On Symmetric Solution Sets

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Abstract
Given an $n \times n$ interval matrix $[A]$ and an interval vector $[b]$ with $n$ components we present an overview on existing results on the solution set $S_{\text{sym}}$ of linear systems of equations $Ax = b$ with symmetric matrices $A \in [A]$ and vectors $b \in [b]$. Similarly we consider the set $E_{\text{sym}}$ of eigenpairs associated with the symmetric matrices $A \in [A]$. We report on characterizations of $S_{\text{sym}}$ by means of inequalities, by means of intersection of sets, and by an approach which is generalizable to more general dependencies of the entries. We also recall two methods for enclosing $S_{\text{sym}}$ by means of interval vectors, and we mention a characterization of $E_{\text{sym}}$.

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1 Introduction

With this paper we intend to give an overview on existing results for the symmetric solution set

$$S_{\text{sym}} := \{ x \in \mathbb{R}^n \mid Ax = b, A = A^T \in [A], b \in [b] \}$$

(1)

where $[A]$ is a given $n \times n$ interval matrix with $[A] = [A]^T$, and $[b]$ is a given interval vector with $n$ components. This set obviously is a subset of the general solution set

$$S := \{ x \in \mathbb{R}^n \mid Ax = b, A \in [A], b \in [b] \}$$

(2)

where the restriction $A = A^T$ on $A \in \mathbb{R}^{n \times n}$ is not required. Knowing $S$ and $S_{\text{sym}}$ is particularly interesting in the following situations:

a) Assume that one has to solve a linear system $Ax = b$ on a computer using floating point arithmetic. Due to rounding errors, the computed result $\tilde{x}$ normally will not fulfill $A\tilde{x} = \tilde{b}$. If $\Delta A \in \mathbb{R}^{n \times n}$, $\Delta b \in \mathbb{R}^n$ are given nonnegative tolerances one may view $\tilde{x}$ as an acceptable solution whenever $\tilde{x} \in S$ with $S$ formed as in (2) with respect to $[A] := A + [\Delta A, \Delta A]$, $[b] := b + [-\Delta b, \Delta b]$; in this case $\tilde{x}$ can be interpreted as exact solution of a linear system $Ax = b$ with some $A \in [A]$, $b \in [b]$.

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b) In contrast to a) where the linear system is known we assume now that one has to solve a linear system \( Ax = b \) where \( A, b \) are not given exactly, but they are known to differ from some \( A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n \) by at most \( \Delta A \in \mathbb{R}^{n \times n} \) and \( \Delta b \in \mathbb{R}^n \), respectively (\( \Delta A, \Delta b \) nonnegative). Then \( A \in [A] := A + \begin{bmatrix} -\Delta A, \Delta A \end{bmatrix}, \quad [b] := b + \begin{bmatrix} -\Delta b, \Delta b \end{bmatrix} \). Compute a solution \( x^* \) of \( Ax = b \). Since \( x^* \in S \) one can accept \( x^* \) as a good approximation for the unknown solution of \( Ax = b \). This situation can occur due to

- conversion errors (from decimal to binary or vice versa),
- errors in measurements,
- errors in adjusting the technical devices.

As we shall see \( S \) and \( S_{sym} \) are not so easy to handle. Therefore, enclosures of \( S \) and \( S_{sym} \) are important. For \( S \) such enclosures can be computed by means of interval arithmetic. Since such methods are contained in textbooks like [1], [19], e.g., we will omit them here. They trivially deliver also enclosures for \( S_{sym} \subseteq S \). But there are also methods to enclose \( S_{sym} \) without bounding \( S \) at the same time. We will study such methods later on. Although we shall concentrate on \( S_{sym} \) in this paper we will give a short glance at \( S \) in order to work out particularities of \( S_{sym} \). So we start in Section 3 with several equivalent statements for \( x \in S \) and list some properties of \( S \). In Section 4 we characterize the boundary \( \partial S_{sym} \) of \( S_{sym} \) by means of parts of hyperplanes and quadrics. In Section 5 we introduce two methods for enclosing \( S_{sym} \) and in Section 6 we report on the eigenpair set

\[
E := \{ (x^T, \lambda)^T \in \mathbb{R}^{n+1} | Ax = \lambda x, \ x \neq 0, \ A \in [A] \}.
\]

and the symmetric eigenpair set

\[
E_{sym} := \{ (x^T, \lambda)^T \in \mathbb{R}^{n+1} | Ax = \lambda x, \ x \neq 0, \ A = A^T \in [A] = [A]^T \}.
\]

It turns out that quadrics are needed in order to describe \( E \) and algebraic inequalities of order at most three in order to describe \( E_{sym} \).

## 2 Notations

In the sequel we denote intervals in square brackets, i.e., \([a] = [a, a]\), and identify point intervals \([a, a]\) by their unique element omitting the brackets. We assume that the reader is familiar with the elementary rules and basic facts of interval arithmetic as introduced in the first chapters of [1] or [19], e.g. We will write \( \mathbb{IR}, \mathbb{IR}^n, \mathbb{IR}^{n \times n} \) for the set of real compact intervals, interval vectors with \( n \) components and \( m \times n \) interval matrices, respectively. We apply the notation \([A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}])\) simultaneously for interval matrices and have a similar notation for interval vectors, real vectors and real matrices. An interval matrix \([A]\) is called regular if each \( A \in [A] \) has this property. By \( \bar{A} \) we denote the midpoint of \([A] \), i.e., \( \bar{A} := \frac{1}{2}(\underline{A} + \overline{A}) \), and by \( \text{rad} [A] := \frac{1}{2}(\overline{A} - \underline{A}) \) its radius which we wrote as \( \Delta A \) in Section 1. In connection with interval vectors and intervals this notation is used analogously. If \( T \) is any bounded subset of \( \mathbb{R}^n \) the symbol \( \square T \) denotes the interval hull of \( T \), i.e., the smallest enclosure of \( T \) by an interval vector. For real matrices \( A \in \mathbb{R}^{n \times n} \)
we write \( \rho(A) \) for the spectral radius of \( A \), and define the absolute value \(|A| = (|a_{ij}|) \in \mathbb{R}^{n \times n}\). In addition, we introduce the entrywise defined partial ordering \( \preceq \), and proceed similarly with vectors. For intervals \([a],[c]\) we define \([a] := \max \{|a| \mid a \in [a]\}, \langle [a] \rangle := \min \{|a| \mid a \in [a]\}\) and \(q([a],[b]) := \max \{|a - c|,|a - c|\}\) (Hausdorff distance). For interval matrices \([A],[B] \in \mathbb{R}^{n \times n}\) we introduce the distance \(q([A],[B]) := q([a_{ij}],[b_{ij}]) \in \mathbb{R}^{n \times n}\) and the comparison matrix \(\langle [A] \rangle = (c_{ij}) \in \mathbb{R}^{n \times n}\) which we define by

\[
c_{ij} := \begin{cases} 
|a_{ij}|, & \text{if } i = j, \\
|[a_{ij}]|, & \text{if } i \neq j.
\end{cases}
\]

We call a regular matrix \( A \in \mathbb{R}^{n \times n}\) an \( M \) matrix if \( a_{ij} \leq 0 \) for \( i \neq j \) and \( A^{-1} \geq 0 \). An \( n \times n \) interval matrix \([A]\) is termed an interval \( M \) matrix if each element \( A \in [A]\) is an \( M \) matrix. In particular, an interval \( M \) matrix is always regular, and it is an interval \( H \) matrix, i.e., it satisfies \( ([A])^{-1} \geq 0 \).

From interval analysis (cf. [1], e.g.) we use the following result.

**Theorem 1**

Let \( f(x_1, \ldots, x_n) \) be a rational expression in \( x_1, \ldots, x_n \) in which each variable \( x_i \) occurs at most once. If \([x_1], \ldots, [x_n]\) are given intervals then

\[
f([x_1], \ldots, [x_n]) = \{ f(x_1, \ldots, x_n) \mid x_1 \in [x_1], \ldots, x_n \in [x_n] \},
\]

i.e., replacing \( x_1, \ldots, x_n \) in \( f \) by \([x_1], \ldots, [x_n]\) yields to the range of \( f \) restricted to \([x_1] \times [x_2] \times \cdots \times [x_n]\).

By \( O \) we denote an orthant of \( \mathbb{R}^n \) and by \( O_1 \) the first orthant, i.e.,

\[
O_1 = \{ x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1, \ldots, n \}.
\]

**3 The solution set \( S \)**

In this section we collect some well-known results on \( S \). We begin with some equivalent formulations of the statement ‘\( x \in S \)’ which we prove for completeness.

**Theorem 2**

For \([A] \in \mathbb{R}^{n \times n}\) and \([b] \in \mathbb{R}^n\) the following assertions are equivalent:

a) \( x \in S \);

b) \( \|b - Ax\| \leq (\text{rad} \left[ A \right]) \cdot |x| + \text{rad} \left[ b \right] \); \hspace{1cm} (Oettli and Prager 1964; [20])

c) \( \exists D \in \mathbb{R}^{n \times n} : |D| \leq I \wedge Ax = b = D \left((\text{rad} \left[ A \right]) |x| + \text{rad} \left[ b \right] \right) \);

\hspace{1cm} (Rohn 1984; [21])

d) \( [b] \cap [A]x \neq \emptyset \); \hspace{1cm} (Beeck 1972; [10])

e) \( 0 \in [b] - [A]x \); \hspace{1cm} (Beeck 1972; [10])

3
\[ b - \sum_{j=1}^{n} a_{ij}^+ x_j \leq 0 \leq \bar{b} - \sum_{j=1}^{n} a_{ij}^- x_j, \quad i = 1, \ldots, n, \]

where \( a_{ij}^-, a_{ij}^+ \) are defined by \([a]_{ij} = \begin{cases} [a_{ij}^-, a_{ij}^+] & \text{if } x_j \geq 0, \\ [a_{ij}^+, a_{ij}^-] & \text{if } x_j < 0. \end{cases}\)

\( (Hartfiel 1980; [12]) \)

**Proof.**

a) \( \Rightarrow \) b)

Since \( x \in S \) there are \( A \in [A], \ b \in [b] \) such that \( Ax = b \). Therefore,

\[ 0 = Ax - b \begin{cases} \leq Ax + (\text{rad } [A])|x| - b + \text{rad } [b], \\ \geq Ax - (\text{rad } [A])|x| - b - \text{rad } [b], \end{cases} \]

whence \(-(\text{rad } [A])x + \text{rad } [b]) \leq b - Ax \leq (\text{rad } [A])x + \text{rad } [b] \). This implies b).

b) \( \Rightarrow \) c)

is seen immediately.

c) \( \Rightarrow \) d)

Let \( D_x \) be such that \( |D_x| = I \) and \( D_x x = |x| \). From c) follows \( (A - D(\text{rad } [A]) D_x)x = b + D(\text{rad } [b]) \in [b] \). Since \( A - D(\text{rad } [A]) D_x \in [A] \), the assertion d) follows.

d) \( \Rightarrow \) e)

From the assumption d) and Theorem 1 we obtain \( A \in [A], \ b \in [b] \) such that \( Ax = b \) holds. Hence \( 0 = b - Ax \in [b] - [A]x \).

e) \( \Rightarrow \) f)

From e) and the definition of \( a_{ij}^\pm \) it follows

\[ \inf([A]x - [b]) = -\bar{b} + \sum_{j=1}^{n} a_{ij}^- x_j \leq 0 \leq \sum_{j=1}^{n} a_{ij}^+ x_j = \sup([A]x - [b]), \]

\[ i = 1, \ldots, n. \]

f) \( \Rightarrow \) a)

From f) we obtain

\[ ([A]x)_i = \sum_{j=1}^{n} a_{ij}^- x_j \leq \bar{b}_i, \quad ([A]x)_i = \sum_{j=1}^{n} a_{ij}^+ x_j \geq \underline{b}_i. \]

Define \( \alpha_{ij}(t) := ta_{ij}^+ + (1-t)a_{ij}^- \), \( \beta_i(t) := \sum_{j=1}^{n} \alpha_{ij}(t)x_j, \ t \in [0,1], \ i, j = 1, \ldots, n. \) Since \( \beta_i(t) \) is continuous it assumes for \( 0 \leq t \leq 1 \) all values between \( \beta_i(0) \) and \( \beta_i(1) \). From \( \beta_i(0) \leq \underline{b}_i, \ \underline{b}_i \leq \beta_i(1) \) and \( \bar{b}_i \leq \bar{b}_i \) there is some value \( t = t_i \in [0,1] \) such that \( \beta_i(t_i) \in [b_i] \). With \( \hat{A} := A(t_1, \ldots, t_n) := (a_{ij}(t_i)), \ \hat{b} := (\beta_i(t_i)) \) we get \( \hat{A}x = \hat{b}, \ \hat{A} \in [A], \ \hat{b} \in [b] \), hence \( x \in S \).

\( \square \)

Note that the equivalence a) \( \Leftrightarrow \) b) is the famous Oettli–Prager criterion which was generalized by Fischer and Heindl in [13] (cf. also [14]). In the equivalence a) \( \Leftrightarrow \) f) the coefficients of the inequalities remain fix as long as one stays in a fixed orthant. Therefore, this equivalence shows that the intersection of S with any fixed
closed orthant $O$ is the intersection of finitely many half-spaces. In particular, the boundary $\partial S$ of $S$ is composed of finitely many pieces of hyperplanes. As long as $[A]$ is regular, $S$ is connected and compact which follows from the continuous mapping $f(A, b) = A^{-1}b$ on the connected and compact set $[A] \times [b]$. If $[A]$ contains a singular matrix, connectivity may be lost as Jansson’s example $[A] = [-1, 1]$, $[b] = 1$, $S = (-\infty, -1] \cup [1, \infty)$ in [17] shows. On the other hand it can be present as the example $[A] = [-1, 1]$, $[b] = [0, 1]$, $S = \mathbb{R}$ illustrates. Compactness is always absent if $[A]$ is not regular and $S \neq \emptyset$. In this case $S$ is unbounded even if the linear systems with singular matrices $A \in [A]$ are not solvable for any $b \in [b]$. Use Cramer’s rule, e.g., to prove this statement. Since half-spaces are convex and intersections of convex sets share this property one obtains it for $S \cap O$. The following example shows, however, that convexity need not hold for the whole solution set $S$.

**Example 1**

Let

$$[A] = \begin{pmatrix} 1 & 0 \\ [-1, 1] & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} [-1, 1] \\ 0 \end{pmatrix}. $$

From Theorem 2 we get the inequalities $|x_2| \leq |x_1| \leq 1$ which characterize the set $S$ completely. Its position in $\mathbb{R}^2$ can be seen from the subsequent Figure 1.

![Figure 1](image)

**Fig. 1** The solution set $S$ of Example 1
4 The symmetric solution set $S_{\text{sym}}$

4.1 Historical remarks and properties

In contrast to $S$ the symmetric solution set $S_{\text{sym}}$ as defined in (1) is much more
difficult to characterize. It occurs when the coefficients of a fixed linear system with
a symmetric matrix $A$ is perturbed by symmetric matrices only. Since $S_{\text{sym}} \subseteq S$
Neumaier advised in a letter to J. Rohn [22] in 1986 to consider this set for its
own. In his book [19] which appeared in 1990 he mentioned $S_{\text{sym}}$ in Example 3.4.2.
In the same year, Jansson gave a talk in Alberna in which he presented inner and
outer enclosures for $S_{\text{sym}}$ (cf. Section 5 of the present paper). For more general
dependencies such enclosures were given by Rump in his survey article [23] which
appeared in 1994. In 1993 an interval version of the Cholesky method was introduced
by the authors in [8] in order to enclose $S_{\text{sym}}$. Criteria of feasibility were proved there
and a corresponding perturbation result was given in [9]. This method was recently
extended to a block version by Schäfer in [25]. It was proved for the first time in 1995
in [9] that the boundary $\partial S_{\text{sym}}$ can be curvilinear – in contrast to the boundary $\partial S$
of $S$. The proof covered only $2 \times 2$ matrices and could unfortunately not be generalized
to the $n$–dimensional case. By means of the Fourier–Motzkin elimination process of
linear programming this was possible in 1996 in [2]; cf. also [3]. These results were
even generalized two years later in [4] where now affine dependencies were allowed.
In [7] the elimination process introduced in [2] and [4] was generalized in order to
handle particular dependencies such as Toeplitz and Hankel matrices. While the
approaches always took place in a fixed orthant it was possible in [18] to modify the
elimination process by an approach similar to the equivalence a) $\Leftrightarrow$ e) in Theorem 2
due to Beeck; cf. Section 4.3 of the present paper.

We start by an instructive $2 \times 2$ example whose features remain true also in the
$n \times n$ case.

Example 2

Let

$$[A] = \begin{pmatrix}
  1 & [1,2] \\
  [1,2] & [-1,0]
\end{pmatrix},
[b] = \begin{pmatrix}
  4 \\
  [1,2]
\end{pmatrix}.$$ 

Considering $A^{-1}b$ one easily recognizes $S_{\text{sym}} \subseteq S \subseteq O_1$. Anticipating Section 4.2 we
list the inequalities by which $S_{\text{sym}}$ is described:

$$4 - x_1 - x_2 \leq 0, $$

$$-4 + x_1 + 2x_2 \leq 0,$$

$$-2 + x_1 - x_2 \leq 0,$$

$$1 - 2x_1 \leq 0,$$

$$x_1^2 - 4x_1 + x_2 \leq 0,$$

$$x_1^2 + x_2^2 - 4x_1 + 2x_2 \geq 0.$$ 

The first four inequalities characterize $S$ while the last two restrict $x$ to be contained
in $S_{\text{sym}}$. 

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From Figure 2 one sees that \( S_{\text{sym}} \) is connected and compact. This remains true also in the general case as long as \( [A] \) is regular. The counterexamples for non-regular matrices are the same as in Section 3. As we shall see (Section 4.2 and 4.4) the boundary \( \partial S_{\text{sym}} \) of \( S_{\text{sym}} \) is composed by pieces of hyperplanes and quadrics; in particular, \( \partial S_{\text{sym}} \) can be curvilinear. In general, \( S_{\text{sym}} \) is not convex, and the same holds true for its intersection with an orthant. Up to now it is unknown whether \( S_{\text{sym}} \cap O \) is always connected.

4.2 Characterization of \( S_{\text{sym}} \) by means of a Fourier–Motzkin elimination

In this section we show that \( S_{\text{sym}} \cap O \) can be characterized by means of quadrics and hyperplanes. To this end we aim at describing \( S_{\text{sym}} \cap O \) by means of linear and quadratic inequalities whose coefficients consist of the bounds \( \underline{b}_i, \overline{b}_i \) and \( \underline{a}_{ij}, \overline{a}_{ij} \). Without loss of generality we choose the orthant \( O = O_1 \). We only indicate the algorithm by replacing \( b \) and \( a_{12} \) by \( \underline{b}, \overline{b} \) and \( \underline{a}_{12}, \overline{a}_{12} \), respectively. We get the following equivalences:

\[
x \in S_{\text{sym}} \cap O_1 \iff x \in O_1 \land \exists A = A^T \in [A], b \in [b] : Ax = b
\]

\[
\iff x \in O_1 \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n :
\]

\[
\left( a_{ij} = a_{ji} \land \left\{ \begin{array}{l}
\frac{b_i}{a_{ij}} \leq \sum_{j=1}^{n} a_{ij} x_j \leq \frac{\overline{b}_i}{a_{ij}} \\
\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}
\end{array} \right. \right)
\]
\( x \in S \cap O_1 \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n : \)

\[
\begin{align*}
(a_{ij} = a_{ji} \land \left\{ \begin{array}{l}
\begin{array}{l}
\text{if } x_k \neq 0, x_l \neq 0, \\
\text{if } x_k = 0 \text{ and } x_l \neq 0, \\
\text{if } x_k \neq 0 \text{ and } x_l = 0, \\
\text{if } x_k = 0 \text{ and } x_l = 0.
\end{array}
\end{array}
\right.
\right\}
\end{align*}
\]

Only the implication \( \Leftarrow \) of the last assertion requires a proof since \( x_i = 0 \) or \( x_j = 0 \) is possible. Let \( A = A^T \) be the matrix whose entries fulfill, by assumption, the inequalities in (5). If \( x_k \neq 0, x_l \neq 0 \) then obviously \( a_{kl} \in [a]_{kl} \). Otherwise nothing can be said on \( a_{kl} \) in this respect. From \( x \in S \cap O_1 \) we get \( a_{ij} \in [a]_{ij} \) such that \( \sum_{j=1}^{n} a_{ij} x_j = b_i \in [b]_i \). Define \( \tilde{A} \) by

\[
\tilde{a}_{kl} := \left\{ \begin{array}{l}
a_{kl} \text{ if } x_k \neq 0, x_l \neq 0, \\
\hat{a}_{kl} \text{ if } x_k = 0 \text{ and } x_l \neq 0, \\
\hat{a}_{lk} \text{ if } x_k \neq 0 \text{ and } x_l = 0, \\
\bar{a}_{kl} \text{ if } x_k = 0 \text{ and } x_l = 0.
\end{array}\right.
\]

Since we assumed \([A] = [A]^T\) when considering \( S_{\text{sym}} \), the midpoint \( \tilde{A} \) of \([A]\) is symmetric, and the definition of \( \tilde{A} \) shows that \( \tilde{A} \) has this property, too. Moreover we get \( \tilde{A} \in [A] \) and \( b \leq \tilde{A} x \leq \tilde{b} \) for the vector \( x \) under consideration. Note that

a) whenever \( a_{ij} \) occurs in (5) it has the same factor \( x_i x_j \);

b) trivial inequalities like \( 0 \leq 0 \) can be omitted;

c) there are only linear inequalities (cf. Theorem 2f) and quadratic ones;

d) no additional multiplication will be needed in the further stages of the algorithm.

We next isolate the \( a_{12} \)-term:

\[
x \in S_{\text{sym}} \cap O_1 \iff x \in S \cap O_1 \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n :
\]

\[
\left\{ \begin{array}{l}
\{b_1 - \sum_{j=1}^{n} a_{1j} x_j\} x_1 \leq a_{12} x_1 x_2 \\
\leq \{\tilde{b}_1 - \sum_{j=1}^{n} \hat{a}_{1j} x_j\} x_1 \\
\{b_2 - \sum_{j=1}^{n} a_{2j} x_j\} x_2 \leq a_{12} x_1 x_2 \\
\leq \{\tilde{b}_2 - \sum_{j=2}^{n} \hat{a}_{2j} x_j\} x_2 \\
a_{12} x_1 x_2 \leq a_{12} x_1 x_2 \leq \bar{a}_{12} x_1 x_2
\end{array} \right. .
\]

remaining \( a_{12} \)-free inequalities
Now those inequalities written down explicitly in (6) hold if and only if the maximum of their left-hand sides is less or equal than the minimum of their right-hand sides and this again holds if and only if each of these left-hand sides is less or equal than each of the right-hand sides. Therefore we get

\[ x \in S_{\text{sym}} \cap O_1 \iff x \in S \cap O_1 \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n, (i, j) \not\in \{(1, 2), (2, 1)\} : \]

\[
\begin{align*}
\left\{ \begin{array}{l}
\{ b_1 - \sum_{j=1}^{n} a_{1j} x_j \} x_1 \leq \{ b_{12} x_1 x_2 \} \\
\{ b_2 - \sum_{j=2}^{n} a_{2j} x_j \} x_2 \leq \{ b_{12} x_1 x_2 \}
\end{array} \right. \\
\{ b_1 - \sum_{j=1}^{n} a_{1j} x_j \} x_1 \leq \{ b_2 - \sum_{j=2}^{n} a_{2j} x_j \} x_2 \\
a_{12} x_1 x_2 \leq \{ b_1 - \sum_{j=1}^{n} a_{1j} x_j \} x_1 \\
a_{12} x_1 x_2 \leq \{ b_2 - \sum_{j=2}^{n} a_{2j} x_j \} x_2 \\
\{ b_2 - \sum_{j=2}^{n} a_{2j} x_j \} x_2 \leq \{ b_1 - \sum_{j=1}^{n} a_{1j} x_j \} x_1 \\
\text{remaining } a_{12}\text{-free inequalities}
\end{align*}
\]

No \( a_{12} \)–term appears now any longer in the set of inequalities, and the isolation and elimination process can be repeated for the other entries \( a_{ij} \). At the end we are left with a set of inequalities as indicated above. This process which was given in a slightly modified form in [2] must be repeated for any of the \( 2^n \) orthants, so the amount of inequalities can increase tremendously with \( n \) although sometimes inequalities can be combined and turn out to be contained in other ones.

We end this section with another example whose inequalities were derived by the method above.

**Example 3**

Let

\[
[A] := \begin{pmatrix}
1 \\
0, 1 \\
0, 1
\end{pmatrix}, \quad [b] := \begin{pmatrix}
[0, 2] \\
[0, 2]
\end{pmatrix}.
\]

Then \( S \cap O_1 \) is characterized by

\[
x_1 \leq 2, \quad x_2 \geq -0.5, \quad x_2 \geq -x_1, \quad x_1 \geq x_2,
\]

while \( S \cap O_4 \) with \( O_4 := \{ (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 \geq 0, \ x_2 \leq 0 \} \) can be described by

\[
x_1 \geq 0, \quad x_2 \geq -2, \quad x_2 \leq 2 - x_1, \quad x_2 \leq 0.25 x_1.
\]
For $S_{\text{sym}}$ the following additional inequalities are needed:

\[
\begin{cases}
4x_1^2 + (4x_2 + 1)^2 \geq 1, & (x_1 - 1)^2 + x_2^2 \leq 1 \quad \text{in the case } S_{\text{sym}} \cap O_1, \\
x_1^2 + 4x_2^2 \geq 0, & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2 \quad \text{in the case } S_{\text{sym}} \cap O_4.
\end{cases}
\]

The following Figure 3 shows $S$ and $S_{\text{sym}}$.

![Figure 3: The solution sets $S$ and $S_{\text{sym}}$ of Example 3](image)

**Fig. 3** The solution sets $S$ and $S_{\text{sym}}$ of Example 3

### 4.3 Characterization of $S_{\text{sym}}$ by means of sets

In this section we do not stick to a fixed orthant but describe $S_{\text{sym}}$ as a whole. This decreases the amount of relations drastically. For the algorithm we need a simple lemma which we formulate first.

**Lemma 1**

a) Let $[a]_i \in \mathbb{IR}$ for $i = 1, \ldots, n$, $n \geq 3$. Then

\[
\bigcap_{i=1}^{n} [a]_i \neq \emptyset \iff [a]_i \cap [a]_j \neq \emptyset \quad \text{for } i < j, \ i, j = 1, \ldots, n.
\]
b) Let $[a]$, $[b]$, $[c] \in \mathbb{R}$. Then

$$([a] + [b]) \cap [c] \neq \emptyset \iff [a] \cap ([c] - [b]) \neq \emptyset.$$ 

The proof is easy. Like the whole approach in this section it can be found in [18].

Now we show how to replace the parameters $[a]_{ij}$, $[b]_i$ by their lower and upper bounds.

$$x \in S_{\text{sym}} \iff \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n :$$

$$\left( a_{ij} = a_{ji} \land \begin{cases} \{ \sum_{j=1}^{n} a_{ij} x_j \} \cap [b]_i \neq \emptyset \\ \{ a_{ij} \} \cap [a]_{ij} \neq \emptyset \end{cases} \right)$$

$$\iff x \in S \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n :$$

$$\left( a_{ij} = a_{ji} \land \begin{cases} \{ a_{12} x_1 x_2 \} \cap ([b]_2 x_2 - \sum_{j=2}^{n} a_{2j} x_2 x_j) \neq \emptyset \\ \{ a_{12} x_1 x_2 \} \cap ([b]_2 x_2 - \sum_{j=2}^{n} a_{2j} x_2 x_j) \neq \emptyset \\ \{ a_{12} x_1 x_2 \} \cap [a]_{12} x_1 x_2 \neq \emptyset \end{cases} \right)$$

$$\iff x \in S \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n :$$

$$\left( a_{ij} = a_{ji} \land \begin{cases} [a]_{12} x_1 x_2 \cap ([b]_1 x_1 - \sum_{j=1}^{n} a_{1j} x_1 x_j) \\ [a]_{12} x_1 x_2 \cap ([b]_1 x_1 - \sum_{j=1}^{n} a_{1j} x_1 x_j) \end{cases} \right).$$

The multiplication by $x_i$ and $x_j$, respectively, is used as in Section 4.2. No additional multiplication is needed in the further course of the algorithm. In order to continue it is necessary to resolve the triple intersection by intersections between the two operands. The key to this is given by Lemma 1. We leave the details to the reader. The quadrics are seen here only at a second glance. They are hidden behind the intersections. However, since these intersections can be expressed by inequalities the approach in this section can be reformulated by formulas as in Section 4.2. Both approaches lack, however, a generalization to more than dependencies as they occur in symmetric ($a_{ij} = a_{ji}$), skew-symmetric ($a_{ij} = -a_{ji}$), and persymmetric matrices.
\( \langle a_{ij} \rangle = a_{n+1-j,n+1-i} \) respective. In order to generalize the elimination process in Section 4.2 we need a modification presented in Section 4.4.

We conclude the present section by the general \( 3 \times 3 \) case.

**Example 4**

Let \( [A] = [A]^T \in \mathbb{R}^{3 \times 3} \), \( [b] \in \mathbb{R}^3 \) and consider again \( S_{\text{sym}} \). Eliminating \( b_i \), \( a_{ii} \) for \( i = 1,2,3 \) and \( a_{12} \) leads to

\[
\begin{align*}
\text{x} \in S_{\text{sym}} & \iff x \in S \land \exists a_{13}, a_{23} \in \mathbb{R} : \left( a_{13} = a_{31} \land a_{23} = a_{32} \right) \\
& \land \left\{ \\
& \{a_{12} x_1 x_2 \} \cap \left( [b]_{1x_1} - [a]_{11} x_1^2 - [a]_{12} x_1 x_2 \right) \\
& \cap \left( [b]_{2x_2} - [a]_{22} x_2^2 - [a]_{23} x_2 x_3 \right) \neq \emptyset
\end{align*}
\]

After having resolved the triple intersection in (7) into simple intersections the isolation of \( a_{13} \) and its elimination yield to

\[
\begin{align*}
\text{x} \in S_{\text{sym}} & \iff x \in S \land \exists a_{23} \in \mathbb{R} : \left( a_{23} = a_{32} \right) \\
& \land \left\{ \\
& \{a_{13} x_1 x_2 \} \cap \left( [b]_{1x_1} - [a]_{11} x_1^2 - [a]_{12} x_1 x_2 \right) \\
& \cap \left( [b]_{3x_3} - a_{23} x_2 x_3 - [a]_{33} x_3^3 \right) \neq \emptyset
\end{align*}
\]

Here again we omitted an intersection which can be deduced from \( x \in S \). eliminating the last parameter \( a_{23} \) the fourfold intersection produces \( \binom{4}{2} \) simple intersections among which only one is \( a_{23} \)-free. This one turns out to be a consequence of \( x \in S \). Therefore, it can be omitted, and we are finally left with the equivalence

\[
\begin{align*}
\text{x} \in S_{\text{sym}} & \iff x \in S \\
& \land \{a\}_{23} x_2 x_3 \cap \left( [b]_{2x_2} - [a]_{12} x_1 x_2 - [a]_{22} x_2^2 \right) \\
& \cap \left( [b]_{3x_3} - [a]_{13} x_1 x_3 - [a]_{33} x_3^3 \right) \neq \emptyset
\end{align*}
\]

Using Lemma 1 a) this multiple intersection can equivalently be written as \( \binom{7}{2} = 21 \) simple ones. Two of them can be deduced from the enquiry \( x \in S \). The remaining 19 ones together with the three occuring from Beeck’s criterion in Theorem 2 d) result
in a total of 22 which all can be rewritten in the form \(0 \in \ldots\). Introducing
the interval bounds, fixing an orthant \(O\) and taking into account
\[
0 \in [a] \iff a \leq 0 \leq \bar{a}
\]
finally yields to 44 inequalities for the characterization of \(S_{\text{sym}} \cap O\) as was already
mentioned without proof in [2].

4.4 A generalizable approach for characterizing \(S_{\text{sym}}\)
The subsequent modification of Section 4.2 is based on the following theorem.

**Theorem 3**

Let \(f_{\lambda \mu}, g_{\lambda}, \lambda = 1, \ldots, k\ (\geq 2), \mu = 1, \ldots, m\), be real valued functions of \(x = (x_1, \ldots, x_n)^T\) on some subset \(D \subseteq \mathbb{R}^n\). Assume that there is a positive integer
\(k_1 < k\) such that
\[
\begin{align*}
f_{\lambda_1}(x) & \neq 0 \text{ for all } \lambda \in \{1, \ldots, k\}, \\
f_{\lambda_1}(x) & \geq 0 \text{ for all } x \in D \text{ and all } \lambda \in \{1, \ldots, k\},
\end{align*}
\]
for each \(x \in D\) there is an index \(\beta^* = \beta^*(x) \in \{1, \ldots, k_1\}\) with \(f_{\beta^*}(x) > 0\)
and an index \(\gamma^* = \gamma^*(x) \in \{k_1 + 1, \ldots, k\}\) with \(f_{\gamma^*}(x) > 0\).

For \(m\) parameters \(u_1, \ldots, u_m\) varying in \(\mathbb{R}\) and for \(x\) varying in \(D\) define the sets \(S_1, S_2\) by
\[
\begin{align*}
S_1 & := \{ x \in D \mid \exists u_\mu \in \mathbb{R}, \ \mu = 1, \ldots, m : (11), (12) \text{ hold} \}, \\
S_2 & := \{ x \in D \mid \exists u_\mu \in \mathbb{R}, \ \mu = 2, \ldots, m : (13) \text{ holds} \},
\end{align*}
\]
where the inequalities (11), (12) and (13), respectively, are given by
\[
\begin{align*}
g_\beta(x) + \sum_{\mu=2}^{m} f_{\beta \mu}(x) u_\mu & \leq f_{\beta_1}(x) u_1, \quad \beta = 1, \ldots, k_1, \\
f_{\gamma_1}(x) u_1 & \leq g_\gamma(x) + \sum_{\mu=2}^{m} f_{\gamma \mu}(x) u_\mu, \quad \gamma = k_1 + 1, \ldots, k,
\end{align*}
\]
and
\[
\begin{align*}
g_\beta(f_{\gamma_1}(x)) + \sum_{\mu=2}^{m} f_{\beta \mu}(x) f_{\gamma_1}(x) u_\mu & \leq g_\gamma(x) f_{\beta_1}(x) + \sum_{\mu=2}^{m} f_{\gamma \mu}(x) f_{\beta_1}(x) u_\mu, \\
\beta = 1, \ldots, k_1, \ \gamma = k_1 + 1, \ldots, k.
\end{align*}
\]
(Trivial inequalities such as \(0 \leq 0\) can be omitted.)

Then
\[
S_1 = S_2.
\]
The assertion of Theorem 3 even holds if \( f_{\beta_1}(x), f_{\gamma_1}(x) \) are replaced in (13) by \( \tilde{f}_{\beta_1}(x) \) and \( \tilde{f}_{\gamma_1}(x) \), respectively, where

\[
f_{\beta_1}(x) = h_{\beta_\gamma}(x) \tilde{f}_{\beta_1}(x), \quad f_{\gamma_1}(x) = h_{\beta_\gamma}(x) \tilde{f}_{\gamma_1}(x)
\]

with nonnegative functions \( \tilde{f}_{\beta_1}, \tilde{f}_{\gamma_1}, h_{\beta_\gamma} \) defined on \( D \).

In contrast to the procedure in Section 4.2 there is no need in Theorem 3 to find factors such that the expressions to be eliminated have the same shape in all inequalities after multiplication. This allows the application of this theorem also for more general dependencies of the entries of \( A \) and \( b \), respectively. We apply now the theorem to symmetric matrices with \( D := O_1 \) starting as in Section 4.2. Just for illustration, we choose \( a_{12} \) as the entry to be eliminated.

\[
x \in S_{\text{sym}} \cap O_1 \iff x \in O_1 \land \exists a_{i j} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n : \quad (a_{i j} = a_{j i} \land \left\{ \begin{array}{l} b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \overline{b}_i \\ a_{ij} \leq a_{ij} \leq \overline{a}_{ij} \end{array} \right. )
\]
\[ x \in O_1 \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n: \]

\[
\begin{cases}
  b_1 - \sum_{j=1 \atop j \neq 1}^{n} a_{1j}x_j & \leq a_{12}x_2 \\
  a_{12}x_2 & \leq b_1 - \sum_{j=1 \atop j \neq 1}^{n} a_{1j}x_j \\
  a_{12}x_2 & \leq b_1 - \sum_{j=1 \atop j \neq 2}^{n} a_{1j}x_j \\
  a_{12}x_1 & \leq b_2 - \sum_{j=2}^{n} a_{1j}x_j \\
  a_{12}x_1 & \leq b_2 - \sum_{j=2}^{n} a_{1j}x_j \\
  a_{12} & \leq a_{12} \\
  a_{12} & \leq \bar{a}_{12}
\end{cases}
\]

\[
\left( a_{ij} = a_{ji} \land \left\{ \begin{array}{l}
  b_1 - \sum_{j=1 \atop j \neq 1}^{n} a_{1j}x_j \leq \bar{a}_{12}x_2, \\
  a_{12}x_2 \leq b_1 - \sum_{j=1 \atop j \neq 1}^{n} a_{1j}x_j, \\
  b_2 - \sum_{j=2}^{n} a_{1j}x_j \leq \bar{a}_{12}x_1, \\
  a_{12}x_2 \leq b_2 - \sum_{j=2}^{n} a_{1j}x_j, \\
  b_1x_1 - \sum_{j=1 \atop j \neq 1}^{n} a_{1j}x_1x_j \leq \bar{b}_2x_2 - \sum_{j=2}^{n} a_{2j}x_2x_j, \\
  b_2x_2 - \sum_{j=2}^{n} a_{2j}x_2x_j \leq \bar{b}_1x_1 - \sum_{j=1 \atop j \neq 1}^{n} a_{1j}x_1x_j
\end{array} \right. \right)
\]

remaining \( a_{12} \)-free inequalities

\[ x \in O_1 \land \exists a_{ij} \in \mathbb{R} \text{ for } i, j = 1, \ldots, n, (i, j) \not\in \{(1, 2), (2, 1)\}: \]

The first four inequalities coincide with those in the elimination process for \( S \cap O_1 \).

The next two are new; they are apparently caused by the symmetry and contain quadratic polynomials. When eliminating \( a_{ij} \) for \((i, j) \not\in \{(1, 2), (2, 1)\}\) according to the remark below Theorem 3, no additional multiplication is needed in inequalities which contain quadratic polynomials. This is true because the function \( f_{\lambda, \mu} \) in front of \( a_{ij} \) reads \( f_{\lambda, \mu}(x) = x_i x_j \) in these inequalities, and in the remaining (non–quadratic) inequalities they are given by \( f_{\lambda, \mu}(x) = x_i \), \( f_{\lambda, \mu}(x) = x_j \) and \( f_{\lambda, \mu}(x) = 1 \), respectively. Note that the sign of these functions remains constant over the orthant \( O_1 \). This is the reason, why no splitting is needed for \( D = O_1 \) during the elimination process.
Pursuing this process shows that the final inequalities for $S_{\text{sym}} \cap O_1$ consist of the inequalities which characterize $S \cap O_1$, and quadratic inequalities.

5 Enclosures for $S_{\text{sym}}$

As the preceding sections show it is generally not easy to decide whether a given vector $x$ belongs to $S_{\text{sym}}$ for given $[A] = [A]^T \in \mathbb{R}^{n \times n}$, $[b] \in \mathbb{R}$. Therefore, it is interesting to look for bounds of $S_{\text{sym}}$. We restrict ourselves to bounds which are not at the same time bounds of $S$, i.e., we look for bounds which should be better than the latter ones. Up to now there are essentially two methods with this property: a direct one and an iterative one. The first generalizes the well-known Cholesky method and uses the definitions

$$[a]^2 := \{ a^2 \mid a \in [a] \}, \quad \sqrt{|a|} := \{ \sqrt{a} \mid a \in [a] \},$$

where $a \geq 0$ is assumed for the square root. The algorithm which was first considered in [8] reads

Step 1: ‘$LL^T$–decomposition’

\begin{verbatim}
for j := 1 to n do
    [l]_{jj} := ( [a]_{jj} - \sum_{k=1}^{j-1} [l]_{jk}^2 )^{1/2} ;
for i := j + 1 to n do
    [l]_{ij} := ( [a]_{ij} - \sum_{k=1}^{j-1} [l]_{ik}[l]_{jk} ) / [l]_{jj} ;
\end{verbatim}

Step 2: Forward substitution

\begin{verbatim}
for i := 1 to n do
    [y]_i := ( [b]_i - \sum_{j=1}^{i-1} [l]_{ij}[y]_j ) / [l]_{ii} ;
\end{verbatim}

Step 3: Backward substitution

\begin{verbatim}
for i := n downto 1 do
    [x]^C_i := ( [y]_i - \sum_{j=i+1}^{n} [l]_{ij}[x]^C_j ) / [l]_{ii} .
\end{verbatim}

It is easy to see by the inclusion monotonicity of the interval arithmetic operations that $S_{\text{sym}} \subseteq [x]^C$ holds. One also sees directly that the feasibility of the preceding interval Cholesky method does not depend on $[b]$ and requires that each matrix $A \in [A]$ is symmetric and positive definite. Unfortunately, this property is not sufficient. As was shown in [8] the feasibility is guaranteed if $[A]$ is a symmetric
$H$ matrix satisfying $a_{ii} > 0$ for $i = 1, \ldots, n$. Interval matrices $[A]$ with a strictly diagonally dominant comparison matrix $([A])$ and $a_{ii} > 0$, $i = 1, \ldots, n$, belong to this class. Moreover, if the algorithm is feasible for $[A]$ and if $\rho([A]^{C}[q([A],[B])) < 1$ then it is feasible for $[B]$, too, as was proved in [9]. Here, $[A]^{C} := ([L]^{T})^{-1}([L])^{-1}$ where $[L] = ([l_{ij}]$ is the lower triangular interval matrix whose non-zero entries $l_{ij}$, $i \geq j$, are defined by the algorithm above. Replacing $[A]$, $[B]$ by $A$, $[A]$, respectively, proves that the interval Cholesky method is feasible whenever $A$ is positive definite and satisfies $\rho([A]^{C}[\text{rad} ([A])) < 1$. The following example shows that the interval Cholesky method is tailored to enclose $S_{\text{sym}}$, but not $S$. However, it does not always yield to enclosures for $S_{\text{sym}}$ which are better than general enclosures for $S$. In particular, for larger $n$ it may show a bad behavior due to rounding errors and data dependencies.

**Example 5**

Let

$$[A] := \begin{pmatrix} 4 & -1,1 \\ -1,1 & 4 \end{pmatrix}, \quad [b] := \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$  

Setting $A := \begin{pmatrix} 4 & \alpha \\ \beta & 4 \end{pmatrix}$ for $A \in [A]$, we get

$$A^{-1}b = \frac{6}{16 - \alpha \beta} \begin{pmatrix} 4 - \alpha \\ 4 - \beta \end{pmatrix} \quad \text{with } \alpha, \beta \in [-1,1].$$

Since $\beta = \alpha$ in the case $A = A^{T} \in [A]$ one obtains

$$\mathcal{G} \mathcal{S}_{\text{sym}} = ([\frac{18}{17}, 2], [\frac{18}{17}, 2])^{T}, \quad \mathcal{G} S = ([\frac{18}{17}, 2], [\frac{18}{17}, 2])^{T},$$

$$[x]^{G} = ([1, 2], [\frac{18}{17}, 2])^{T}, \quad [x]^{G} = ([1, 2], [\frac{18}{17}, 2])^{T},$$

where $[x]^{G}$ denotes the vector resulting from the interval Gaussian algorithm (cf. [1]). The sets

$$S_{\text{sym}} = \left\{ \frac{6}{4 + \alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid -1 \leq \alpha \leq 1 \right\} = \left\{ \gamma \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid \frac{6}{5} \leq \gamma \leq 2 \right\}$$

and (see [12])

$$S = \text{convex hull} \left\{ \left[ \frac{6}{5}, \frac{6}{5} \right]^{T}, \ (2, 2)^{T}, \ [\frac{18}{17}, \frac{30}{17}]^{T}, \ [\frac{30}{17}, \frac{18}{17}]^{T} \right\}$$

can be seen in Figure 4.
Example 5 illustrates that the following properties can occur.

1) $\mathcal{D} S_{\text{sym}} \neq \mathcal{D} S$ (cf. also [19], p. 95),

2) $\mathcal{D} S_{\text{sym}} \neq [x]^C$,

3) $\mathcal{D} S \neq [x]^G$;

4) $S \not\subseteq [x]^C$ (but $S_{\text{sym}} \subseteq [x]^C$),

5) $[x]^C \subseteq [x]^G$ with $[x]^C \neq [x]^G$.

Unfortunately, $[x]^G \subseteq [x]^C$ with $[x]^G \neq [x]^C$ can also occur as was shown by an example in [8].

The second method which we want to recall was introduced by Jansson [15], [16] and is a modification of the following well-known iterative method for general linear interval systems

$$[x]^{k+1} = \bar{x} + C([b] - [A]\bar{x}) + (I - C[A])([x]^k - \bar{x}), \quad k = 0, 1, \ldots$$

where $C \in \mathbb{R}^{n \times n}$ is any preconditioning matrix and $\bar{x}$ is any fixed vector from $\mathbb{R}^n$. Usually, $C$ is an approximate inverse of the midpoint $\bar{A}$ and $\bar{x}$ is an approximate solution of the midpoint equation $\bar{A}x = \bar{b}$. Exploiting the symmetry of $A \in [A]$ and of $[A] = [A]^T$ and substituting $[x]^k_\Delta := [x]^k - \bar{x}$ finally yields to

$$[x]^{k+1}_\Delta = [z]^{\text{sym}} + (I - C[A])[x]^k_\Delta, \quad k = 0, 1, \ldots$$

(14)
with

\[ [z]_{i}^{\text{sym}} := \sum_{j=1}^{n} c_{ij} ([b]_{j} - [a]_{jj} \hat{x}_{j}) - \sum_{j=1}^{n} \sum_{i=1}^{j-1} (c_{ij} \hat{x}_{i} + c_{i} \hat{x}_{j}) [a]_{ji} \]

\[ = \{ (C(b - A \hat{x}))_{i} \mid A = A^T \in [A], b \in [b] \} \quad \text{for } i = 1, \ldots, n. \]

Note that the last equality holds by virtue of Theorem 1. Thus \([z]^{\text{sym}}\) is the interval hull of the set \( \{ (C(b - A \hat{x}))_{i} \mid A = A^T \in [A], b \in [b] \} \) and is therefore optimal. The subsequent results were proved in [15], [16].

**Theorem 4**

Let \((S_{\text{sym}})_{i}\) denote the projection of \(S_{\text{sym}}\) onto the \(x_{i}\) coordinate axis and let

\[ [\Delta]^{k} := (I - C[A])[x]^{k}_{\Delta}. \]

If

\[ ([x]^{k_{0}+1}_{\Delta})_{i} \subseteq ([x]^{k_{0}}_{\Delta})_{i} \quad \text{for } i = 1, \ldots, n \]

then the following assertions hold.

a) \([A]\) and \(C\) are regular.

b) \(S_{\text{sym}} \subseteq \hat{x} + [x]^{k_{0}+1}_{\Delta} \subseteq \hat{x} + [x]^{k}_{\Delta} \quad \text{for } k \geq k_{0}.\)

c) \(\bar{x}_{i} + \bar{z}_{i}^{\text{sym}} + \bar{\Delta}_{i}^{k_{0}} \leq \min (S_{\text{sym}})_{i} \leq \bar{x}_{i} + \bar{z}_{i}^{\text{sym}} + \bar{\Delta}_{i}^{k_{0}}, \]

\(\bar{x}_{i} + \bar{z}_{i}^{\text{sym}} + \bar{\Delta}_{i}^{k_{0}} \leq \max (S_{\text{sym}})_{i} \leq \bar{x}_{i} + \bar{z}_{i}^{\text{sym}} + \bar{\Delta}_{i}^{k_{0}}.\)

\[ \square \]

By Theorem 4 one sees at once that the relations

\[ \inf S_{\text{sym}} \in \hat{x} + \bar{z}_{i}^{\text{sym}} + [\Delta]^{k_{0}} \]

\[ \sup S_{\text{sym}} \in \hat{x} + \bar{z}_{i}^{\text{sym}} + [\Delta]^{k_{0}} \]

hold. Therefore, if \(\text{rad} [\Delta]^{k_{0}}\) is small then the enclosure for \(S_{\text{sym}}\) is good. This can be expected, in particular, if \(\text{rad} [A], \text{rad} [b]\) are small, \(C \approx A^{-1}\) and \(\hat{x} \approx A^{-1}b\). Then

\(\|\text{rad} [x]^{k_{0}}_{\Delta}\|_{\infty} \ll 1\) is possible whence \([\Delta]^{k_{0}}\) is quadratically small.

**6 The symmetric eigenpair set**

Our final section is devoted to the eigenpair sets \(E\) and \(E_{\text{sym}}\) from (3) and (4), respectively. In order to characterize \(E\) we apply the Oettli–Prager theorem to the equation \((A - \lambda I)x = 0\) which leads to

\((x^T, \lambda)^T \in E \iff \| (\hat{A} - \lambda I) x \| \leq (\text{rad} [A]) \| x \|. \) (15)

There is a slight difficulty in comparison with Section 3 in so far as the matrix \(A - \lambda I\) is not fixed but depends on the last component of the vector \((x^T, \lambda)^T\) tested to belong to \(E\). This difficulty can be overcome thinking \(\lambda\) to be arbitrary but fixed. Then \(A - \lambda I\) is a fixed matrix for which Theorem 2 can be applied without any
restrictions. Rewriting (15) as in Theorem 2f) by means of inequalities shows that 
\( E \cap O \) can be characterized by a variety of inequalities which – by virtue of the 
\( \lambda x \) term are at most quadratic. That means that \( E \) is the union of finitely many intersections of sets whose boundaries are pieces of hyperplanes and quadrics. We
mention that the inequalities for \( E \) can also be obtained by applying Theorem 3.
Using any approach of Section 4 shows that the symmetric eigenpair set \( E_{\text{sym}} \) can
be described by means of inequalities with polynomials of order three at most. The
starting point is now formed by the double inequality

\[
(A - \lambda I)x \leq 0 \leq (A - \lambda I)x.
\]

which is equivalent to \( (A - \lambda I)x = 0 \). For details we refer to [6]. We conclude our
paper with an illustrative example for \( E \) and \( E_{\text{sym}} \).

Example 6

Let

\[
[A] = \begin{pmatrix}
1 & [-1, 1] \\
[-1, 1] & 1
\end{pmatrix}.
\]

Then \( E \) is characterized by the inequalities

\[
\begin{cases}
  x_1 - |x_2| \leq \lambda x_1 \leq x_1 + |x_2|, \\
  -|x_1| + x_2 \leq \lambda x_2 \leq |x_1| + x_2.
\end{cases}
\]

If \( \lambda = 1 \) then any vector \((x_1, x_2)^T \neq 0\) occurs as an eigenvector hence \( E \subseteq \mathbb{R}^3 \)
contains the plane \( \lambda = 1 \) punctured at \((0, 0, 1)^T\). If \( \lambda \neq 1 \) then \( x_1 \cdot x_2 \neq 0 \) and
\((x^T, \lambda)^T \in E\) if and only if the subsequent inequalities hold.

\[
\begin{cases}
  1 - \frac{|x_2|}{x_1} \leq \lambda \leq 1 + \frac{|x_2|}{x_1}, \\
  1 - \frac{x_1}{|x_2|} \leq \lambda \leq 1 + \frac{x_1}{|x_2|}.
\end{cases}
\]

Restricting \( A \in [A] \) to be symmetric yields to any pair \((x_1, x_2)^T \neq 0\) as eigenvector
in the case \( \lambda = 1 \). This is true by virtue of \( I = I^T \in [A] \). If \( \lambda \neq 1 \) then \( E_{\text{sym}} \) is
described by \( x_1 = |x_2| > 0, -1 \leq 1 - \lambda \leq 1 \) which together with \( E \) leads to the
following Figure 5 in which we intersected these sets with the plane \( P_1 : x_2 = 1 \).
Fig. 5  \( E_1 := E \cap P_1 \) and \( E_{\text{sym}} \cap P_1 \) (dashed) of Example 6

References


