,y) = f_\&(f_\vee(f_\neg(A_1(x)), B_1(y)), \ldots, f_\vee(f_\neg(A_N(x)), B_N(y))), \tag{13}$$

$$R_{\rm CNF}(x,y) = f_\&[f_\vee(f_\neg(A_{11}(x)), \ldots, f_\neg(A_{1n}(x_n)), B_1(y)), \ldots,$$
$$f_\vee(f_\neg(A_{N1}(x_1)), \ldots, f_\neg(A_{Nn}(x_n)), B_N(y))]. \tag{14}$$

1.4 Relation between DNF and CNF approaches

For each of these two methodologies, it is desirable to check that this methodology is *universal*, i.e., that if we use this methodology, then, for an arbitrary control function $f : X \to Y$, and for an arbitrary accuracy, there exist appropriate if-then rules for which the resulting control strategy represented by $\widetilde{f}(x)$ approximates the original control function $f(x)$ within a given accuracy.

There exists many universal approximation results for approximations which are derived from Mamdani-style DNF formulas; first such results were formulated and proved, almost simultaneously, in 1990–92 papers by J. Buckley, Z. Cao, E. Czogala, D. Dubois, M. Grabisch, J. Han, Y. Hayashi, C.-C. Jou, A. Kandel, B. Kosko, J. Mendel, H. Prade, and L.-X. Wang; for a recent survey, see, e.g., [6] and references therein. There also exist several universal approximation results for implication-style CNF formulas [1, 2, 3, 11, 13].

These results are usually proved separately and provide two different (seemingly unrelated) approximations. In logic, however, CNF and DNF forms are related. In classical (2-valued) logic, every propositional formula F can be represented in both DNF and CNF forms $F_{\rm DNF}$ and $F_{\rm CNF}$; for every input x, these forms lead to exactly the same truth value: $F_{\rm CNF}(x) = F(x) = F_{\rm DNF}(x)$. In fuzzy logic, each propositional formula can also be transformed (generally, non-equivalently, see [11]) into CNF and DNF forms, so that using $f_\& = \min$, $f_\vee = \max$, and $f_\neg(a) = 1 - a$, we get $F_{\rm CNF}(x) \le F_{\rm DNF}(x)$ (to be more precise, $F_{\rm CNF}(x) \le F(x) \le F_{\rm DNF}(x)$; see, e.g., [12, 14]). In view of this relation, it is desirable to have a universal approximation result for CNF and DNF formulas which is consistent with this "fuzzy duality", i.e., in which there is a similar relation between the fuzzy relations $R_{\rm DNF}(x,y)$ and $R_{\rm CNF}(x,y)$ which approximate the desired function f. Such a result is presented in this paper.

In proving this duality-related result, we also somewhat generalize the known CNF and DNF universal approximation theorems.

2 General Case: Functions Defined on an Arbitrary Compact Set

Definition 1. *Let $k = 1$ or $k = 2$, and let $\oplus$ be a propositional k-ary operation in classical (2-valued) logic (e.g., &, $\vee$, $\neg$, $\rightarrow$). We say that an operation $f_\oplus : [0,1]^k \rightarrow [0,1]$ is consistent with classical logic if it coincides with $\oplus$ when all its inputs are 0's and 1's (corresponding to "false" and "true").*

Please note that we did not require that $f_\oplus$ is continuous (as a function), or that $f_\&(a,b)$ is commutative or associative, etc.

Definition 2. *Let X be a compact metric space with a metric d_X, Y be a complete metric space with a metric d_Y, $f : X \rightarrow Y$ be a continuous function from X to Y, and $\varepsilon > 0$ be a real number. We say that a fuzzy relation $R : X \times Y \rightarrow [0,1]$ ε-approximates a function $f : X \rightarrow Y$ if the following two conditions hold:*

- *for every $x \in X$, $R(x, f(x)) > 0$, and*
- *for every $x \in X$ and $y \in Y$, if $R(x,y) > 0$, then $d_Y(y, f(x)) \le \varepsilon$.*

Theorem 1. *Let operations $f_\&$, $f_\vee$, and $f_\neg$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \rightarrow Y$ into a complete metric space Y, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which:*

- *both fuzzy relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using the interpretation determined by $f_\&$, $f_\vee$, and $f_\neg$) ε-approximate f, and*
- *$R_{\rm CNF}(x,y) \le R_{\rm DNF}(x,y)$ for all x and y.*

Comment. For the convenience of the readers, all the proofs are placed in the special Proofs section.

A similar result holds for $R_\rightarrow$ instead of $R_{\rm CNF}$:

Theorem 1′. *Let operations $f_\&$, $f_\vee$, and $f_\rightarrow$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \rightarrow Y$ into a complete metric space Y, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which:*

- *both fuzzy relations $R_\rightarrow$ and $R_{\rm DNF}$ (obtained using the interpretation determined by $f_\&$, $f_\vee$, and $f_\rightarrow$) ε-approximate f, and*
- *$R_\rightarrow(x,y) \le R_{\rm DNF}(x,y)$ for all x and y.*

From the fact that a fuzzy relation $R(x,y)$ ε-approximates a function $f(x)$, we can conclude that the result $\widetilde{f}(x) = D(\mu_x)$ of applying, for every $x \in X$, a defuzzification procedure D (see below) to the corresponding membership function $\mu_x(y) = R(x,y)$ is ε-close to $f(x)$:

Definition 3. *By a defuzzification procedure, we mean a mapping $D : [0,1]^Y \to Y$ which maps every membership function $\mu : Y \to [0,1]$ (which is not identically zero) into an element $D(\mu) \in Y$ for which $\mu(D(\mu)) > 0$.*

Proposition 1. *If a fuzzy relation $R(x,y)$ ε-approximates a function $f(x)$, then, for every defuzzification procedure D, the result $\widetilde{f}(x) = D(\mu_x)$ of applying this defuzzification procedure D to the corresponding membership function $\mu_x(y) = R(x,y)$ is ε-close to $f(x)$, i.e., $d_Y\left(\widetilde{f}(x), f(x)\right) \le \varepsilon$.*

Corollary 1. *Let operations $f_\&$, $f_\vee$, and $f_\neg$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \to Y$ into a complete metric space Y, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which, for each defuzzification procedure D, the results $\widetilde{f}_{\rm DNF}(x)$ and $\widetilde{f}_{\rm CNF}(x)$ of defuzzifying the relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\neg$) are ε-close to f.*

Corollary 1′. *Let operations $f_\&$, $f_\vee$, and $f_\to$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \to Y$ into a complete metric space Y, for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which, for each defuzzification procedure D, the results $\widetilde{f}_{\rm DNF}(x)$ and $\widetilde{f}_{\rm CNF}(x)$ of defuzzifying the relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\to$) are ε-close to f.*

3 Towards a More Realistic Situation: Case When $Y = \mathbb{R}$

In the case when Y is a real line ($Y = \mathbb{R}$), we can use a different class of possible "defuzzification procedures" and still get the same universal approximation result. Namely, we can use the following definition:

Definition 3′. ($Y = \mathbb{R}$) *By a defuzzification procedure, we mean a mapping D which maps every non-zero membership function $\mu : \mathbb{R} \to [0,1]$ into an real number $D(\mu)$ in such a way that for an arbitrary interval $[a,b]$, if a membership function $\mu(x)$ is equal to 0 for all values x outside an interval $[a,b]$, then $D(\mu) \in [a,b]$.*

Comment. Both centroid and center-of-maximum are defuzzification procedures in this sense.

Proposition 1′. ($Y = \mathbb{R}$) *If a fuzzy relation $R(x,y)$ ε-approximates a function $f(x)$, then, for every defuzzification procedure D, the result $\widetilde{f}(x) = D(\mu_x)$ of applying this defuzzification procedure D to the corresponding membership function $\mu_x(y) = R(x,y)$ is ε-close to $f(x)$, i.e., $d_Y\left(\widetilde{f}(x), f(x)\right) \le \varepsilon$.*

4 Realistic Case: Functions From $\mathbb{R}^n$ to $\mathbb{R}$

For the case when $X = \mathbb{R}^n$, we can prove similar results with rules of type (2):

Theorem 2. *Let operations $f_\&$, $f_\vee$, and $f_\neg$ be consistent with classical logic. Then, for every integer $n > 0$, for every compact set $X \subset \mathbb{R}^n$, for every continuous function $f : X \to \mathbb{R}$, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (2) for which:*

- *both fuzzy relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\neg$) ε-approximate f, and*
- *$R_{\rm CNF}(x,y) \le R_{\rm DNF}(x,y)$ for all $x = (x_1, \ldots, x_n) \in X$ and y.*

Theorem 2′. *Let operations $f_\&$, $f_\vee$, and $f_\to$ be consistent with classical logic. Then, for every integer $n > 0$, for every compact set $X \subset \mathbb{R}^n$, for every continuous function $f : X \to \mathbb{R}$, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (2) for which:*

- *both fuzzy relations $R_\to$ and $R_{\rm DNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\to$) ε-approximate f, and*
- *$R_\to(x,y) \le R_{\rm DNF}(x,y)$ for all $x(x_1, \ldots, x_n) \in X$ and y.*

In our universal approximation result, we prove that for every function $f : X \to Y$, we can select the rules and the membership functions for which we get the desired approximation property.

For Mamdani's (DNF) case, a stronger statement is true: that whatever "realistic" membership function $\mu_0(x)$ we choose, we can always find rules in which all the membership functions $A_{ik}(x_k)$ and $B_i(y)$ are *of the type* μ_0, i.e., they all have the form $\mu(x) = \mu_0(a \cdot x + b)$ for some real numbers $a \neq 0$ and b (see, e.g., [7, 8]).

From the proof of Theorems 2 and 2′, we see that all the membership functions used in the approximation have the same type, i.e., that there is a type μ_0 which provides a universal approximation property both for DNF and for CNF forms. We do not know whether a similar result is true for an arbitrary given type.

5 Proofs

5.1 Proof of Theorem 1

This proof is similar to the original Kosko's proof [5] of a universal approximation result for DNF (i.e., for Mamdani methodology), and to our own proofs from [7, 8, 11].

1°. Let us take $\varepsilon_1 = \varepsilon/2$. Since a function f is continuous on a compact set X, it is also uniformly continuous. Therefore, there exists $\delta > 0$ such that if $d_X(x, x') \le \delta$, then $d_Y(f(x), f(x')) \le \varepsilon_1$.

Since X is a compact metric space, there exists a finite δ-net for X, i.e., a finite set of elements $x^{(1)}, \ldots, x^{(N)} \in X$ for which, for every $x \in X$, there exists an i for which $d_X\left(x, x^{(i)}\right) \le \delta$. For each of these elements $x^{(i)}$, we can find $y^{(i)} = f\left(x^{(i)}\right)$. We will show that Theorem 1 holds for N rules of type (1) where for every i,

- $A_i(x) = 1$ if $d_X\left(x, x^{(i)}\right) \le \delta$, and $A_i(x) = 0$ otherwise;
- $B_i(y) = 1$ if $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$, and $B_i(y) = 0$ otherwise.

All these fuzzy sets A_i and B_i are crisp: indeed, $A_i(x)$ is a characteristic function of the inequality $d_X\left(x, x^{(i)}\right) \le \delta$, and $B_i(y)$ is a characteristic function of the inequality $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$.

Since all the $f_\&$, $f_\vee$, and $f_\neg$ are consistent with classical logic and A_i, B_i are crisp, we conclude that the relations $R_{\rm CNF}$ and $R_{\rm DNF}$ can be obtained using classical logical connectives. Namely,

$$R_{\rm DNF}(x, y) = 1 \iff \exists i \left(d_X\left(x, x^{(i)}\right) \le \delta \,\&\, d_Y\left(y, y^{(i)}\right) \le \varepsilon_1\right); \tag{15}$$

$$R_{\rm CNF}(x, y) = 1 \iff \forall i \left(d_X\left(x, x^{(i)}\right) > \delta \vee d_Y\left(y, y^{(i)}\right) \le \varepsilon_1\right). \tag{16}$$

2°. Let us first show that the relation $R_{\rm DNF}$ ε-approximates the given function f.

2.1°. In accordance with the definition of ε-approximation, we first prove that for every $x \in X$, we have $R_{\rm DNF}(x, f(x)) > 0$.

Indeed, let x be an arbitrary element of the set X. Since $x^{(1)}, \ldots, x^{(N)}$ is a δ-net, there exists an i for which $d_X\left(x, x^{(i)}\right) \le \delta$. Due to our choice of δ, we conclude that $d_Y\left(f(x), y^{(i)}\right) \le \varepsilon_1$. Thus, (15) is true, hence, $R_{\rm DNF}(x, f(x)) > 0$.

2.2°. Let us now prove that for every $x \in X$ and $y \in Y$, if $R_{\rm DNF}(x, y) > 0$, then $d_Y(y, f(x)) \le \varepsilon$.

Indeed, since $R_{\rm DNF}$ is a crisp relation, the only possibility for $R_{\rm DNF}(x, y) > 0$ is to have $R_{\rm DNF}(x, y) = 1$, i.e., $R_{\rm DNF}(x, y)$ to be true. This means that there exists an i for which $d_X\left(x, x^{(i)}\right) \le \delta$ and $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$. Due to our choice of δ, from $d_X\left(x, x^{(i)}\right) \le \delta$, we can conclude that $d_Y\left(f(x), f\left(x^{(i)}\right)\right) = d_Y\left(f(x), y^{(i)}\right) \le \varepsilon_1$. Thus, from the triangle inequality, we conclude that $d_Y(y, f(x)) \le d_Y\left(y, y^{(i)}\right) + d_Y\left(y^{(i)}, f(x)\right) \le \varepsilon_1 + \varepsilon_1 = \varepsilon$. The statement is proven.

3°. Let us now show that the relation R_{CNF} also ε-approximates the given function f.

3.1°. In accordance with the definition of ε-approximation, we first prove that for every $x \in X$, we have $R_{\mathrm{CNF}}(x, f(x)) > 0$.

Indeed, let x be an arbitrary element of the set X. For every i, we have $d_X\left(x, x^{(i)}\right) \leq \delta$ or $d_X\left(x, x^{(i)}\right) > \delta$. By definition of δ, the inequality $d_X\left(x, x^{(i)}\right) \leq \delta$ implies that $d_Y\left(f(x), f\left(x^{(i)}\right)\right) = d_Y\left(f(x), y^{(i)}\right) \leq \varepsilon_1$. Therefore, we have $d_Y\left(f(x), y^{(i)}\right) \leq \varepsilon_1$ or or $d_X\left(x, x^{(i)}\right) > \delta$. Hence, (16) is true, and $R_{\mathrm{CNF}}(x, f(x)) > 0$.

3.2°. Let us show that for every $x \in X$ and $y \in Y$, if $R_{\mathrm{CNF}}(x, y) > 0$, then $d_Y(y, f(x)) \leq \varepsilon$.

Indeed, since R_{CNF} is a crisp relation, the only possibility for $R_{\mathrm{CNF}}(x, y) > 0$ is to have $R_{\mathrm{CNF}}(x, y) = 1$, i.e., $R_{\mathrm{CNF}}(x, y)$ to be true. This means that for every i, either $d_X\left(x, x^{(i)}\right) > \delta$ or $d_Y\left(y, y^{(i)}\right) \leq \varepsilon_1$. Equivalently, this disjunction means that the inequality $d_X\left(x, x^{(i)}\right) \leq \delta$ implies (in the logical sense, where implication is material!) that $d_Y\left(y, y^{(i)}\right) \leq \varepsilon_1$.

Since $x^{(1)}, \ldots, x^{(N)}$ is a δ-net, there exists an i for $d_X\left(x, x^{(i)}\right) \leq \delta$. For this i, we thus $d_Y\left(y, y^{(i)}\right) \leq \varepsilon_1$. On the other hand, due to our choice of δ, from $d_X\left(x, x^{(i)}\right) \leq \delta$, we can conclude that $d_Y\left(f(x), f\left(x^{(i)}\right)\right) = d_Y\left(f(x), y^{(i)}\right) \leq \varepsilon_1$. Thus, from the triangle inequality, we conclude that $d_Y(y, f(x)) \leq d_Y\left(y, y^{(i)}\right) + d_Y\left(y^{(i)}, f(x)\right) \leq \varepsilon_1 + \varepsilon_1 = \varepsilon$. The statement is proven.

4°. To complete the proof of the theorem, we must now show that $R_{\mathrm{CNF}}(x, y) \leq R_{\mathrm{DNF}}(x, y)$ for all x and y.

Since both relations $R_{\mathrm{CNF}}(x, y)$ and $R_{\mathrm{DNF}}(x, y)$ are crisp, the desired inequality is equivalent to saying that for every x and y, if $R_{\mathrm{CNF}}(x, y)$ is true, then $R_{\mathrm{DNF}}(x, y)$ should also be true. Indeed, let $R_{\mathrm{CNF}}(x, y)$ hold for some x and y. This means that for every i, the inequality $d_X\left(x, x^{(i)}\right) \leq \delta$ implies (in the logical sense) that $d_Y\left(y, y^{(i)}\right) \leq \varepsilon_1$. Since $x^{(1)}, \ldots, x^{(N)}$ is a δ-net, there exists an i for which $d_X\left(x, x^{(i)}\right) \leq \delta$. Thus, for this i, we have $d_Y\left(y, y^{(i)}\right) \leq \varepsilon_1$. So, for this i, both inequalities $d_X\left(x, x^{(i)}\right) \leq \delta$ and $d_Y\left(y, y^{(i)}\right) \leq \varepsilon_1$ are true, hence the formula (16) is true. The statement is proven, and so is the theorem.

Comment. We have already mentioned that in crisp (2-valued) propositional logic, CNF and DNF forms represent the same function. It deserves mentioning that although in our proof, the relations $R_{\mathrm{CNF}}(x, y)$ and $R_{\mathrm{DNF}}(x, y)$ are both crisp, they do not necessarily the same. Indeed, let us consider the simplest case when $X = Y = [0, 1]$ with a normal metric $d_X(x, x') = d_Y(x, x') = |x - x'|$, and $f(x) = x$. Then, $\delta = \varepsilon_1 = \varepsilon/2$. As a δ-net, we can select the points $x^{(i)} = (2i - 1) \cdot \delta$, i.e., $x^{(1)} = \delta$, $x^{(2)} = 3\delta$, etc.; then, $y^{(i)} = x^{(i)}$. Here, for $x = 2\delta$ and $y = 0$, we have $d_X\left(x, x^{(1)}\right) \leq \delta$ and $d_Y\left(y, y^{(1)}\right) \leq \varepsilon_1$, so $R_{\mathrm{DNF}}(x, y)$

is true. However, the property $R_{\mathrm{CNF}}(x, y)$ is not true, because for $i = 2$, we have $d_X\left(x, x^{(2)}\right) \le \delta$, but $d_Y\left(y, y^{(2)}\right) = 3\delta = 3\varepsilon_1 > \varepsilon_1$.

5.2 Proof of Theorem 1′.

For this proof, we can use the exact same crisp sets A_i and B_i as in the proof of Theorem 1.

5.3 Proof of Proposition 1

By definition, $\widetilde{f}(x) = D(\mu_x)$, where the membership function $\mu_x : Y \to [0, 1]$ is defined as $\mu_x(y) = R(x, y)$. By the definition of a defuzzification procedure, for every $x \in X$, we have $\mu_x(D(\mu_x)) > 0$, i.e., by definition of μ_x, $R\left(x, \widetilde{f}(x)\right) > 0$. From the definition of ε-approximation, we can now conclude that $d_Y\left(\widetilde{f}(x), f(x)\right) \le \varepsilon$. The proposition is proven.

5.4 Proof of Proposition 1′

By definition, $\widetilde{f}(x) = D(\mu_x)$, where the membership function $\mu_x : Y \to [0, 1]$ is defined as $\mu_x(y) = R(x, y)$. From the definition of ε-approximation, we conclude that if $R(x, y) > 0$, then $d_Y(y, f(x)) = |y - f(x)| \le \varepsilon$. Thus, if $|y - f(x)| > \varepsilon$, we have $R(x, y) = \mu_x(y) = 0$. Hence, the function $\mu_x(y)$ is equal to 0 outside the interval $[f(x) - \varepsilon, f(x) + \varepsilon]$. By definition of a defuzzification procedure, we can now conclude that the result $\widetilde{f}(x)$ of its defuzzification also belongs to the same interval, i.e., that $\left|\widetilde{f}(x) - f(x)\right| \le \varepsilon$. The proposition is proven.

5.5 Proof of Theorems 2 and 2′

This proof is similar to the proofs of Theorems 1 and 1′. Indeed, for $X \subset \mathbb{R}^n$, continuity of a function $f : X \to \mathbb{R}$ with respect to a normal (Euclidean) metric is equivalent to its continuity with respect to the uniform metric $d_X(x, x') = \max_i |x_i - x'_i|$. Thus, from the proofs of Theorems 1 and 1′, we conclude that there exist appropriate rules of type (1), with $A_i(x) = 1 \iff d_X\left(x, x^{(i)}\right) \le \delta$ and $A_i(x) = 0$ for all other x, and $B_i(y) = 1 \iff \left|y - y^{(i)}\right| \le \varepsilon_1$ and $B_i(y) = 0$ for all other y. By the definition of the uniform metric d_X, the inequality $d_X\left(x, x^{(i)}\right) \le \delta$ is equivalent to $A_{i1}(x_1) \& \ldots \& A_{in}(x_n) = 1$, where $A_{ik}(x_k) = 1 \iff \left|x_k - x_k^{(i)}\right| \le \delta$ and $A_{ik}(x_k) = 0$ for all other x_k. Thus, rules of type (1) can be reformulated in the desired form (2). The theorems are thus proven.

Acknowledgments

This work was supported in part by NASA under cooperative agreement NCC5-209, NSF grants No. DUE-9750858 and CDA-9522207, by the United Space Alliance, grant No. NAS 9-20000 (PWO C0C67713A6), by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, and by the National Security Agency under Grants No. MDA904-98-1-0561.

The authors are greatly thankful to Vilém Novák and Ronald R. Yager for formulating the problem, for the encouragement, and for valuable discussions.

References

[1] D. Dubois, M. Grabisch, and H. Prade, "Gradual rules and the approximation of functions", *Proceedings of the 2nd International Conference on Fuzzy Logic and Neural Networks*, Iizuka, Japan, July 17–22, 1992, pp. 629–632.

[2] D. Dubois, M. Grabisch, and H. Prade, "Synthesis of real-valued mappings based on gradual rules and interpolative reasoning", *Proc. of the IJCAI'93 Fuzzy Logic in AI Workshop*, Chaméry, France, Aug. 28–Sep. 3, 1993, pp. 29–40.

[3] D. Dubois, H. Prade, and M. Grabisch, "Gradual rules and the approximation of control laws", In: H. T. Nguyen, M. Sugeno, R. Tong, and R. Yager (eds.), *Theoretical aspects of fuzzy control*, J. Wiley, N.Y., 1995, pp. 147–181.

[4] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[5] B. Kosko, "Fuzzy systems as universal approximators", *Proceedings of the IEEE International Conference on Fuzzy Systems 1992*, San Diego, CA, 1992, pp. 1153–1162.

[6] V. Kreinovich, G. C. Mouzouris, and H. T. Nguyen, "Fuzzy rule based modeling as a universal approximation tool", In: H. T. Nguyen and M. Sugeno (eds.), *Fuzzy Systems: Modeling and Control*, Kluwer, Boston, MA, 1998, pp. 135–195.

[7] H. T. Nguyen and V. Kreinovich, "On approximations of controls by fuzzy systems", *Fifth International Fuzzy Systems Association World Congress*, Seoul, Korea, July 1993, p. 1414–1417.

[8] H. T. Nguyen, V. Kreinovich, and O. Sirisaengtaksin, "Fuzzy control as a universal control tool", *Fuzzy Sets and Systems*, 1996, Vol. 80, No. 1, pp. 71–86.

[9] H. T. Nguyen and M. Sugeno (eds.), *Fuzzy Systems: Modeling and Control*, Kluwer, Boston, MA, 1998.

[10] H. T. Nguyen and E. A. Walker, *First Course in Fuzzy Logic*, CRC Press, Boca Raton, FL, 1999.

[11] V. Novák and I. Perfilieva, "Evaluating Linguistic Expressions and Functional Fuzzy Theories in Fuzzy Logic", In: L. A. Zadeh and J. Kacprzyk (eds.), *Computing with Words in Information/Intelligent Systems*, Springer-Verlag, Berlin, 1999, pp. 383–406.

[12] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical principles of fuzzy logic*, Kluwer Academic Publ., Boston/Dordrecht, 1999.

[13] I. Perfilieva, "Normal Forms for Fuzzy Logic Relations and the Best Approximation Property", *Proc. Conf. EUSFLAT'99*, Palma de Mallorca 1999.

[14] Q. Zuo, I. B. Türkşen, H. T. Nguyen, and V. Kreinovich, "In expert systems, even if we fix AND/OR operations, a natural answer to a composite query is the interval of possible degrees of belief", *Reliable Computing*, 1995, Supplement (Extended Abstracts of APIC'95: International Workshop on Applications of Interval Computations, El Paso, TX, Febr. 23–25, 1995), pp. 236–240.