Which Truth Values in Fuzzy Logics Are Definable?

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Abstract

In fuzzy logic, every word or phrase describing uncertainty is represented by a real number from the interval $[0, 1]$. There are only denumerable many words and phrases, and continuum many real numbers; thus, not every real number corresponds to some commonsense degree of uncertainty. In this paper, for several fuzzy logic, we describe which numbers are describing such degrees, i.e., in mathematical terms, which real numbers are *definable* in the corresponding fuzzy logic.

1 Introduction: Why Definable Values Are Important

In commonsense reasoning, we use words and phrases from natural language – like "possible", "reasonably possible", etc. – to describe our degree of certainty in different statements. To represent these degrees of certainty in a computer, traditionally, real numbers from the interval $[0,1]$ are used: 1 ("true") means that we are absolutely sure that a statement is true, 0 means that we are absolutely sure that the statement is false, and numbers between 0 and 1 describe partial certainty; see, e.g., [5, 8, 12].

The correspondence between words and phrases describing uncertainty and real numbers is not one-to-one: in every natural language, there are no more than countably many ($\aleph_0$) words and phrases, while there is a continuum ($\aleph_1 > \aleph_0$) numbers in the interval $[0,1]$. From the practical viewpoint, it is therefore desirable to find out which real numbers are actually needed for describing degrees of uncertainty.

Since one of the main reasons for representing degrees by real numbers is is to represent these degrees in a computer, one possible answer to the above question is to look at which real numbers are actually computer represented. At present, a typical computer-represented "real number" is actually a *rational* number (= fraction). Therefore, it may seem reasonable to only consider rational numbers – especially since every real number can be approximated by rational numbers with any given accuracy. However, this answer is not very satisfying because sometimes, irrational numbers from the interval $[0,1]$ are also useful in fuzzy logic. Let us give two examples. These two examples are related to *hedges* like "almost", "very", etc., which constitute an important part of our reasoning about uncertainty [5, 8, 12]:

- A standard representation of a hedge "almost" is that when a statement S has a degree d, the statement "almost S" has a degree $\sqrt{d}$. Thus, even if S has a simple degree of belief, like $d = 1/2$ or $d = 3/4$, the resulting degree of belief for "almost S" will be an irrational number: correspondingly, $\sqrt{2}/2$ or $\sqrt{3}/2$.
- A standard representation of "very" is d^2, so it does not directly lead to an irrational number. However, indirectly it does. For example, in [6], a "perfect" degree is interpreted as a degree p for which a further "intensification" leads to the opposite effect, i.e., for which "very" $p = 1 - p$. For "very" $p = p^2$, the resulting equation leads to the *golden ratio* $p = (\sqrt{5} - 1)/2$, which is an irrational number. This number has a lot of uses, so it is desirable to keep it in our set of possible values.

These examples prompt us to consider not only computer-represented rational numbers, but also more "complicated" numbers. First, we want to include numbers which can be obtained by an explicit application of standard fuzzy logic operations – like "and", "or", and hedges), so as to cover values like $\sqrt{2}/2$. More generally, we want to include numbers p which are uniquely determined by some meaningful conditions. These conditions can be explicit, equating the number p with a basic expression (like in the above example $p =$ "almost" $1/2$). These conditions can be implicit, e.g., as an equality between two meaningful terms – as in the above example "very" $p = 1 - p$.

We can have even more complicated conditions. For example, if in fuzzy logic, we have an "and" operation (t-norm) $a \mathbin{\widetilde{\&}} b$, then a natural definition of a fuzzy implication $a \mathbin{\widetilde{\rightarrow}} b$ is $a \mathbin{\widetilde{\rightarrow}} b \stackrel{\text{def}}{=} \sup \left\{c \mid \left(c \mathbin{\widetilde{\&}} a\right) \le b\right\}$. By definition of the least upper bound sup, this means that the value $p \stackrel{\text{def}}{=} a \mathbin{\widetilde{\rightarrow}} b$ is an upper bound – i.e., it is greater than or equal to any c for which $\left(c \mathbin{\widetilde{\&}} a\right) \le b$ – and that it is the *least* upper bound, i.e., that it does not exceed any other upper bound q. In formal terms, this definition takes the following form:

$$\forall c \left(\left(c \mathbin{\widetilde{\&}} a \le b\right) \rightarrow (p \ge c)\right) \;\&\; \forall q \left(\forall c \left(\left(c \mathbin{\widetilde{\&}} a \le b\right) \rightarrow (q \ge c)\right) \rightarrow (p \le q)\right).$$

Summarizing these examples, we can say that from all the numbers from the interval $[0, 1]$, we want to use only those numbers which are uniquely determined by some reasonable conditions.

In logic, elements of a set which are uniquely determined by some condition are called *definable*. In these terms, our original problem of selecting truth values which are really needed can be reformulated as follows: describe all definable truth values.

In this paper, we formalize this problem, and show how its solution depends on the particular selection of operations in fuzzy logic.

2 Definitions

Let us first define what a condition can look like. In mathematical logic, formal expressions which describe conditions are called *formulas*, so we want to define the notion of a formula. We will give sketchy definitions here; readers who are interested in technical details can look, e.g., in [3, 4, 9].

Let us fix a set of constants (e.g., 0 and 1), and a set of operations on the interval $[0,1]$; this set can include an "and"-operation (t-norm) $\widetilde{\&}$, an "or"-operation $\widetilde{\vee}$, a fuzzy negation $\widetilde{\neg}$, hedge operations, etc. Some of these operations are binary (like t-norm and t-conorm), some are unary (like negation), we may also have ternary operations, etc. We also have a sequence of variables $x_1, \ldots, x_n, \ldots$

Then, we define the notion of a *term*. Every constant is a term, every variable is a term, and if $f(x_1, \ldots, x_m)$ is an operation and $t_1, \ldots, t_m$ are terms, then the expression $f(t_1, \ldots, t_m)$ is also a term. For example, $x_1 \mathbin{\widetilde{\&}} \left(x_2 \mathbin{\widetilde{\vee}} x_3\right)$ is a term, "very" p is a term, $\neg p$ is a term, etc.

Next, we define the notion of an *elementary formula* as an expression of the type $t_1 = t_2$, $t_1 < t_2$, $t_1 \le t_2$, $t_1 > t_2$, $t_1 \ge t_2$, or $t_1 \ne t_2$, where t_i are terms. For example, $\left(c \mathbin{\widetilde{\&}} a\right) \le b$, $q \ge c$, and $p \le q$ are elementary formulas.

The notion of a *formula* is defined as follows:

- Every elementary formula is a formula.
- If F and G are formulas, then the expressions (F), $F \& G$, $F \vee G$, $\neg F$, and $F \rightarrow G$ are formulas.
- If F is a formula and v is a variable, then expressions $\forall v\, F$ and $\exists v\, F$ are formulas.

It is easy to check that all the above conditions are formulas in this sense.

Finally, a truth value v is called *definable* if there exists a logical formula $F(x)$ with a single free variable x such that this formula is only true for $x = v$.

3 Results

What are the operations normally used in fuzzy logic? Some of the operations are *polynomial*: e.g., $a \mathbin{\widetilde{\&}} b = a \cdot b$ and $a \mathbin{\widetilde{\vee}} b = a + b - a \cdot b$. Some of the operations, like $q =$ "almost"$(p) = \sqrt{p}$, are not polynomial, but are solutions of a polynomial equation – in the above example, the equation $q^2 - p = 0$. Such functions are called *algebraic* (or, to be more precise, *semialgebraic*; see, e.g., [2]).

To be even more precise, a set $S \subseteq \mathbb{R}^q$ is called *semialgebraic* if it is a finite union of subsets, each of which is defined by a finite system of polynomial equations $P_r(x_1, \ldots, x_q) = 0$ and inequalities of the types $P_s(x_1, \ldots, x_q) > 0$ and $P_t(x_1, \ldots, x_q) \ge 0$ – for some polynomials P_i with integer coefficients.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *semialgebraic* if its graph $\{(x, f(x))\}$ is a semialgebraic set.

For example, the graph of the function $z = \min(x, y)$ is a union of two pieces of planes $z - x = 0$ and $z - y = 0$, each piece is described by a polynomial equation (of the plane) and of polynomial inequalities (describing this particular part of the plane): e.g., for $z - x = 0$, the inequalities are: $y - x \ge 0$ (meaning that $x \le y$ and $x \ge 0$, $y \ge 0$, $1 - x \ge 0$, and $1 - y \ge 0$ (these four inequalities mean that both x and y belong to the interval $[0, 1]$).

It turns out that for such operations, every definable truth value x is an *algebraic number*, i.e., a solution of a polynomial equation $a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_k \cdot x^k = 0$ with integer coefficients a_i:

Theorem 1. *When all logical operations are semialgebraic, then every definable truth value is algebraic.*

(For reader's convenience, all the proofs are given in the last Proofs section).

We know that every definable truth value is algebraic. The next natural question is: is the inverse also true, i.e., is every algebraic number from the interval $[0,1]$ definable? The following two results show that the answer to this question depends on the specific choice of the logical operations. First, let us give an example where every algebraic number is definable:

Theorem 2. *For* $a \mathbin{\tilde{\&}} b = a \cdot b$, $a \mathbin{\tilde{\vee}} b = \min(a+b,1)$, *and* $\tilde{\neg} a = 1-a$, *every algebraic number from the interval* $[0,1]$ *is definable.*

One can see, from the proof, that not only every algebraic number is definable, but is can be defined by a quantifier-free formula $F(x)$.

Next, comes an example when some algebraic numbers are not definable. To describe this example in its utmost generality, we need to introduce a new definition.

A set $S \subseteq \mathbb{R}^q$ is called *semilinear* if it is a finite union of subsets, each of which is defined by a finite system of linear equations $P_r(x_1,\ldots,x_q) = 0$ and inequalities of the types $P_s(x_1,\ldots,x_q) > 0$ and $P_t(x_1,\ldots,x_q) \geq 0$ – for some linear functions P_i with integer coefficients. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *semilinear* if its graph $\{(x, f(x))\}$ is a semilinear set.

For example, the text before Theorem 1 shows that the simplest t-norm $\min(a,b)$ is a semilinear operation; similarly, the simplest t-conorm $\max(a,b)$, the "bold" t-conorm $\min(a+b,1)$, and the bold t-norm $\max(a+b-1,0)$ are semilinear operations.

Theorem 3. *When all logical operations are semilinear, then every definable truth value is rational.*

Since not every algebraic number is rational, we thus conclude that for such logics, not every algebraic number is definable. The next natural question for such operations is: are all rational numbers definable? The answer depends on a specific choice of semilinear operations. First, let us give an example of a logic in which every rational number is definable:

Theorem 4. *For* $a \mathbin{\tilde{\vee}} b = \min(a+b,1)$ *and* $\tilde{\neg} a = 1-a$, *every rational number from the interval* $[0,1]$ *is definable.*

One can see, from the proof, that not only every rational number is definable, but it can be defined by a quantifier-free formula $F(x)$.

As an example of a semilinear logic in which not all rational numbers are definable, we give the simplest fuzzy logic, for which, as it turns out, we only have three definable values:

Theorem 5. *For* $a \mathbin{\tilde{\&}} b = \min(a,b)$, $a \mathbin{\tilde{\vee}} b = \max(a,b)$, *and* $\tilde{\neg} a = 1-a$, *the only three definable truth values are* 0, 1, *and* $1/2$.

4 Proofs

Proof of Theorem 1. Since all the operations are semi-algebraic, then, for every definable real number v, the defining relation $F(x)$ is obtained from a semi-algebraic relation by using quantifiers. According to the famous Tarski-Seidenberg theorem [10, 11] (see also [2]), every relation that is obtained from a semialgebraic relation by adding quantifiers $\forall x$, $\exists x$ (that run over all real numbers x), is still semialgebraic. Thus, the condition $F(x)$ is itself semialgebraic. In other words, the relation $F(x)$ can be described as $P_1(x) = 0$, $P_2(x) \geq 0$, etc. for some polynomials P_i with integer coefficients. The definable number v satisfies this condition, hence $P_1(v) = 0$, i.e., the number v is algebraic. Q.E.D.

Proof of Theorem 2. Let v be an algebraic real number from the interval $[0, 1]$; let us show that it is definable. If $v = 0$, then the defining condition is that $x \mathbin{\widetilde{\vee}} x = x$ and $x < \widetilde{\neg}\, x$. Similarly, if $v = 1$, then the defining condition is that $x \mathbin{\widetilde{\vee}} x = x$ and $x > \widetilde{\neg}\, x$. To complete the proof, we must consider the case when $0 < v < 1$.

By definition of an algebraic number, there exists a polynomial $P(x) = a_0 + a_1 \cdot x + \ldots + a_k \cdot x^k$ with integer coefficients for which $P(x) = 0$ for $x = v$. We want to "translate" this equation into a fuzzy logic formula.

The first obstacle to this translation is that in fuzzy logic, we only consider non-negative real numbers, which the coefficients a_i can be negative and thus, the value $P(x)$ can be negative (thus difficult to interpret) for some $x \in [0, 1]$. To overcome this obstacle, we move negative terms to the other side of the equation $P(x) = 0$. As a result, we get an equality of two polynomials $\sum b_i \cdot x^i = \sum c_j \cdot x^j$ with natural (integer non-negative) coefficients b_i and c_j. The values on both side of this equation are now non-negative.

However, this equation is not yet ready for the fuzzy logic interpretation, because the values $\sum b_i \cdot x^i$ and $\sum c_j \cdot x^j$ can exceed 1. How can we overcome this second obstacle? Since all the coefficients b_i and c_j are non-negative, both functions $\sum b_i \cdot x^i$ and $\sum c_j \cdot x^j$ are increasing with x. Thus, their largest values are attained when $x = 1$, and equal to, correspondingly, $\sum b_i$ and $\sum c_j$. Without losing generality, let us assume that $\sum b_i \geq \sum c_j$.

Since $v < 1$, we have $v^n \to 0$ as $n \to \infty$. Thus, there exists an n for which $v^n \cdot \sum b_i < 1$; then, $v^n \cdot \left(\sum b_i \cdot v^i\right) \leq v^n \cdot \sum b_i < 1$. Since $\sum c_j \leq \sum b_i$, we also have $v^n \cdot \sum c_j < 1$ and $v^n \cdot \left(\sum c_j \cdot v^j\right) \leq v^n \cdot \sum c_j < 1$. Hence, the desired real number v satisfies the following two conditions: $\sum b_i \cdot x^{i+n} = \sum c_j \cdot x^{j+n}$ and $\sum b_i \cdot x^{i+n} < 1$.

These conditions can already be interpreted in fuzzy logical terms: indeed, x^k means $x \mathbin{\widetilde{\&}} \ldots \mathbin{\widetilde{\&}} x$ (k times), $b \cdot x$ means $x + \ldots + x$ (b times), and $x + y$ means $x \mathbin{\widetilde{\vee}} y$ – as long as $x + y < 1$ (and all the sums in the above formula are less than 1).

We thus get a fuzzy logic condition $F(x)$ which is equivalent to the original polynomial equation $P(x) = 0$. We are almost done, the only remaining problem is that the equation may have several different roots, and we want a formula which is true for only one real number. If this is the case, then we must add, to the condition $F(x)$, additional conditions which separate v from other roots of the equation $P(x) = 0$. Indeed, let v' be a different root. Without losing generality, let us assume that $v < v'$. To get the desired additional condition, we would like to find the natural numbers n and m for which $n \cdot v^m < 1$ and $n \cdot (v')^m > 1$. If we find such values, then the desired separating condition is $z \widetilde{\vee} \ldots \widetilde{\vee} z$ (n times) < 1, where z denotes $x \mathbin{\widetilde{\&}} x \ldots \mathbin{\widetilde{\&}} x$ (m times). The desired condition on n and m is equivalent to $(1/v)^m > n > (1/v')^m$. When $m \to \infty$, the difference $(1/v)^m - (1/v')^m$ tends to ∞; thus, for large enough m, the length of the interval $[(1/v')^m, (1/v)^m]$ exceeds 1 and hence, this interval contains at least one integer n. The existence of the desired m and n is proven. Q.E.D.

Proof of Theorem 3. The proof is similar to the proof of Theorem 1. Namely, each semilinear set is a polytope with rational-coordinate vertices. The quantifier $\exists x_i$ corresponds to projecting this polytope unto a space of one fewer dimension. Once can easily prove that this projection transforms polytopes into polytopes, and vertices with rational coordinates are transformed into vertices with rational coordinates. (This reduction procedure can also be described in purely algebraic terms; see, e.g., [1].) Thus, for semilinear conditions, the condition $F(x)$ is itself semilinear. In other words, the relation $F(x)$ can be described as $P_1(x) = 0$, $P_2(x) \geq 0$, etc. for some linear function $P_i(x)$ with integer coefficients. The definable number v satisfies this condition, hence $P_1(v) = 0$, i.e., $a \cdot v + b = 0$ for some integers a and b; hence, the number $v = -b/a$ is rational. Q.E.D.

Proof of Theorem 4. The proof is similar to the one presented in [7].

We want to prove that every rational number m/n from the interval $[0, 1]$ is definable. In the proof of Theorem 2, we already proved that 0 and 1 are definable, so it is sufficient to prove that rational numbers between 0 and 1 are definable; thus, it is sufficient to consider the case when $0 < m < n$. Without losing generality, we can assume that the numbers m and n have no common divisors: otherwise, we can divide both n and m by their common divisor, and get a simpler fraction representing the same rational number.

Let us prove this result by induction over m. The base is easy to prove: for $m = 1$, the value $1/n$ can be defined as the only value x for which

$$x \widetilde{\vee} \ldots \widetilde{\vee} x \ (n-1 \text{ times}) = \widetilde{\neg}\, x.$$

Let us now prove the induction step. Assume that for for some m, we have already proved the definability of all the fractions with $m' < m$, and we want to prove that m/n is definable. To prove it, let us divide n by $m < n$. Since m and n have no common divisors, we have a non-zero remainder r: $n = k \cdot m + r$, with $0 < r < m$. Dividing both sides of this equation by n, we conclude that $1 - k \cdot (m/n) = r/n$. Since $r < m$, by induction assumption, the value r/n is definable, so there exists a formula $F(x)$ which is only true when $x = r/n$. To get a formula $G(y)$ which defines m/n, all we need to do is substitute $x = \tilde{\neg}\left(y \tilde{\vee} \ldots \tilde{\vee} y \text{ (k times)}\right)$ instead of x into the formula $F(x)$. When $G(y)$ is true, then $x = r/n$, hence $1 - k \cdot y = r/n$ and $y = m/n$. The induction step is proven. Q.E.D.

As an example, let us show how $2/5$ will be defined. To define $1/5$, we have a formula $x \tilde{\vee} x \tilde{\vee} x \tilde{\vee} x = \tilde{\neg} x$. To define $2/5$, we divide 5 by 2, getting $5 = 2 \cdot 2 + 1$ hence $1 = 2 \cdot (2/5) = 1/5$. So, to get the condition for $2/5$, we substitute $x = \tilde{\neg}\left(y \tilde{\vee} y\right)$ into the above formula instead of x. As a result, we get the following condition:

$$(\tilde{\neg}\left(y \tilde{\vee} y\right)) \tilde{\vee} (\tilde{\neg}\left(y \tilde{\vee} y\right)) \tilde{\vee} (\tilde{\neg}\left(y \tilde{\vee} y\right)) \tilde{\vee} (\tilde{\neg}\left(y \tilde{\vee} y\right)) = \tilde{\neg}\left(\tilde{\neg}\left(y \tilde{\vee} y\right)\right).$$

Proof of Theorem 5. Let us first show that all three values are indeed definable. Indeed, 0 is the only value which does not exceed everyone else, i.e., it is the only value x which satisfies the condition $\forall y\,(x \leq y)$. Similarly, 1 is the only value which is greater than or equal to everyone else, i.e., it is the only value x which satisfies the condition $\forall y\,(x \geq y)$. Finally, $1/2$ is the only value which satisfies the condition $\tilde{\neg} x = x$.

To complete the proof, we must show that no other value v from the interval $[0,1]$ is definable. Let us prove this by reduction to a contradiction. Assume that v is definable, which means that there is a formula $F(x)$ be a formula which defines v – i.e., which is true for $x = v$ and false for all other values x. Without losing generality, we can assume that $0 < v < 1/2$. Let us build a piece-wise linear transformation $f : [0,1] \to [0,1]$ by taking $f(0) = 0$, $f(v) = v/2$, $f(1/2) = 1/2$, $f(1-v) = 1 - v/2$, $f(1) = 1$, and by making f linear on the intervals $[0,v]$, $[v,1/2]$, $[1/2, 1-v]$, and $[1-v, 1]$. It is easy to check that thus defined function f transforms v into $v/2$ and preserves all three logical operations (i.e., is an *isomorphism*) of the corresponding fuzzy logic. Thus, a formula $F(x)$ is true if and only if the formula $F(f(x))$ is also true. Since $F(v)$ is true, we conclude that the formula $F(f(v)) = F(v/2)$ is also true, which contradicts to our assumption that $F(x)$ is false for every $x \neq v$. This contradiction shows that the value v is *not* definable. Q.E.D.

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