Optimal Finite Characterization of Linear Problems with Inexact Data

Vladik Kreinovich

Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
email vladik@cs.utep.edu

Abstract

For many linear problems, in order to check whether a certain property is true for all matrices A from an interval matrix $\mathbf{A}$, it is sufficient to check this property for finitely many "vertex" matrices $A \in \mathbf{A}$. J. Rohn has discovered that we do not need to use all 2^{n^2} vertex matrices, it is sufficient to only check these properties for $2^{2n-1} \ll 2^{n^2}$ vertex matrices of a special type A_{yz}. In this paper, we show that a further reduction is impossible: without checking all 2^{2n-1} matrices A_{yz}, we cannot guarantee that the desired property holds for all $A \in \mathbf{A}$. Thus, these special vertex matrices provide an *optimal* finite characterization of linear problems with inexact data.

1 Introduction

Many practical problems are described by systems of linear equations and/or inequalities, i.e., as *linear problems.* The components A_{ij} of the corresponding matrices A are often not exactly known; for each of these components, we only know the interval $[\underline{A}_{ij}, \overline{A}_{ij}]$ of possible values. The class of all matrices A which are consistent with this information is called an *interval matrix*

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A : \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}$ is a matrix with components $\underline{A}_{ij}$, $\overline{A}$ is a matrix with components $\overline{A}_{ij}$, and $A \le B$ means that $A_{ij} \le B_{ij}$ for all i and j.

It is often convenient to represent an interval matrix by its midpoint $\widetilde{A} \stackrel{\rm def}{=} \frac{1}{2}(\underline{A}+\overline{A})$ and its radius $\Delta \stackrel{\rm def}{=} \frac{1}{2}(\overline{A}-\underline{A})$. In these notations, $\mathbf{A} = [\widetilde{A}-\Delta, \widetilde{A}+\Delta]$.

In practical applications, all the elements of the matrices are computer-represented numbers and thus, rational numbers; it is worth mentioning that our results hold for arbitrary real numbers as well.

We say that an interval matrix $\mathbf{A}$ *satisfies* a property $\mathcal{P}$ (e.g., is non-singular or positive definite) if all matrices $A \in \mathbf{A}$ satisfy this property. It is known that for many such properties, an interval matrix satisfies the property $\mathcal{P}$ if and only if all its *vertex matrices*, i.e., matrices for which $A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ for all i and j, satisfy this property. Thus, in order to check whether a given interval matrix satisfies the property $\mathcal{P}$, it is sufficient to check this property for a finite set of vertex matrices.

This set is finite but huge: e.g., for square matrices (of size $n \times n$), we have 2^{n^2} possible vertex matrices; as a result, for large n, checking all such matrices requires an unrealistic amount of computation time.

In [3, 6], it was shown that for many properties $\mathcal{P}$, we do not need to check all these matrices: it is sufficient to use vertex matrices from the following special class. Namely, let us define $e \stackrel{\text{def}}{=} (1, \ldots, 1)^T$,

$$Y \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n \,:\, |y| = e\} = \text{ the set of all } \pm 1 - \text{vectors.}$$

For every $y, z \in Y$, we can can define a matrix A_{yz} as $\widetilde{A} - y\Delta z^T$. In other words, for every i and j,

$$(A_{yz})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if } y_i \cdot z_j = -1 \\ \underline{A}_{ij} & \text{if } y_i \cdot z_j = 1 \end{cases} .$$

(these matrices were first introduced in [3], p. 43). Each such matrix is a vertex matrix, but there are only 2^{2n-1} matrices A_{yz} compared to 2^{n^2} vertex matrices ($2n-1$ since $A_{yz} = A_{-y,-z}$). For some problems, it is sufficient to check only some of such matrices, e.g., only matrices A_{yy} or only matrices $A_{y,-y}$ (in both cases, we need only 2^{n-1} vertex matrices).

For such problems, a natural question is: can we further decrease the set of checked matrices? In this paper, we show that for most problems described in [3, 6], further decrease is impossible: all 2^{2n-1} (corr., 2^{n-1}) vertex matrices A_{yz} (corr., A_{yy}) are needed. To be more precise: there exist cases when the property $\mathcal{P}$ holds for all but one of these matrices and still does not hold for the corresponding interval matrix $\mathbf{A}$. In this sense, finite characterizations presented in [3, 6] are optimal.

These results are in good accordance with the fact that many of the corresponding problems are NP-hard (see, e.g., [2]) and therefore, less than exponential finite characterizations are not to be expected.

Comment. The fact that a exponential $\approx 2^n$ finite characterization cannot be decreased is not as pessimistic as it may seem:

- First, NP-hardness means that we cannot expect less than exponential-time algorithms for solving the corresponding problems. Of course, this does not necessarily mean that the algorithms based on checking all 2^{n-1} vertex matrices are necessarily optimal; we may have faster – although still exponential-time – algorithms based on different ideas.

- Second, the fact that we need to check all 2^{n-1} matrices does not necessarily mean that the computation time of the corresponding algorithm for checking the property $\mathcal{P}$ for an interval matrix is 2^{n-1} times larger than the computation time t of checking this property for a single matrix. For some properties, it was shown that many of these 2^{n-1} checkings contain the exact same computational steps; so, when we need to check all these matrices, we can perform the common steps only once. As a result, the total computational time for all the checkings is much smaller than $2^{n-1} \times t$ [7].

2 Interval Matrix Properties

Definition 2.1.

- *A square matrix A (not necessarily symmetric) is called* positive (semi)definite *if its symmetric part* $\frac{1}{2}(A + A^T)$ *is positive (semi)definite.*
- *A square matrix is called* stable *if* $\mathrm{Re}\ \lambda < 0$ *for each eigenvalue* λ *of* A.

Definition 2.2.

- *A square interval matrix* $\mathbf{A}$ *is called* regular *if each* $A \in \mathbf{A}$ *is regular.*
- *A square interval matrix* $\mathbf{A}$ *is called* positive (semi)definite *if each* $A \in \mathbf{A}$ *is positive (semi)definite.*
- *A square symmetric interval matrix* $\mathbf{A}$ *(i.e., both* $\underline{A}$, $\overline{A}$ *symmetric) is called* stable *if each* $A \in \mathbf{A}$ *is stable.*

For each of these four properties – regularity, positive definiteness, positive semidefiniteness, and stability – the problem of checking whether a given interval matrix has this property is known to be NP-hard (see, e.g., [2]).

Theorem 2.1. [1, 3, 5]

- $\mathbf{A}$ *is regular if and only if for all the matrices* A_{yz}, *the determinant* $\det A_{yz}$ *has the same sign.*
- $\mathbf{A}$ *is positive (semi)definite if and only if* A_{yy} *is positive (semi)definite for each* $y \in Y$.
- $\mathbf{A}$ *is stable if and only if* $A_{y,-y}$ *is stable for each* $y \in Y$.

The following result shows that for these properties, all the above matrices $A_{y,z}$ are needed for this characterization:

- all 2^{2n-1} different matrices A_{yz} are needed for checking regularity;
- all 2^{n-1} different matrices A_{yy} are needed for checking positive definiteness, and
- all 2^{n-1} different matrices $A_{y,-y}$ are needed for checking stability.

Theorem 2.2.

- *For every n, and for every pair $\langle \widetilde{y}, \widetilde{z}\rangle$, $\widetilde{y}, \widetilde{z} \in Y$, there exists an interval matrix $\mathbf{A}$, for which*
  - *for all pairs $\langle y, z\rangle \neq \langle \widetilde{y}, \widetilde{z}\rangle, \langle -\widetilde{y}, -\widetilde{z}\rangle$, all the values* $\det A_{yz}$ *have the same sign;*
  - $\mathbf{A}$ *is not regular.*
- *For every n, and for every $\widetilde{y} \in Y$, there exists an interval matrix $\mathbf{A}$, for which*
  - *the matrix A_{yy} is positive (semi)definite for all $y \neq \widetilde{y}, -\widetilde{y}$, and*
  - $\mathbf{A}$ *is not positive (semi)definite.*
- *For every n, and for every $\widetilde{y} \in Y$, there exists an interval matrix $\mathbf{A}$, for which*
  - *the matrix $A_{y,-y}$ is stable for all $y \neq \widetilde{y}, -\widetilde{y}$, and*
  - $\mathbf{A}$ *is not stable.*

Proof. We will show that for all four properties, we can select the desired interval matrix in the form $\mathbf{A}(a, b, \delta)$, where δ is a real number, a and b are vectors, $\widetilde{A}(a, b, \delta) = (2n + \delta)D_a D_b - ab^T$, $\Delta(a, b) = ee^T$, and $D_a \stackrel{\text{def}}{=} \text{diag}(a_1, \ldots, a_n)$ is the diagonal matrix built from the vector a. Specifically:

- for regularity, we can take $\mathbf{A}(\widetilde{y}, \widetilde{z}, 0)$;
- for positive definiteness, we can take $\mathbf{A}(\widetilde{y}, \widetilde{y}, 0)$;
- for positive semi-definiteness, we can take $\mathbf{A}(\widetilde{y}, \widetilde{y}, -\varepsilon)$ for a small $\varepsilon > 0$;
- for stability, we can take $\mathbf{A}(\widetilde{y}, -\widetilde{y}, 0)$.

For the case of regularity, due to our choice of the interval matrix $\mathbf{A}(\tilde{y}, \tilde{z}, 0)$, we have

$$A_{yz} = 2nD_{\tilde{y}}D_{\tilde{z}} - \tilde{y}\tilde{z}^T - yz^T.$$

Let us show that the problem corresponding to arbitrary $\tilde{y}$ and $\tilde{z}$ can be reduced to the case when $\tilde{y} = \tilde{z} = e$.

Indeed, if we denote $p_i \stackrel{\text{def}}{=} y_i \cdot \widetilde{y}_i$ and $q_i \stackrel{\text{def}}{=} z_i \cdot \widetilde{z}_i$, then $y_i = \widetilde{y}_i \cdot p_i$ and $z_i = q_i \cdot \tilde{z}_i$, hence $A_{yz} = D_{\tilde{y}}A^*_{pq}D_{\tilde{z}}$, where $A^*_{pq} \stackrel{\text{def}}{=} 2nI - ee^T - pq^T$. Hence, $\det A_{yz} = \det D_{\tilde{y}} \cdot \det A^*_{pq} \cdot \det D_{\tilde{z}}$. Since $D_{\tilde{y}}$ and $D_{\tilde{z}}$ are diagonal matrices with ± 1 on diagonals, they are non-degenerate. Hence, to prove that all the matrices A_{yz}, $\langle y, z\rangle \neq \langle \widetilde{y}, \widetilde{z}\rangle, \langle -\widetilde{y}, -\widetilde{z}\rangle$, have the determinants of the same sign, it is sufficient to prove that all the matrices A^*_{pq}, $\langle p, q\rangle \neq \langle e, e\rangle, \langle -e, -e\rangle$, have the determinants of the same sign; specifically, we will prove that all these matrices A^*_{pq} are positive definite and therefore, they all have positive determinants. We will also prove that A^*_{ee} (hence, $A_{\tilde{y}\tilde{z}}$) is a singular matrix.

Indeed, for every vector x, we have

$$x^TA^*_{pq}x = 2nx^Tx - (e^Tx)^2 - (x^Tp)(qx^T) = 2n\cdot\|x\|^2 - (e,x)^2 - (p,x)\cdot(q,x)\,, \quad (2.1)$$

where (a, b) denotes a scalar (dot) product of the two vectors. For the scalar product, we have the well-known Cauchy-Schwartz inequality $|(e, x)| \le \|e\|\cdot\|x\|$, in which the equality is possible only if vectors e and x are collinear: $e\|x$. Here, $e = (1, \ldots, 1)^T$, so $\|e\| = \sqrt{n}$, $|(e, x)| \le \sqrt{n}\cdot\|x\|$, and

$$(e, x)^2 \le n \cdot \|x\|^2, \quad (2.2)$$

and the equality is possible only if $x\|e$.

Similarly, $|(p, x)| \le \sqrt{n}\cdot\|x\|$ and $|(q, x)| \le \sqrt{n}\cdot\|x\|$. Hence,

$$(p, x) \cdot (q, x) \le n \cdot \|x\|^2, \quad (2.3)$$

and the equality is possible only if $p\|x$, $q\|x$, and $p = \alpha \cdot q$ for $\alpha > 0$ (else we would have $(p, x)\cdot(q, x) = -n\cdot\|x\|^2$). Substituting (2.2) and (2.3) into (2.1), we conclude that

$$x^TA^*_{pq}x \ge 2n\|x\|^2 - n\|x\|^2 - n\|x\|^2 = 0, \quad (2.4)$$

and the equality is possible only when $x\|e$, $x\|p$, and $x\|q$ (hence $p\|e$ and $q\|e$), and $p = \alpha\cdot q$ for $\alpha > 0$. Since $p, q \in Y$, the only possibility for equality is, hence, when either $p = q = e$, or $p = q = -e$. So, for all other pairs, the equality is impossible, and the matrix A^*_{pq} is indeed positive definite.

For $p = q = e$, we have $A^*_{pq}e = (2nI - 2ee^T)e = 0$, so A^*_{ee} is a singular matrix; thus, $A_{\tilde{y}\tilde{y}}$ is also singular. The regularity part is proven.

For positive definiteness, we take $\mathbf{A}(\tilde{y}, \tilde{y}, 0)$, so

$$A_{yy} = 2nI - \tilde{y}\tilde{y}^T - yy^T = D_{\tilde{y}}A^*_{pp}D_{\tilde{p}}. \quad (2.5)$$

Hence, $x^T A_{yy} x = z^T A^*_{pp} z$, where $z \stackrel{\text{def}}{=} D_{\tilde{y}} x$; thence, A_{yy} is positive definite if and only if A^*_{pp} is positive definite. We can therefore conclude that for all $y \neq \widetilde{y}, -\widetilde{y}$, the matrix A_{yy} is positive definite, while for $y = \widetilde{y}$, it is only positive semi-definite and not positive definite. Thus, for positive definiteness, the theorem is also proven.

To prove a similar result for positive semi-definiteness, we consider an interval matrix $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{A}(\tilde{y}, \tilde{y}, -\varepsilon) = \mathbf{A}(\tilde{y}, \tilde{y}, 0) - \varepsilon I$ for some small $\varepsilon > 0$. For this interval matrix, for every $y \in Y$, we have $B_{yy} = A_{yy} - \varepsilon I$. Since all the matrices A_{yy} for $y \neq \widetilde{y}, -\widetilde{y}$ were positive definite, for sufficiently small ε, the new matrices B_{yy} are still positive definite. On the other hand, since the matrix $A_{\tilde{y}\tilde{y}}$ was positive semi-definite, with one of the eigenvalues 0, the new matrix $B_{\tilde{y}\tilde{y}} = A_{\tilde{y}\tilde{y}} - \varepsilon I$ has a negative eigenvalue $-\varepsilon$ and hence, is not positive semi-definite. So, for positive semi-definiteness, the theorem is also proven.

For stability, we take $\mathbf{A}(\tilde{y}, -\tilde{y}, 0)$. For this interval matrix, $A_{y,-y} = -2nI + \tilde{y}\tilde{y}^T + yy^T$. This matrix is equal to (2.5) times -1, so for all $y \neq \widetilde{y}, -\widetilde{y}$, this matrix $A_{y,-y}$ is negative definite (hence stable), while for $y = \widetilde{y}$, the corresponding matrix has a 0 eigenvalue (and is, hence, not stable). Q.E.D.

3 Linear Interval Equations and Inverse Interval Matrices

Definition 3.1.

- *For an interval matrix* $\mathbf{A}$ *and an interval vector* $\mathbf{b}$*, we define* $[\underline{x}, \overline{x}]$ *as the interval hull of the solution set*

$$X = \{x : Ax = b \text{ for some } A \in \mathbf{A}, b \in \mathbf{b}\}.$$

- *For a regular* $\mathbf{A}$*, we define* $[\underline{B}, \overline{B}]$ *as the interval hull of the set*

$$\{A^{-1} : A \in \mathbf{A}\}.$$

The problems of computing each of these interval hulls are known to be NP-hard (see, e.g., [2]).

Both interval hulls can be characterized in terms of the matrices A_{yz} and – for solution set – vectors b_y, which are defined, for every $y \in Y$, as follows:

$$(b_y)_i = \begin{cases} \overline{b}_i & \text{if } y_i = 1 \\ \underline{b}_i & \text{if } y_i = -1 \end{cases}.$$

Theorem 3.1. [3, 4]

- *If* $\mathbf{A}$ *is regular, then we have:*

$$\underline{x} = \min_{y,z \in Y} A_{yz}^{-1} b_y; \quad \overline{x} = \max_{y,z \in Y} A_{yz}^{-1} b_y.$$

- *For a regular* $\mathbf{A}$*, we have*

$$\underline{B} = \min_{y,z \in Y} A_{yz}^{-1}; \quad \overline{B} = \max_{y,z \in Y} A_{yz}^{-1}.$$

The following result shows that all the combinations $\langle y, z \rangle$ are needed for this characterization:

- all 2^{2n} different pairs $\langle y, z \rangle$ are needed for describing the solution set hull;
- all 2^{2n-1} different matrices A_{yz} are needed for describing the inverse matrix hull.

Theorem 3.2.

- *For every* n*, and for every pair* $\langle \widetilde{y}, \widetilde{z} \rangle$*,* $\widetilde{y}, \widetilde{z} \in Y$*, there exists a regular interval matrix* $\mathbf{A}$ *and an interval vector* $\mathbf{b}$*, for which either*

$$\underline{x} \neq \min_{\langle y,z \rangle \neq \langle \tilde{y}, \tilde{z} \rangle} A_{yz}^{-1} b_y$$

or

$$\overline{x} \neq \max_{\langle y,z \rangle \neq \langle \tilde{y}, \tilde{z} \rangle} A_{yz}^{-1} b_y.$$

- *For every* n*, and for every pair* $\langle \widetilde{y}, \widetilde{z} \rangle$*,* $\widetilde{y}, \widetilde{z} \in Y$*, there exist:*
  - *a regular interval matrix* $\mathbf{A}$ *for which*

$$\underline{B} \neq \min_{\langle y,z \rangle \neq \langle \tilde{y}, \tilde{z} \rangle, \langle -\tilde{y}, -\tilde{z} \rangle} A_{yz}^{-1};$$

  - *a regular interval matrix* $\mathbf{A}$ *for which*

$$\overline{B} \neq \max_{\langle y,z \rangle \neq \langle \tilde{y}, \tilde{z} \rangle, \langle -\tilde{y}, -\tilde{z} \rangle} A_{yz}^{-1}.$$

Proof. Let us start with the case of an interval linear system. We will show that in this case, for an arbitrary positive real number $\varepsilon > 0$, the desired property is true for the interval matrix $\mathbf{A}(\tilde{y}, \tilde{z}, \varepsilon)$ (as described in the proof of Theorem 2.2) and for the interval vector $\mathbf{b} = e[-1, 1]$, i.e., the vector for which $\mathbf{b}_i = [-1, 1]$ for all i. For this vector, $b_y = y$ for every $y \in Y$.

To prove that this pair of an interval matrix and an interval vector always satisfy the desired property, let us first reduce the case of the general vectors $\tilde{y}, \tilde{z} \in Y$ to the case when $\tilde{y} = \tilde{z} = e$. Indeed, in this case, similarly to the reduction with which we started the proof of Theorem 2.2, we have

$$A_{yz} = (2n + \varepsilon)D_{\tilde{y}}D_{\tilde{z}} - \tilde{y}\tilde{z}^T - yz^T = D_{\tilde{y}}A^*_{pq}D_{\tilde{z}},$$

where $A^*_{pq} \stackrel{\text{def}}{=} (2n + \varepsilon)I - ee^T - pq^T$ corresponds to the case when $\tilde{y} = \tilde{z} = e$. Since each diagonal matrix $D_{\tilde{y}}$ has ± 1 on the diagonal, we have $D_{\tilde{y}}^{-1} = D_{\tilde{y}}$ hence $A_{yz}^{-1} = D_{\tilde{z}}{A^*_{pq}}^{-1}D_{\tilde{y}}$. Since, as one can easily check, $b_y = D_{\tilde{y}}b_p$, we can conclude that $A_{yz}^{-1}b_y = D_{\tilde{z}}({A^*_{pq}}^{-1}b_p)$. Since $D_{\tilde{z}}$ is a diagonal matrix consisting of ± 1, we get the desired reduction.

Because of this reduction, we can, without losing generality, consider only the case when $\tilde{y} = \tilde{z} = e$. In this case, $A_{yz} = (2n + \varepsilon)I - ee^T - yz^T$. In the proof of Theorem 2.2, we have shown that for $\varepsilon = 0$, this matrix is semi-definite, hence, when we add $\varepsilon \cdot I$, we get a positive definite matrix – which is thus regular. We need to describe the vector $A_{yz}^{-1}b_y$, i.e., the solution x to the linear system $A_{yz}x = b_y$.

For $y = z = e$, one can easily check that the vector $x = (1/\varepsilon)e$ with components $x_i = 1/\varepsilon$ is the desired solution. Let us show that for every index i and for every pair $\langle y, z\rangle \neq \langle e, e\rangle$, the i-th component x_i of the corresponding vector x is smaller than $1/\varepsilon$. Thus, the maximum in x_i is attained only for $\langle y, z\rangle = \langle e, e\rangle$, and so, if we omit this pair, we do not get the correct interval hull of the solution of the system of linear equations. Hence, for solving an interval linear system, the theorem will be proven.

Indeed, for $\langle y, z\rangle \neq \langle e, e\rangle$, the vector x is a solution to the linear system $A_{yz}x = b_y = y$, i.e., to the system:

$$(2n + \varepsilon) \cdot x_i - (x, e) - (x, z) \cdot y_i = y_i, \tag{3.1}$$

Moving the term $2(x, e)$ to the right-hand side and dividing both sides by $2n+\varepsilon$, we conclude that

$$x_i = \frac{y_i + (x, e) + (x, z) \cdot y_i}{2n + \varepsilon}. \tag{3.2}$$

By definition, $(x, e) = \sum_{i=1}^{n} x_i$, hence, $|(x, e)| \leq \sum_{i=1}^{n} |x_i|$; the equality is attained only in two cases:

- if every component of x_i is non-negative (i.e., has the same sign as e_i), or
- if every component of x_i is non-positive (i.e., has the same sign as $-e_i$).

Similarly, $|(x, z)| \leq \sum_{i=1}^{n} |x_i|$, and the equality is attained only in two cases:

- if every component of x_i has the same sign as z_i, or

- if every component of x_i has the same sign as $-z_i$.

Thus,

$$|y_i + (x, e) + y_i \cdot (x, z)| \le 1 + 2 \cdot \sum_{i=1}^{n} |x_i|, \tag{3.3}$$

with the equality possible only if all the values y_i have the same sign, same as (x, e), and all the values (x, z) are positive (hence, all the components of x_i and e_i have the same sign, and so do x_i and z_i). Applying the inequality (3.3) to the formula (3.2), we conclude that

$$|x_i| \le \frac{1 + 2 \cdot \sum\limits_{i=1}^{n} |x_i|}{2n + \varepsilon.} \tag{3.4}$$

Adding these inequalities for $i = 1, \ldots, n$, we conclude that

$$\sum_{i=1}^{n} |x_i| \le \frac{n}{2n+\varepsilon} \cdot \left(1 + 2 \cdot \sum_{i=1}^{n} |x_i|\right) = \frac{n}{2n+\varepsilon} + \frac{2n}{2n+\varepsilon} \cdot \sum_{i=1}^{n} |x_i|, \tag{3.5}$$

hence

$$\left(1 - \frac{2n}{2n+\varepsilon}\right) \cdot \sum_{i=1}^{n} |x_i| \le \frac{n}{2n+\varepsilon}, \tag{3.6}$$

$$\frac{\varepsilon}{2n+\varepsilon} \cdot \sum_{i=1}^{n} |x_i| \le \frac{n}{2n+\varepsilon} \tag{3.7}$$

and

$$\sum_{i=1}^{n} |x_i| \le \frac{n}{\varepsilon}. \tag{3.8}$$

From (3.4) and (3.8), we can now conclude that

$$|x_i| \le \frac{1 + 2n/\varepsilon}{2n+\varepsilon} = \frac{1}{\varepsilon}, \tag{3.9}$$

hence

$$x_i \le \frac{1}{\varepsilon}, \tag{3.10}$$

and the equality is only possible if all the components of the vectors e, y, and z have the same signs, i.e., if $e = y = z$. For linear interval systems, the theorem is proven.

For matrix inversion, we take the same interval matrix $\mathbf{A}(\tilde{y}, \tilde{z}, \varepsilon)$ as in solving interval linear systems. Due to the above-mentioned properties $A_{yz} = D_{\tilde{y}} A^*_{pq} D_{\tilde{z}}$ and $A^{-1}_{yz} = D_{\tilde{z}} {A^*_{pq}}^{-1} D_{\tilde{y}}$, we can reduce the general case to the case when $\tilde{y} = \tilde{z} = e$; for each (ij)-component of the resulting bounds $\overline{B}$ and $\underline{B}$, we may have to switch minimum to maximum, depending on the sign of the product $\tilde{z}_i\tilde{y}_j$.

Therefore, it is sufficient to prove this theorem for the case when $\tilde{y} = \tilde{z} = e$. In this case, $A_{ee} = (2n+\varepsilon)I - 2ee^T$. One can easily check that the inverse matrix has the form

$$A_{ee}^{-1} = \frac{1}{2n+\varepsilon}I + \frac{2}{\varepsilon\cdot(2n+\varepsilon)}ee^T,$$

i.e., it has entries $\dfrac{2+\varepsilon}{\varepsilon\cdot(2n+\varepsilon)}$ on the diagonal and $\dfrac{2}{\varepsilon\cdot(2n+\varepsilon)}$ off the diagonal.

Let us show that such high values cannot be attained for all other inverse matrices $M = A_{yz}^{-1}$. Indeed, the matrix M satisfies the equation $A_{yz}M = I$, i.e.,

$$(2n+\varepsilon)\cdot m_{ik} - \sum_{j=1}^{n} m_{jk} - y_i \cdot \sum_{j=1}^{n} z_j \cdot m_{jk} = \delta_{ik}.$$

Therefore, we have

$$m_{ik} = \frac{\sum\limits_{j=1}^{n} m_{jk} + y_i \cdot \sum\limits_{j=1}^{n} z_j \cdot m_{ij} + \delta_{ik}}{2n+\varepsilon}.$$

Here, $\sum\limits_{j=1}^{n} m_{jk} \le \sum\limits_{j=1}^{n} |m_{jk}|$, and also $\sum\limits_{j=1}^{n} y_j \cdot m_{jk} \le \sum\limits_{j=1}^{n} |y_j|\cdot|m_{jk}| = \sum\limits_{j=1}^{n} |m_{jk}|$; therefore,

$$m_{ik} \le \frac{2\sum\limits_{j=1}^{n} |m_{jk}| + \delta_{ik}}{2n+\varepsilon}. \tag{3.11}$$

Since the value m_{jk} cannot exceed a non-negative number, its absolute value $|m_{jk}|$ cannot exceed the same number:

$$|m_{ik}| \le \frac{2\sum\limits_{j=1}^{n} |m_{jk}| + \delta_{ik}}{2n+\varepsilon}.$$

Adding the corresponding inequalities for all i from 1 to n and taking into consideration that $\sum\limits_{i=1}^{n} \delta_{ik} = 1$, we conclude that

$$\sum_{i=1}^{n} |m_{ik}| \le \frac{2n\cdot\sum\limits_{j=1}^{n} |m_{jk}| + 1}{2n+\varepsilon}.$$

Moving all the terms proportional to the sum $\sum\limits_{i=1}^{n}|m_{jk}|$ to the left-hand side, we conclude that

$$\frac{\varepsilon}{2n+\varepsilon}\cdot\sum_{j=1}^{n}|m_{jk}|\le\frac{1}{2n+\varepsilon},$$

i.e., that $\sum\limits_{j=1}^{n}|m_{jk}|\le 1/\varepsilon$. Thus, from (3.11), we can conclude that $m_{ik}\le \dfrac{2/\varepsilon+\delta_{ik}}{2n+\varepsilon}$. By recalling the expression for A_{ee}^{-1}, we can see that the right-hand side of the last inequality is exactly the entries for A_{ee}^{-1}, so we can conclude that $A_{yz}^{-1}\le A_{ee}^{-1}$.

Similarly to the linear equation case, we can see that the only possibility for the equality is when $y=z=e$. The theorem is proven.

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