Computing Higher Central Moments for Interval Data

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Abstract

Higher central moments are very useful in statistical analysis: the third moment $M_3$ characterizes asymmetry of the corresponding probability distribution, the fourth moment $M_4$ describes the size of the distribution’s tails, etc. When we know the exact values $x_1, \ldots, x_n$, we can use the known formulas for computing the corresponding sample central moments. In many practical situations, however, we only know intervals $x_1, \ldots, x_n$ of possible values of $x_i$; in such situations, we want to know the range of possible values of $M_n$. In this paper, we propose algorithms that compute such ranges.

1 Formulation of the Problem

In engineering and science, when we have $n$ measurement results $x_1, \ldots, x_n$, traditional statistical approach (see, e.g., [1, 4]) usually starts with computing their (sample) average

\[ E = \bar{x} = \frac{x_1 + \ldots + x_n}{n} \]

and their (sample) variance

\[ V = \frac{(x_1 - E)^2 + \ldots + (x_n - E)^2}{n}. \]

(Often, a standard deviation $\sigma \overset{\text{def}}{=} \sqrt{V}$ is used instead of $V$.) Variance is a particular case of a central moment

\[ M_m = \frac{(x_1 - E)^m + \ldots + (x_n - E)^m}{n} \tag{1} \]
corresponding to \( m = 2 \). Higher moments – i.e., moments corresponding to \( m = 3, 4, \ldots \) – are also used in engineering and science. For example, the third central moment \( M_3 \) is used to describe the asymmetry of the corresponding probability distribution, and the fourth central moment \( M_4 \) is used to describe the size of the distribution’s tails. To be more precise, skewness \( M_3 / \sigma^3 \) is used to characterize asymmetry, and kurtosis \( M_4 / \sigma^4 - 3 \) is used to characterize the size of the tails (3 is subtracted so that kurtosis be equal to 0 for the practically frequent case of a normal distribution).

In addition to central moments, sometimes, non-central moments are also used:

\[
M'_m = \frac{x_1^m + \ldots + x_n^m}{n}.
\]

When we know the exact values of \( x_i \), then we can compute each moment by using the explicit formula (1). In some practical situations, however, we only have intervals \( x_i = [x_i, \bar{x}_i] \) of possible values of \( x_i \). This happens, for example, if instead of observing the actual value \( x_i \) of the random variable, we observe the value \( \bar{x}_i \) measured by an instrument with a known upper bound \( \Delta_i \) on the measurement error (and no information about probabilities of different possible values of measurement error); then, the actual (unknown) value is within the interval \( x_i = [\bar{x}_i - \Delta_i, \bar{x}_i + \Delta_i] \).

As a result, the sets of possible values of \( E, V \), and \( M_4 \) are also intervals. Since \( E \) is an increasing function of each of the variables \( x_i \), it is easy to compute the interval \( E = [E, \bar{E}] \) of possible values of \( E \):

\[
E = \frac{\bar{x}_1 + \ldots + \bar{x}_n}{n}; \quad \bar{E} = \frac{x_1 + \ldots + x_n}{n}.
\]

Similarly, we can easily compute the exact bounds for non-central moments \( M'_m \). What is the interval \([M_m, \bar{M}_m]\) of possible values for the central moment \( M_m \)? In [2, 3], we have analyzed this problem for the case of variance (\( m = 2 \)). In this case, we have shown the that the general problem of computing the interval \([\underline{\underline{L}}, \overline{\overline{V}}]\) exactly is NP-hard. We also showed:

- that there exists a quadratic-time algorithm for computing \( \overline{\overline{V}} \); and
- that there exists a quadratic-time algorithm for computing \( \overline{\overline{V}} \) for the case when the intervals do not all group together, i.e., crudely speaking, when for some \( C \), every \( C \) different intervals \( x_{i_1}, \ldots, x_{i_c} \) have an empty intersection.

In this paper, we extend these algorithms to higher central moments.

2 First Result: Computing \( M_m \) for Even \( m \)

**Theorem 1.** For every even \( m \), there exists an algorithm that computes \( M_m \) in quadratic time.
This algorithm is as follows:

- First, we sort all $2n$ values $x_i, x_i'$ into a sequence $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(2n)}$.
  This sequence divides the real line into $2n+1$ segments $[x_{(k)}, x_{(k+1)}]$, where
  $k = 0, \ldots, 2n$, $x_{(0)} \overset{\text{def}}{=} -\infty$, and $x_{(2n+1)} \overset{\text{def}}{=} +\infty$.

- For each of these segments $[x_{(k)}, x_{(k+1)}]$, we do the following:
  - First, we define $x_1, \ldots, x_n$ as the following linear functions of $\alpha$:
    - if $x_i \geq x_{(k+1)}$, we take $x_i = x_i'$ (independent of $\alpha$);
    - if $x_i \leq x_{(k)}$, we take $x_i = x_i'$ (independent of $\alpha$);
    - in all other cases, we take $x_i = x_i$.
  - Based on these expressions for $x_i$, we find the expression for
    \[ E = \frac{x_1 + \ldots + x_n}{n} \]
    as a linear function of $\alpha$.
  - Then, we substitute these expressions for $x_i$ and $E$ into the equation
    \[ \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - E)^{m-1} = (\alpha - E)^{m-1}. \]
    This equation is a polynomial equation of order $\leq m - 2$ in terms of the unknown $\alpha$ (terms proportional to $\alpha^{m-1}$ cancel out), so it has $\leq m - 2$ solutions. We compute these solutions.
  - For each of the solutions that is inside the segment $[x_{(k)}, x_{(k+1)}]$, we substitute the corresponding value $\alpha$ into the formulas for $x_i$ and $E$; thus, we compute $x_i$ and $E$; based on these values, we compute
    \[ M_m = \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - E)^{m}. \]

- The smallest of thus computed values $M_m$ is the desired lower bound $M_n$.

For reader’s convenience, the proof that this algorithm indeed computes $V$ in quadratic time (and all other proofs) is placed in the special Proofs section at the end of this paper.

### 3 Second Result: Computing $\overline{M}_n$ for Even $m$ in Time $2^n$

For small number of measurements, we can use the following algorithm to compute $\overline{M}_m$:

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Theorem 2. For every even $m$, there exists an algorithm that computes $\bar{M}_m$ in time $O(2^n)$.

The algorithm is as follows: for each $i$, we select either $x_i = \underline{z}_i$ or $x_i = \overline{z}_i$. For each $i$ from 1 to $n$, there are two options, so totally, we have $2^n$ combinations to try. For each of these combinations, we compute $M_m$; the largest of the resulting $2^n$ values is the desired upper bound $\bar{M}_m$.

4 Third Result: Computing $\bar{M}_m$ for Even $m$ in Quadratic Time (Case When Intervals Do Not Group Together)

Sets $S_1, \ldots, S_n$ are called pairwise disjoint if every pair has an empty intersection, i.e., if $S_i \cap S_j = \emptyset$ for all $i \neq j$. We can generalize this definition from pairs to tuples of arbitrary size $C$:

Definition 1. Let $C \geq 2$ be an integer. We say that a sequence of sets $S_1, \ldots, S_n$ is $C$-wise disjoint if for every $C$ different indices $i_1, \ldots, i_C$, we have $S_{i_1} \cap \cdots \cap S_{i_C} = \emptyset$.

Theorem 3. For every even $m$ and for every $C \geq 2$, there exists an algorithm that computes $\bar{M}_m$ in quadratic time when the input intervals $x_1, \ldots, x_n$ are $C$-wise disjoint.

This algorithm is as follows:

- First, we sort all $2n$ values $\underline{z}_i, \overline{z}_i$ into a sequence $x^{(1)} \leq x^{(2)} \leq \ldots \leq x^{(2n)}$.
  This sequence divides the real line into $2n+1$ segments $[x^{(k)}, x^{(k+1)}]$, where $k = 0, \ldots, 2n$, $x^{(0)} \equiv -\infty$, and $x^{(2n+1)} \equiv +\infty$.
- For each of these segments $[x^{(k)}, x^{(k+1)}]$, we do the following:
  - First, we describe several combinations $(x_1, \ldots, x_n)$ as follows:
    * if $\overline{z}_i < x^{(k)}$, we take $x_i = \underline{z}_i$;
    * if $\underline{z}_i > x^{(k+1)}$, we take $x_i = \overline{z}_i$;
    * for all other indices $i$ (there are $\leq C$ of them), we consider all possible combinations of $x_i = \underline{z}_i$ and $x_i = \overline{z}_i$.
  As a result, we get $\leq 2^C$ different combinations.
  - For each resulting combination $(x_1, \ldots, x_n)$, we compute $E$ as the average of all the values $x_i$, then we compute
    \[
    M_{m-1} = \frac{1}{n} \sum_{i=1}^{n} (x_i - E)^{m-1},
    \]
    and $\alpha = E + M_{m-1}^{1/(m-1)}$. 

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For each combination for which the resulting value $\alpha$ is within the segment $[x_{(k)}, x_{(k+1)}]$, we compute

$$M_m = \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - E)^m.$$  

- The largest of thus computed values $M_m$ is the desired upper bound $\overline{M}_m$.

5 Fourth Result: Computing $\underline{M}_m$ and $\overline{M}_m$ for Odd $m$ in Cubic Time (Case When Intervals Do Not Group Together)

**Theorem 4.** For every odd $m$ and for every $C \geq 2$, there exists an algorithm that computes $\overline{M}_m$ in cubic time when the input intervals $x_1, \ldots, x_n$ are $C$-wise disjoint.

The algorithm for computing $\underline{M}_m$ is as follows:

- First, we sort all $2n$ values $x_i, \overline{x}_i$ into a sequence $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(2n)}$.
  This sequence divides the real line into $2n+1$ segments $[x_{(k)}, x_{(k+1)}]$, where $k = 0, \ldots, 2n$, $x_{(0)} \overset{\text{def}}{=} -\infty$, and $x_{(2n+1)} \overset{\text{def}}{=} +\infty$.

- For each pair of segments $[x_{(k)}, x_{(k+1)}]$ and $[x_{(l)}, x_{(l+1)}]$, $k \leq l$, we do the following:
  - First, we describe several combinations $(x_1, \ldots, x_n)$ as linear functions of $\alpha^-$ and $\alpha^+$ as follows:
    * if $x_i < x_{(k)}$, we take $x_i = \overline{x}_i$;
    * if $x_i > x_{(k+1)}$, we take $x_i = \overline{x}_i$;
    * for all other indices $i$ (there are $\leq 2C$ of them), we consider all possible combinations of $x_i = \overline{x}_i, x_i = x_{(k)}, x_i = \alpha^-$ (if $[x_{(k)}, x_{(k+1)}] \subseteq x_i$) and $x_i = \alpha^+$ (if $[x_{(l)}, x_{(l+1)}] \subseteq x_i$).
  As a result, we get $\leq 4^2C$ different combinations.

- For each resulting combination $(x_1, \ldots, x_n)$, we find the expression for $E$ as the average of all the values $x_i$. Then, we substitute the expressions for $x_i$ and $E$ into the system of equations

$$\frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - E)^{m-1} = (E - \alpha^-)^{m-1};$$

$$\frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - E)^{m-1} = (\alpha^+ - M)^{m-1}.$$
We compute all solutions of this system of polynomial equations with
unknowns $\alpha^-$ and $\alpha^+$.

- For each of the solutions for which $\alpha^- \in [x_{(k)}, x_{(k+1)}]$ and $\alpha^+ \in [x_{(l)}, x_{(l+1)}]$, we substitute the corresponding values $\alpha^-$ and $\alpha^+$ into the formulas for $x_i$ and $E$, thus, we compute $x_i$ and $E$; based on these values, we compute

$$M_m = \frac{1}{n} \sum_{i=1}^{n} (x_i - E)^m.$$ 

- The smallest of thus computed values $M_m$ is the desired lower bound $\underline{M}_m$.

**Theorem 5.** For every odd $m$ and for every $C \geq 2$, there exists an algorithm that computes $\underline{M}_m$ in cubic time when the input intervals $x_1, \ldots, x_n$ are $C$-wise disjoint.

Since $m$ is odd, the $m$-th central moment of the values $x_1, \ldots, x_n$ is equal to minus the $m$-th moment of the values $-x_1, \ldots, -x_n$. Turning to $-x_i$ changes largest and smallest values and vice versa. Thus, to compute $\underline{M}_m$ for the intervals $x_i = [z_i, \overline{z}_i]$, it is sufficient to compute the lower bound $\underline{M}_m$ for the intervals $-x_i = [-\overline{z}_i, -z_i]$, and then change the sign of the resulting bound. Since we can use the above cubic-time algorithm to compute $\underline{M}_m$, we thus get a cubic-time algorithm for computing $\underline{M}_m$.

# 6 Proofs

## 6.1 Proof of Theorem 1

The central moment $M_m$ is a continuous function of $n$ variables; thus, its smallest possible value on a compact box $x_1 \times \ldots \times x_n$ is attained at some point $(x_1^{(0)}, \ldots, x_n^{(0)})$. Since the function $M_m$ is also smooth, for each variable $i$ for which the interval $x_i$ is non-degenerate (i.e., of finite width), the minimum is attained either when $x_i$ is inside the corresponding interval $(z_i, \overline{z}_i)$ and the derivative $\partial M_m / \partial x_i$ is equal to 0, or when $x_i^{(0)}$ coincides with one of the endpoints of this interval. To be more precise, we must have one of the following three situations:

- either $x_i^{(0)} \in (z_i, \overline{z}_i)$ and $\partial M_m / \partial x_i = 0$;
- or $x_i^{(0)} = \underline{z}_i$ and $\partial M_m / \partial x_i \geq 0$;
- or $x_i^{(0)} = \overline{z}_i$ and $\partial M_m / \partial x_i \leq 0$. 

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Differentiating \( M_m \) w.r.t. \( x_i \), and taking into consideration that \( \partial E / \partial x_i = 1/n \), we conclude that

\[
\frac{\partial M_m}{\partial x_i} = \frac{1}{n} \cdot m \cdot (x_i - E)^{m-1} + \frac{1}{n} \sum_{j=1}^{n} m \cdot (x_j - E)^{m-1} \cdot \left(-\frac{1}{n}\right).
\]

The sum in this formula is proportional to the \((m-1)\)-st central moment \( M_{m-1} \), so the above formula can be simplified into:

\[
\frac{\partial M_m}{\partial x_i} = \frac{m}{n} \cdot ((x_i - E)^{m-1} - M_{m-1}).
\]  \hspace{1cm} (2)

Due to this formula:

- if \( \partial M_m / \partial x_i = 0 \), then \((x_i - E)^{m-1} = M_{m-1} \), hence \( x_i = \alpha \), where \( \alpha \) is the \( m \)-th root of \( M_{m-1} \);
- if \( \partial M_m / \partial x_i \geq 0 \), then \((x_i - E)^{m-1} \geq M_{m-1} \), hence (since the function \( z \rightarrow z^{1/(m-1)} \) is increasing for even \( m \)) \( x_i \geq \alpha \);
- if \( \partial M_m / \partial x_i \leq 0 \), then \((x_i - E)^{m-1} \leq M_{m-1} \), hence \( x_i \leq \alpha \).

Therefore, the above conditions on \( x_i^{(0)} \) can be reformulated as follows:

- either \( x_i^{(0)} \in (z_i, \bar{x}_i) \) and \( x_i^{(0)} = \alpha \); in this case, \( z_i < \alpha < \bar{x}_i \);
- or \( x_i^{(0)} = z_i \) and \( x_i^{(0)} = \bar{x}_i \); hence \( \alpha \); in this case, \( z_i \leq x_i \);
- or \( x_i^{(0)} = \bar{x}_i \) and \( x_i^{(0)} = \alpha \); hence \( \alpha \); in this case, \( \alpha \leq \bar{x}_i \).

Hence, once we know \( \alpha \), we can determine all \( n \) values \( x_i^{(0)} \) as follows:

- if \( x_i \leq \alpha \), then we cannot have the first case (when \( \alpha < \bar{x}_i \)) or the second case (when \( \alpha \leq \bar{x}_i \); hence \( \alpha < \bar{x}_i \)); therefore, we can only have the third case, when \( x_i^{(0)} = \bar{x}_i \);
- similarly, if \( \alpha \leq z_i \), then we must have \( x_i^{(0)} = z_i \);
- finally, if \( z_i < \alpha < \bar{x}_i \), then we must have \( x_i^{(0)} = \alpha \).

The only thing that remains is to find \( \alpha \). Once we know to which of \( 2n + 1 \) segments \([x_k, x_{k+1}]\) the value \( \alpha \) belongs, we can uniquely describe all the values \( x_i \) as linear functions of \( \alpha \), and then define \( \alpha \) from the condition that \( \alpha = E + M_{m-1}^{1/(m-1)} \), i.e., equivalently, that \( M_{m-1} = (\alpha - E)^{m-1} \). This is exactly what our algorithm does.

This proves that our algorithm is correct. To complete the proof, we must also show that this algorithm requires quadratic time.

Indeed, sorting requires \( O(n \cdot \log(n)) \) steps, and the rest of the algorithm requires linear time \( (O(n)) \) for each of \( 2n \) segments, i.e., the total quadratic time. The theorem is proven.
6.2 Proof of Theorem 2

Similarly to the proof of Theorem 1, we can conclude that for every \( i \), the maximum of \( M_m \) over the interval \( x_i \) is attained either inside the interval (when the partial derivative is 0) or at one of the endpoint of this interval. Thus, to prove that our algorithm is correct, we must show that the maximum of \( M_m \) cannot be attained for \( x_i \in (Z_i, X_i) \), when \( \partial M_m / \partial x_i = 0 \). Indeed, in the maximum point, the second derivative \( \partial^2 M_m / \partial x_i^2 \) must be non-positive. In the proof of Theorem 1, we have already derived an explicit formula (2) for \( \partial M_m / \partial x_i \). The formula (2) describes this derivative in terms of \( M_{m-1} \), so when we differentiate both sides of the formula (2), we can use the same expression for the derivative of \( M_{m-1} \). As a result, we get the following:

\[
\frac{\partial^2 M_m}{\partial x_i^2} = \frac{m}{n} \cdot T,
\]

where

\[
T = (m - 1) \cdot (x_i - E)^{m-2} - (m - 1) \cdot (x_i - E)^{m-2} \cdot \frac{1}{n}.
\]

\[
\frac{m - 1}{n} \cdot (x_i - E)^{m-2} + \frac{m - 1}{n} \cdot M_{m-2} = \frac{m - 1}{n} \cdot ((n - 2) \cdot (x_i - E)^{m-2} + M_{m-2}).
\]

In the trivial case of \( n = 1 \), all central moments are 0. When \( n \geq 2 \), both terms are non-negative, so the second derivative is non-negative. We know that the second derivative must be non-positive, so it must be equal to 0. Since the sum of two non-negative numbers is equal to 0, both numbers are equal to 0, in particular,

\[
M_{m-2} = \frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - E)^{m-2} = 0.
\]

Therefore, all the values \( x_i \) are identically equal to \( E \). In this case, \( M_m = 0 \), so this cannot be where the largest possible value of \( M_m \) is attained. This contradiction shows that the maximum cannot be attained inside the interval \( x_i \), hence it attained at the endpoints. The theorem is proven.

6.3 Proof of Theorem 3

We have already proven, in Theorem 2, that maximum can only be attained at one of the endpoints of the interval \([Z_i, X_i]\), i.e., when \( x_i^{(0)} = Z_i \) or \( x_i^{(0)} = X_i \). Hence, for each \( i \), we have one of the following two situations:

- either \( x_i^{(0)} = Z_i \) and \( \partial M_m / \partial x_i \leq 0 \);
- or \( x_i^{(0)} = X_i \) and \( \partial M_m / \partial x_i \geq 0 \).
We already know, from the proof of Theorem 1, that the condition $\partial M_m / \partial x_i \leq 0$ is equivalent to $x_i \leq \alpha$, and the condition $\partial M_m / \partial x_i \geq 0$ is equivalent to $x_i \geq \alpha$. Thus, the above two situations can be reformulated as follows:

- either $x_i^{(0)} = \underline{x}_i$ and $x_i^{(0)} = \bar{x}_i \leq \alpha$;
- or $x_i^{(0)} = \bar{x}_i$ and $x_i^{(0)} = \underline{x}_i \geq \alpha$.

Hence:

- if $\bar{x}_i < \alpha$, then we cannot have the second case (when $\underline{x}_i \geq \alpha$) and therefore we can only have the first case, when $x_i^{(0)} = \underline{x}_i$;
- similarly, if $\alpha < \underline{x}_i$, then we must have $x_i^{(0)} = \bar{x}_i$.

The only case when the knowledge of $\alpha$ does not help us determine $x_i$ is the case when $\underline{x}_i \leq \alpha \leq \bar{x}_i$, i.e., when $\alpha \in \mathbb{x}_i$.

Since intervals are $C$-wise disjoint, for each $\alpha$, there can be no more than $C$ such intervals, so we can try all $2^C$ possible assignments for each segment. In other words, the time increases by a constant ($\leq 2^C$) over the running time of the algorithm described in Theorem 1. This justifies the algorithm and proves that it runs in quadratic time.

### 6.4 Proof of Theorem 4

Similarly to the proof of Theorem 1, we conclude that for the point where the function $M_m$ attains its minimum, we have:

- either $x_i^{(0)} \in (\underline{x}_i, \bar{x}_i)$ and $\partial M_m / \partial x_i = 0$;
- or $x_i^{(0)} = \underline{x}_i$ and $\partial M_m / \partial x_i \geq 0$;
- or $x_i^{(0)} = \bar{x}_i$ and $\partial M_m / \partial x_i \leq 0$.

Here, the derivative $\partial M_m / \partial x_i$ is described by the same formula (2) as in the proof of Theorem 1. The difference is that $m$ is now odd, so:

- if $\partial M_m / \partial x_i = 0$, then $(x_i - E)^{m-1} = M_{m-1}$, hence $|x_i - E| = M_{m-1}^{1/(m-1)}$, so either $x_i$ is equal to $\alpha^-$ defined as $E - M_{m-1}^{1/(m-1)}$, or $x_i$ is equal to $\alpha^+$ defined as $E + M_{m-1}^{1/(m-1)}$;
- if $\partial M_m / \partial x_i \geq 0$, then $(x_i - E)^{m-1} \geq M_{m-1}$, hence $|x_i - E| \geq M_{m-1}^{1/(m-1)}$, so $x_i \leq \alpha^-$ or $x_i \geq \alpha^+$;
- if $\partial M_m / \partial x_i \leq 0$, then $(x_i - E)^{m-1} \leq M_{m-1}$, hence $|x_i - E| \leq M_{m-1}^{1/(m-1)}$, so $\alpha^- \leq x_i \leq \alpha^+$.
Therefore, the above conditions on $x_{i}^{(0)}$ can be reformulated as follows:

- in the first case, $x_{i}^{(0)} \in (z_{i}, t_{i})$ and either $x_{i}^{(0)} = \alpha^{-}$ or $x_{i}^{(0)} = \alpha^{+}$; in this case, either $z_{i} < \alpha^{-} < t_{i}$ or $z_{i} < \alpha^{+} < t_{i}$;
- in the second case, $x_{i}^{(0)} = z_{i}$ and either $x_{i}^{(0)} = \alpha^{-}$ or $x_{i}^{(0)} = \alpha^{+}$;
- in the third case, $x_{i}^{(0)} = t_{i}$ and $\alpha^{-} \leq x_{i}^{(0)} = t_{i} \leq \alpha^{+}$.

Hence, once we know $\alpha^{-}$ and $\alpha^{+}$, we can determine (at least some) some values $x_{i}^{(0)}$ as follows:

- if $\alpha^{-} < z_{i} < t_{i} < \alpha^{+}$, then we cannot have the first case (when $\alpha^{-} > z_{i}$ or $\alpha^{+} < t_{i}$), and we cannot have the second case (when $t_{i} < \alpha^{-}$ or $t_{i} \geq \alpha^{+}$); therefore, we can only have the third case, when $x_{i}^{(0)} = t_{i}$;
- if $\alpha^{+} \leq z_{i}$, then we cannot have the first case (when either $\alpha^{+} > z_{i}$, or $\alpha^{-} > z_{i}$, hence $\alpha^{+} > z_{i}$), and we cannot have the third case (when $z_{i} \leq \alpha^{+}$, hence $z_{i} < \alpha^{+}$); therefore, we can only have the second case, when $x_{i}^{(0)} = z_{i}$;
- similarly, if $z_{i} < \alpha^{-}$, then we cannot have the first case (when either $\alpha^{-} < z_{i}$, or $\alpha^{+} < z_{i}$, hence $\alpha^{-} < z_{i}$), and we cannot have the third case (when $\alpha^{-} \leq z_{i}$); therefore, we can only have the second case, when $x_{i}^{(0)} = z_{i}$.

We have described all the cases in which neither of the two auxiliary values $\alpha^{-}$ and $\alpha^{+}$ is in the interval $x_{i}$; in all these cases, we can uniquely determine the value $x_{i}^{(0)}$. The only cases when we cannot uniquely determine the value $x_{i}^{(0)}$ are the cases when either $\alpha^{-}$ or $\alpha^{+}$ is within the interval $x_{i}$.

Once we choose segments that contain $\alpha^{-}$ and $\alpha^{+}$, we have no more than $C$ intervals $x_{i}$ that contain $\alpha^{-}$ and no more than $C$ intervals that contain $\alpha^{+}$. Thus, for the remaining $\geq n - 2C$ indices $i$, we can uniquely determine $x_{i}$. For the $2C$ indices, we try all possible combinations. This is exactly what we do in our algorithm. Thus, the algorithm is indeed correct.

The algorithm requires linear time $O(n)$ for each pair of segments; there are $O(n^{2})$ pairs of segments, hence the algorithm requires cubic time. The theorem is proven.

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