Towards Fusing Sophisticated Mathematical Knowledge and Informal Expert Knowledge: An Arbitrary Metric Can Be Naturally Interpreted in Fuzzy Terms

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Abstract

In many practical situations, we are faced with a necessity to combine sophisticated mathematical knowledge about the analyzed systems with informal expert knowledge. To make this combination natural, it is desirable to reformulate the abstract mathematical knowledge in understandable intuitive terms. In this paper, we show how this can be done for an abstract metric.

One way to define a metric is to pick certain properties $P_1, \ldots, P_n$, and to define a similarity between two objects x and y as the degree to which $P_1(x)$ is similar to $P_1(y)$ *and* $P_2(x)$ is similar to $P_2(y)$ etc.

Similarity is naturally described by $1 - |d_1 - d_2|$ (we can use robustness arguments to get this expression). Since we can have infinitely many properties, we should use min for "and". The distance is then $1-$this similarity. The resulting metrics are "natural".

It seems, at first glance, that not all metrics are natural in this sense. Interestingly, an arbitrary continuous metric can be thus described.

Similarly, we can thus describe all "kinematic metrics" (space-time analogues of metrics), while probabilistic explanation is difficult.

Keywords: Metrics, Fuzzy Interpretation.

1 For Data Fusion, It Is Desirable to Express Abstract Mathematical Notions in Natural Terms

In many practical situations, we are faced with a necessity to combine sophisticated mathematical knowledge about the analyzed systems with informal expert knowledge. To make this combination natural, it is desirable to reformulate the abstract mathematical knowledge in understandable intuitive terms.

In this paper, we show how this can be done for a specific mathematical notion: the notion of a metric.

2 Some Metrics Are Natural, But Are All Metric Natural?

The distance between two points is a particular example of a function which describes "closeness" ("similarity") between the two objects. There are many examples of such functions, and mathematicians have developed a general notion of a "metric". For an arbitrary set X, a metric is defined as a real-valued function $d : X \times X \to R$ for which the following three properties hold:

- $d(a,b) = 0$ if and only if $a = b$;
- $d(a,b) = d(b,a)$ (symmetry); and
- $d(a,c) \le d(a,b) + d(b,c)$ (triangle inequality).

Almost every natural notion of a distance satisfies this definition. A natural question is: is the inverse true? In other words,

- is this definition just right – in the sense that every metric satisfying this definition can be naturally interpreted,

- or this definition is too general, and naturally appearing metrics form a proper subclass of the class of all metrics?

3 Our Approach: Using Fuzzy Logic

What does it mean for a metric to be "natural"? "Natural" means the metric can be interpreted in commonsense terms. One big problem with interpreting commonsense knowledge in precise mathematical terms is that the words that experts use to describe their knowledge are not precise, they are "fuzzy". Since fuzzy logic has been invented specifically for describing such "fuzziness" in precise mathematical terms, it is natural to use fuzzy logic as a basis for our definition of naturalness.

The idea of using fuzzy logic to describe naturalness is not only natural itself, it is also known to be successful: in our previous papers [2, 4], we have shown that the use of fuzzy logic makes a special metric used in logic programming very natural. In this paper, we expand on this result and show that an arbitrary metric can be thus interpreted.

4 Motivations of the Following Definitions

What is a natural way to describe the closeness between the two objects? Let us start with maximal closeness, i.e., with identity. How do we know that the two objects a and b are identical? Two objects are identical when, whatever measurements and observations we perform on both of them, we always get the exact same result for both objects. How can we represent these results?

For most existing measuring instruments, the results of the measurement are automatically entered into the computer and thus, are represented as a sequence of 0's and 1's. For the few cases when measurements are manual, we can also easily type their results into a computer, thus transforming these results into a sequence of 0's and 1's. Thus, we can view a sequence of measurements as a sequence of properties, i.e., "measurements" whose results are 0 or 1 (true or false).

This interpretation is not only natural from the viewpoint of the internal computer representation, it is also natural from the commonsense viewpoint. Indeed, e.g., when we have a number from the interval $[0,1]$, then knowing this number means being able to answer binary (yes-no) questions like "is this number smaller than 1/2"? Depending on the answer, the natural next question is "is this number smaller than 1/4?" or "is this number smaller than 3/4?".

From the commonsense viewpoint, however, it is natural, in addition to binary questions, to consider fuzzy questions like "is this number small?". For such fuzzy property P, it is no longer true that for any object a, $P(a)$ is either true or false; the truth value $P(a)$ can take any value from the interval $[0,1]$.

With this interpretation in mind, we can assume that we have a sequence of all possible (fuzzy) properties $P_1, P_2, \ldots, P_n, \ldots$, and we say that the objects a and b are *identical* if for all i, $P_i(a) = P_i(b)$.

It is natural to say that the objects a and b are *close* if for all i, the values $P_i(a)$ and $P_i(b)$ are close. What is the (numerical) degree with which a and b are close?

- To formalize this, we must first describe the closeness $c(p,q)$ between two fuzzy truth values p and q. There are several possible definitions; we select the one which is the least sensitive to the possible uncertainty in p and q; it is $c(p,q) = 1 - |p-q|$ (see, e.g., [5]).
- The quantifier "for all i" is naturally described, in fuzzy logic, as min over all i.

Thus, for every a and b, the degree of closeness can be naturally described as

$$\min_i (1 - |P_i(a) - P_i(b)|).$$

Correspondingly, since the difference is the opposite to closeness, the degree of difference ("metric") can be described as the negation $(1-)$ the degree of closeness, i.e., as

$$d(a,b) = 1 - \min_i (1 - |P_i(a) - P_i(b)|). \qquad (1)$$

Our main result is that an arbitrary metric can be thus represented. In other words, we prove that an arbitrary metric is natural.

5 Definitions and the Main Result

Clearly, the formula (1) can only describe metrics whose values are within the interval $[0,1]$, so we will only consider such metric spaces. We will also restrict ourselves to *separable* metric spaces, i.e., metric spaces X which have a denumerable dense subset $\{x_1, x_2, \ldots\}$; most metric spaces such as the set of all real numbers, the set of all vectors, most function spaces are separable.

Definition.

- *By a fuzzy property on a set X, we mean a function $P : X \to [0,1]$.*
- *We say that a metric $d : X \times X \to [0,1]$ is natural if it can be represented in the form (1) for some fuzzy properties $P_1, \ldots, P_n, \ldots$*

Theorem. *Every separable metric is natural.*

Proof. Let (X,d) be a separable metric space. By definition of a separable metric space, this means that in the set X, there exists a denumerable dense subset $\{x_1, x_2, \ldots\}$. We will show that the formula (1) holds for the fuzzy properties $P_i(x) \stackrel{\text{def}}{=} d(x, x_i)$.

Before we proceed with the proof, let us give an intuitive meaning of the fuzzy property $P_i(x)$: the further away from x_i is the point x, the larger the value $P_i(x)$. Thus, the property $P_i(x)$ describes the property "far away from x_i".

Back to the proof. The value $1-z$ is the smallest when z is the largest. Thus,

$$\min_i (1 - |P_i(a) - P_i(b)|) = 1 - \max_i |P_i(a) - P_i(b)|,$$

and the right-hand side of the formula (1) can be rewritten as follows:

$$1 - \min_i (1 - |P_i(a) - P_i(b)|) = \max_i |P_i(a) - P_i(b)|.$$

Thus, to prove the formula (1), it is sufficient to prove that for all a and b, the following equality holds:

$$d(a,b) = \max_i |P_i(a) - P_i(b)|. \qquad (2)$$

To prove this equality, we will first prove the similar inequality:

$$d(a,b) \geq \max_i |P_i(a) - P_i(b)|. \qquad (3)$$

Indeed, due to triangle inequality, for every i, we have:

$$d(a, x_i) \leq d(b, x_i) + d(a,b).$$

Due to our choice of the fuzzy properties P_i, this means that:

$$P_i(a) \leq P_i(b) + d(a,b).$$

Subtracting $P_i(b)$ from both sides of this inequality, we conclude that

$$P_i(a) - P_i(b) \leq d(a,b). \qquad (4)$$

Similarly, we conclude that

$$P_i(b) - P_i(a) \leq d(a,b). \qquad (5)$$

From the inequalities (4) and (5), we conclude that

$$\max(P_i(a) - P_i(b), P_i(a) - P_i(b)) \leq d(a,b),$$

i.e., that

$$|P_i(a) - P_i(b)| \leq d(a,b). \qquad (6)$$

Since $d(a,b)$ is larger than or equal to each of the absolute values $|P_i(a) - P_i(b)|$, we can thus conclude that $d(a,b)$ is greater than or equal to the maximum of these values, i.e., that the inequality (3) is indeed true.

Now, since the sequence $\{x_i\}$ is dense in X, for the point a, there exists a subsequence x_{i_k} which converges to a. For this subsequence, $d(a, x_{i_k}) \to d(a,a) = 0$ and $d(b, x_{i_k}) \to d(a,b)$. Due to our choice of the fuzzy properties P_i, we thus conclude that $P_{i_k}(a) \to 0$ and $P_{i_k}(b) \to d(a,b)$. Therefore,

$$d(a,b) = \lim_k |P_{i_k}(a) - P_{i_k}(b)|.$$

The maximum cannot be smaller than the limit, hence

$$\lim_k |P_{i_k}(a) - P_{i_k}(b)| \leq \max_i |P_i(a) - P_i(b)|,$$

i.e.,

$$d(a,b) \leq \max_i |P_i(a) - P_i(b)|. \qquad (7)$$

Combining (3) and (7), we conclude that the equality (2) holds. Q.E.D.

6 Another Result: Space-Time Analogues of Metrics

Similarly, we can thus describe all "kinematic metrics" (a version of metric used for geometry of space-time); see, e.g., [1, 6]. Let us briefly describe the main ideas of such metrics.

In normal geometry, we can have several paths connecting two points a and b. A distance $d(a,b)$ between the two points a and b on standard geometry can be described as the shortest path between a and b.

In space-time geometry, distance becomes relative, and the only directly measurable quantity is proper time between the two events. The proper time between the events a and b can only be defined when a precedes b. According to special relativity, the faster one travels, the smaller amount of proper time is spent on this travel. When the speed of the traveler approaches the speed of light c, proper time of this travel tends to 0. This is not just a theoretical conclusion, it is an observable fact: e.g., elementary particles whose decay half-time is miniscule at rest, can spend large amounts of time traveling without decay at a speed close to c. As a result, the smallest possible proper time is always 0. A meaningful quantity here is the *largest* proper time $\tau(a,b)$ between the two events. Based on this definition, once can deduce the following properties of the resulting function $\tau : X \times X \to R$ (called "kinematic metric"):

- $\tau(a,b) > 0$ if and only if $a \prec b$ (i.e., if b is inside the future cone for a);
- if $\tau(a,b) > 0$ then $\tau(b,a) = 0$ (antisymmetry);
- if $a \prec b \prec c$, then $\tau(a,c) \geq \tau(a,b) + \tau(b,c)$ (anti-triangle inequality).

To describe such metrics, we can consider "monotonic" fuzzy properties, i.e., properties for which $a \prec b$ implies $P(a) \leq P(b)$. Here, $a \prec b$ if for all monotonic properties, $P(a) \leq P(b)$. As a degree to which a precedes b, one can thus take the degree to which, for all natural monotonic properties P_i, $P_i(a)$ implies $P_i(b)$.

Based on sensitivity considerations (similar to the ones for the standard metrics), we select $\max(q-p,0)$ as the degree for $p \to q$. Then, we get the following class of "natural" kinematic metrics:

$$\tau(a,b) = \min_i \max(P_i(b) - P_i(a), 0).$$

A result similar to the above theorem shows that every kinematic metric can be thus represented. (In the proof, we take $P_i(a) = \tau(a,x_i)$.)

In this case, in addition to the ability to use fuzzy logic to explain a metric, we can show that a similar probabilistic approach does not work: it is known that we cannot use a similar probabilistic interpretation to get an arbitrary kinematic metric (see, e.g., [3]).

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