10th IMEKO TC7 International Symposium
June 30–July 2, 2004, Saint-Petersburg, Russia

MODELLING MEASUREMENTS AS TIMED INFORMATION PROCESSES IN SIMPLEX DOMAINS

Graçaliz P. Dimuro[1], *Antônio Carlos R. Costa*[12] *and Vladik Kreinovich*[3]

[1]Escola de Informática, Universidade Católica de Pelotas, 96010-140, Pelotas, Brazil
[2]PPGC, Universidade Federal do Rio Grande do Sul, 90501-970, Porto Alegre, Brazil
[3]Department of Computer Science, University of Texas at El Paso, El Paso, TX 79968, USA

Abstract – This paper presents a domain-theoretic model for measurements and measuring instruments, by making explicit in simplex-domain structures two important aspects of measurement processes: the notion of standard representation relation, established between the (physical) values that are being measured and the meanings of the readings (semantic values) of the measuring instruments used to measure them, and the time underlying every measurement process, in a way that it is possible to trace the history of every measuring process. We also present the modelling of measurements performed by combined measuring instruments synchronized in time. Finally, the domain-theoretic modelling of a sample measuring process is presented to illustrate the approach.

Keywords: qualitative domains, simplex domains, measurement instruments, fusion processes

1. INTRODUCTION

Scott and Strachey [1] introduced Domain Theory as a mathematical framework for the semantics of programming languages. The main idea is that programming language semantics can be formally specified in terms of objects of domains, conceived as partially ordered sets of objects with certain properties such that the order (called *information order*) models the notions of approximation between objects and the objects themselves model partial results of computation steps. This means that (partially computed) objects can be compared by the quality of information they carry with respect to some totally computed object (a maximal object in the domain, called *total* object). In this sense, if an object x *approximates* another object y, then y contains at least the same information carried by x. The least element or *bottom* (required for every domain) models the absence of information, representing the beginning of any computational step; in contrast, *total objects* represent the final result of a finite computation or a limit, when they are produced by infinite computations.

The main subject addressed by Domain Theory is precisely the modelling of computations performed over objects that can only be produced by infinite processes. Therefore, it became the ideal structure for modelling computations over the real numbers [2] and real intervals [3]. Since then, domain-like structures have been used in several applications, most of them in the context of computation and mathematics [4].

Those features of Domain Theory support the main intuition behind our work, the possibility of developing a domain-theoretical model of uncertainty in measuring instruments and processes. In [5], we introduced the simplex and simplicial complex domains – qualitative domains whose objects are, respectively, simplexes and simplicial complexes that are coherent, in a certain sense –, allowing a domain-theoretical modelling of measurement processes based on those domains. We showed that qualitative domains of coherent simplexes can be used to model the steps of (possibly infinite) measuring processes. In [6], we used the same approach to model a simple competitive sensor data fusion [7]-[8], where several subsequent perceptions obtained by a single perception module are fused. In both papers, time is not considered explicitly in the domain-theoretic model.

In this paper, we extend the initial work [5]-[6] done on simplex domains, where the notions of *qualitative domains* and *simplexes*[1] were firstly combined. We introduce a general model for a *measuring instrument* (measuring device, sensor etc.) as a simplex domain whose objects are constructed from *reading events*, labelled with *temporal* information and interpreted over a *semantic domain*[2], allowing for the definition of a temporal order between them, in addition to the information order. A *standard representation relation* between sources of physical values and semantic domains is defined, so that the notion of a calibrated measuring instrument for each set of physical values can be defined, showing constructively (step-by-step) how information events about a physical value are gathered and interpreted in the semantic domain. An exponential constructor is introduced for modelling the histories of measuring processes.

The new features introduced in this paper can support the definition of some timed simplex-domain

---

[1] See [8] for standard definitions and results concerning Domain Theory, [10] for qualitative domains and [11] for simplexes.

[2] The elements of the semantic domain model the set of possible final measurements that shall be taken into consideration by the user. For instance, in agents (or robots), they model the internal representations (internal data structures) used for making decisions or coming to conclusions [7]-[8].

constructors that allow the modelling of composed measurement instruments that are able to synchronize and to produce fused measurements with less measurement uncertainty.

The paper is organized as follows. Section 2 presents basic definitions and summarizes previous work. Section 3 introduces the notion of measuring instrument and the calibration procedure. Reading processes are discussed in Section 4, allowing temporal information. The exponential constructor is introduced in Section 5. Section 6 discusses measuring processes performed by combined instruments. A sample of a measuring process is presented in Section 7 and Section 8 comes with the Conclusion.

2. SIMPLEX DOMAINS

A *qualitative domain* $\mathbf{D} = (\mathbf{D}, \subseteq)$ is a collection of sets $\mathbf{D}$, ordered under the inclusion relation $\subseteq$, such that: (i) $\mathbf{D}$ is non-empty: the empty set $\varnothing \in \mathbf{D}$ is the bottom of $\mathbf{D}$; (ii) $\mathbf{D}$ is closed under directed[3] unions: $\forall S \subseteq \mathbf{D}(S \text{ directed} \Rightarrow \bigcup S \in \mathbf{D})$; (iii) D is downward closed: $\forall a,b(a \in \mathbf{D} \wedge b \subseteq a \Rightarrow b \in \mathbf{D})$. The *information order* is the inclusion relation. The information unity is called *token* and the *set of tokens* is given by $|\mathbf{D}| = \{\alpha | \{\alpha\} \in \mathbf{D}\} = \bigcup \mathbf{D}$. A set $x \subseteq |\mathbf{D}|$ is said to be *coherent* if $x \in \mathbf{D}$.

A *simplex* σ is a countable set. A simplex σ^p of dimension p is a finite set of $p+1$ elements. σ^ω represents a simplex of infinite dimension and σ^ε is the empty non-dimensional simplex[4]. A *simplicial complex* is a countable set K of simplexes such that if σ is in K so are all its faces (subsets). The dimension of K is the largest of the dimensions of its the simplexes.

Let B be a countable set of tokens having an associated interpretation in a *semantic domain* given by a complete lattice $\mathbb{S} = (\mathbb{S}, \sqsubseteq_{\mathbb{S}}, \bot_{\mathbb{S}}, \top_{\mathbb{S}})$, where $\bot_{\mathbb{S}}$ is the least element, $\top_{\mathbb{S}}$ is the greatest element and $\bot_{\mathbb{S}} \neq \top_{\mathbb{S}}$.

$i : B \to \mathbb{S}$ is said to be a *token interpretation function* if and only if $\forall \beta \in B(i(\beta) \neq \bot_{\mathbb{S}} \wedge i(\beta) \neq \top_{\mathbb{S}})$. The notion of *coherence* in B is defined in terms of a token interpretation function i, that is, a simplex of tokens $\sigma = \{\beta_1, \beta_2, ...\} \subseteq B$ is said to be *coherent* if and only if $\sqcup\{i(\beta) \in \mathbb{S} \geq |\beta \in \sigma\} \neq \top_{\mathbb{S}}$, where $\sqcup$ means the supremum of the indicated set. The collection of (partial or total) coherent simplexes induced on B by the interpretation i is denoted by $CohSimp(B,i)$. The interpretation of a coherent simplex $\sigma \in CohSimp(B,i)$ is given by the *simplex interpretation function* $\mathbf{i} : CohSimp(B,i) \to \mathbb{S}$, defined by $\mathbf{i}(\sigma) = \sqcup\{i(\beta) \in \mathbb{S} \geq |\beta \in \sigma\}$.

Theorem 1. [5] $(CohSimp(B,i), \subseteq)$ is a qualitative domain, called the *simplex domain* induced on B by an interpretation function i. □

3. MEASURING INSTRUMENTS AND CALIBRATION FUNCTIONS

The set of physical values is conceived as cpo[5] $\mathbb{V} = (\mathbb{V}, \sqsubseteq_{\mathbb{V}}, \bot_{\mathbb{V}})$, with elements represented by values laying in a semantic domain $\mathbb{S}$. It is possible to defined many different relations between $\mathbb{S}$ and $\mathbb{V}$, but one of them shall be consider as a standard for calibration procedures. To define such standard relation, denoted by $\lhd\!\rhd \subseteq \mathbb{S} \times \mathbb{V}$, consider the sets:

$$\mathbb{V}_{s \in \mathbb{S}} = \{v \in \mathbb{V} | s \lhd\!\rhd v\}, \mathbb{S}_{v \in \mathbb{V}} = \{s \in \mathbb{S} | s \lhd\!\rhd v\}, \quad (1)$$

$$\mathbb{V}_{X \subseteq \mathbb{S}} = \{v \in \mathbb{V} | \exists s \in X\, (s \lhd\!\rhd v)\}, \quad (2)$$

$$\mathbb{S}_{Y \subseteq \mathbb{V}} = \{s \in \mathbb{S} | \exists v \in Y\, (s \lhd\!\rhd v)\}. \quad (3)$$

Definition 1. A *standard representation relation* for $\mathbb{V}$ in $\mathbb{S}$ is a relation $\lhd\!\rhd \subseteq \mathbb{S} \times \mathbb{V}$ such that:

i. Every well-defined physical value $v \in \mathbb{V}$ has a well-defined representation in $\mathbb{S}$, that is:

$$\forall v \in \mathbb{V}\,(v \neq \bot \Rightarrow \exists s \in \mathbb{S}\,(s \neq \bot_{\mathbb{S}} \wedge s \neq \top_{\mathbb{S}} \wedge s \lhd\!\rhd v)). \quad (4)$$

ii. $\lhd\!\rhd$ satisfies the *strict*-like properties:

$$\forall s\,(s \lhd\!\rhd \bot_{\mathbb{V}} \Rightarrow s = \bot_{\mathbb{S}}); \forall v\,(\bot_{\mathbb{S}} \lhd\!\rhd v \Rightarrow v = \bot_{\mathbb{V}}). \quad (5)$$

iii. $\lhd\!\rhd$ satisfies the *order-preserving* properties:

$$\forall s_1, s_2 \in \mathbb{S}\,(\mathbb{V}_{s_1} \neq \varnothing \wedge \mathbb{V}_{s_2} \neq \varnothing \wedge s_1 \sqsubseteq_{\mathbb{S}} s_2 \Rightarrow \forall v_1 \mathbb{V}_{s_1} \exists v_2 \in \mathbb{V}_{s_2}\,(v_1 \sqsubseteq_{\mathbb{V}} v_2)); \quad (6)$$

$$\forall v_1, v_2 \in \mathbb{V}\,(v_1 \sqsubseteq_{\mathbb{V}} v_2 \Rightarrow \forall s_1 \in \mathbb{S}_{v_1} \exists s_2 \in \mathbb{S}_{v_2}\,(s_1 \sqsubseteq_{\mathbb{S}} s_2)). \quad (7)$$

iv. $\lhd\!\rhd$ satisfies the *continuous-like* properties:

$$\forall X \subseteq \mathbb{S}\,(\sqcup X \neq \top_{\mathbb{S}} \wedge \forall s \in X\,(\mathbb{V}_s \neq \varnothing) \Rightarrow \mathbb{V}_X \text{ directed} \wedge \sqcup X \lhd\!\rhd \sqcup \mathbb{V}_X); \quad (8)$$

$$\forall Y \subseteq \mathbb{V}\,(\sqcup \mathbb{S}_Y \lhd\!\rhd \sqcup Y). \quad (9)$$

Given a cpo of physical values $\mathbb{V}$ and a semantic domain $\mathbb{S}$, a *measuring instrument* for $\mathbb{V}$ in $\mathbb{S}$, denoted by $\mathbf{M}_{(\mathbb{S},\mathbb{V})}$, is any simplex domain $\mathbf{M}_{(\mathbb{S},\mathbb{V})} = (CohSimp(M,i), \subseteq)$, where the countable set M is called the *token internal scale* with respective interpretation function $i : M \to \mathbb{S}$. A coherent simplex

[3] A directed set is a partially ordered set where each two elements always have a supremum.

[4] The notion of simplicial complex [11] was extended in [5] to consider both denumerable and non-dimensional simplexes.

[5] A cpo (or a complete partial order) is a partially ordered set having a least element, where each directed subset has a supremum.

$\sigma \in \mathbf{M}_{(\mathbb{S},\mathbb{V})}$ is called *coherent simplex of readings*, or *coherent reading*, for short. The set of all (partial or total) coherent readings is denoted by $CohSimp(M,i)$. The coherent readings are interpreted in the semantic domain $\mathbb{S}$ by the simplex interpretation function $\mathbf{i}: CohSimp(M,i) \to \mathbb{S}$. If σ is a possible coherent reading in $\mathbf{M}_{(\mathbb{S},\mathbb{V})}$, then its interpretation $\mathbf{i}(\sigma) \in \mathbb{S}$ is called the measurement given by σ.

An important task is to know weather or not a measuring instrument is calibrated so that a measurement can be reliable associated to a physical value. In this sense, a measuring instrument is considered calibrated if it is capable to interpret its coherent readings according to a standard representation relation.

Definition 2: A measuring instrument is *calibrated* for the standard $\lhd\rhd \subseteq \mathbb{S} \times \mathbb{V}$ if and only if the following conditions hold:

$$\forall \sigma \in \mathbf{M}_{(\mathbb{S},\mathbb{V})} \exists v \in \mathbb{V}\left(\mathbf{i}(\sigma) \lhd\rhd v\right), \tag{10}$$

$$\forall v \in \mathbb{V} \exists \sigma \in \mathbf{M}_{(\mathbb{S},\mathbb{V})}\left(\mathbf{i}(\sigma) \lhd\rhd v\right). \tag{11}$$

A calibrated measuring instrument is denoted by $\mathbf{M}_{\lhd\rhd}$.

Considering a standard $\lhd\rhd \subseteq \mathbb{S} \times \mathbb{V}$ and given a measuring instrument $\mathbf{M}_{(\mathbb{S},\mathbb{V})} = \left(CohSimp(M,i), \subseteq\right)$, a calibration procedure for $\lhd\rhd$ is any operator $\mathcal{C}_{\lhd\rhd} : [M \to \mathbb{S}] \to [M \to \mathbb{S}]$ that is able to adjust the scale interpretation scale *i*, such that

$$\forall \sigma \in \mathbf{M}_{\lhd\rhd} \exists v \in \mathbb{V}\left(\mathbf{i}_{\lhd\rhd}(\sigma) \lhd\rhd v\right), \tag{12}$$

$$\forall \sigma \in \mathbf{M}_{\lhd\rhd} \exists v \in \mathbb{V}\left(\mathbf{i}_{\lhd\rhd}(\sigma) \lhd\rhd v\right), \tag{13}$$

where $\mathbf{i}_{\lhd\rhd}(\sigma) = \sqcup\left\{\left(\mathcal{C}_{\lhd\rhd} \circ i(\beta)\right) \in \mathbb{S} \geq \middle| \beta \in \sigma\right\}$ is the calibrated simplex interpretation function. Any measuring instrument is liable to be calibrated if and only one succeeds in finding such a calibration procedure.

4. READING PROCESSES

The two fundamental notions of Domain Theory are those of *partial objects* and the *approximation relation*. Partial objects represent partial information about a subject, and the approximation relation orders such objects according to the degree of completeness of their information content. Given an approximation order between the partial coherent readings in a measuring instrument, a directed set of such objects can be understood as a *reading process*, where each reading in the directed set can be seen as resulting from a particular step in the progressive process of accumulating information about the physical value v.

Definition 3. Let $T = \{t_0, t_1, \ldots\}$ be a (possibly infinite) sequence of discrete instants of time. A *reading process* of physical value $v \in \mathbb{V}$ with time duration *T*, performed by a measuring instrument $\mathbf{M}_{\lhd\rhd}$, is a function $\tau_M^T : T \to \left(M \cup \{\varepsilon\}\right)$, with $\tau_T^M(t) = \varepsilon \Leftrightarrow t = t_0$, where *M* is the token internal scale and t_0 is the initial time of the process.

We use the notation β_{kt} to say that $\tau_T^M(t) = \beta_k$. β_{kt} is called the *reading event* that occurs at the time $t \neq t_0$ and ε is the *null information event* that occurs at the beginning of every reading process at the initial time t_0. The set of readings events that occur at any time $t \neq t_0$ is denoted by M_τ.

The reading process interpretation function $i_\tau : M_\tau \to \mathbb{S}$, induced by the interpretation function $i: M \to \mathbb{S}$ of the measuring instrument $\mathbf{M}_{\lhd\rhd}$, on a reading process τ_T^M, is given by $i_\tau(\beta_{kt}) = i(\beta_k)$. It follows that a simplex of readings $\sigma_\tau \subseteq M_\tau$ is coherent with respect to a reading process interpretation i_τ if and only if $\left\{\beta_k \in M \middle| \exists t \in T\left(\beta_{kt} \in \sigma_\tau\right)\right\}$ is a coherent reading with respect to *i*. Then, it is reasonable to use the standard representation relation of the measuring instrument $\mathbf{M}_{\lhd\rhd}$ to obtain a subset of partial coherent readings that are relevant for the reading process τ_T^M about a given physical value $v \in \mathbb{V}$:

Definition 4. A non-empty coherent reading $\sigma_\tau \subseteq M_\tau$ is said to be a *v*-reading if $\mathbf{i}(\sigma) \lhd\rhd v$, for $\sigma = \left\{\beta_k \in M \middle| \exists t \in T\left(\beta_{kt} \in \sigma_\tau\right)\right\}$. The empty simplex σ^ε is called a null-reading. They are denoted by σ^v.

The subset of reading events that are relevant for the reading process of *v* is given by $M_\tau^v = \left\{\beta_{kt} \in M_\tau \middle| \exists \sigma^v\left(\beta_{kt}\right) \in \sigma^v\right\} \subseteq M_\tau$. The set of *v*-readings together with the null-reading is denoted by $CohSimp\left(M_\tau^v, i_\tau\right)$. The time duration of any partial *v*-reading $\sigma^v \in CohSimp\left(M_\tau^v, i_\tau\right)$ is then given by

$$\Gamma_\tau\left(\sigma^v\right) = \begin{cases} 0 & \text{if } \sigma = \sigma^\varepsilon, \\ \max\left\{t_n \in T \middle| \beta_{kt} \in \sigma^v\right\} - t_0 & \text{if } \sigma \neq \sigma^\varepsilon \text{ is finite,} \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{14}$$

Theorem 2. $\mathbf{M}_{i_\tau}^v = \left(CohSimp\left(M_\tau^v, i_\tau\right), \subseteq\right)$ is a qualitative domain – the *Domain of v*-readings induced on the calibrated measuring instrument $\mathbf{M}_{\lhd\rhd}$ by a reading process τ_T^M about a physical value *v*. □

A *v*-reading simplex in the domain $\mathbf{M}_{i_\tau}^v$ is a totally temporally ordered set representing the order of occurrence of its reading events in the reading process *τ*.

Define an equivalence relation ~ on M_τ^v as $\beta_{k't'} \sim \beta_{k''t''} \Leftrightarrow k' = k''$. For each $\beta_k \in M$ in the scale of the calibrated measuring instrument $\mathbf{M}_{\lhd\rhd}$, the set $X_k^v = \left\{ \beta_{kt} \middle| \exists t \in T \left(\beta_{kt} \in M_\tau^v \right) \right\}$ is an equivalence class in M_τ^v. Denote by $\mathbf{X} = \left\{ X_k^v \neq \varnothing \middle| \beta_k \in M \right\}$ the collection of non-empty equivalence classes. Observe that $\mathbf{X} \neq \varnothing$, since $T - \{t_0\} \neq \varnothing$ for any τ ($\tau \neq \tau^\varepsilon$).

The *duration* $\Gamma_{\mathbf{X}} = \mathbf{X} \to T$ of a $X_k^v \in \mathbf{X}$ is defined as $\Gamma\left(X_k^v\right) = \max\left\{ t \in T \middle| \beta_{kt} \in X_k^v \right\} - t_0$. For each $\beta_k \in M$ such that $X_k^v \neq \varnothing$, it is possible to define its *maximal occurrence time* $\Gamma_M : M \to T$ by

$$\Gamma_M\left(\beta_k\right) = \max\left\{ t \in T \middle| \beta_{kt} \in M_\tau^v \right\}. \tag{15}$$

Consider an interpretation function defined on $\mathbf{X}$ as $i_{\mathbf{X}} : \mathbf{X} \to \mathbb{S}$, so that $i_{\mathbf{X}}\left(X_k^v\right) = i\left(\beta_k\right)$. It follows that

Theorem 3. $\mathbf{X}_{i_{\mathbf{X}}}^v = \left(CohSimp\left(\mathbf{X}, i_{\mathbf{X}}\right), \subseteq \right)$ is a qualitative domain of coherent simplexes isomorphic to a sub-domain $\mathbf{N}_{i|_N}^{\{v\}} = \left(CohSimp\left(N \subseteq M, i|_N\right), \subseteq \right)$ of the calibrated measuring instrument $\mathbf{M}_{\lhd\rhd}$. $\mathbf{N}_{i|_N}^{\{v\}} \equiv \mathbf{X}_{i_{\mathbf{X}}}^v$ is called the *v*-measuring instrument induced by a reading process τ_T^M on the measuring instrument $\mathbf{M}_{\lhd\rhd}$. □

$\mathbf{N}_{i|_N}^{\{v\}}$ represents the part of a measuring instrument that shall be involved in a measuring process of a certain physical value *v*. From Theorem 3, it is possible to associate, to each token $\beta_k \in N$, an information about the time of its latest occurrence, using (15).

5. TRACING MEASURING PROCESSES

The *exponential constructor* is used to model the measuring processes in time, generating the set of all possible histories of all measuring processes. It is essential, e.g., to model sensor data fusion [6][7][8], in order to allow the synchronization of successive and/or parallel sensing processes performed by several sensors.

For a calibrated measuring instrument $\mathbf{M}_{\lhd\rhd}$, consider a finite $K \subseteq CohSimp_{fin}\left(M, i\right)$. A simplicial complex K is said to be coherent if and only if $\{\beta\}, \{\beta'\} \in K \Rightarrow i\left(\beta\right) \sqcup i\left(\beta'\right) \neq \top_{\mathbb{S}}$. The set of coherent complexes is denoted by $CohComp\left(M, i\right)$.

Thorem 4. $!\mathbf{M}_{\lhd\rhd} = \left(CohComp\left(M, i\right), \subseteq \right)$ is a qualitative domain, called the exponential of $\mathbf{M}_{\lhd\rhd}$. □

The exponential domain $!\mathbf{M}_{\lhd\rhd}$ of coherent simplicial complexes represents the measuring processes of physical values $\mathbb{V}$. Any coherent simplicial complex of $CohComp\left(M, i\right)$ shows the *history of the approximation* of its maximal elements in $\mathbf{M}_{\lhd\rhd}$. The total objects in $!\mathbf{M}_{\lhd\rhd}$ are coherent complexes representing complete histories of measurements; partial objects indicates measurements partially performed.

6. COMBINING INSTRUMENTS

In this section, we briefly show some examples of how to obtain a modelling for measuring processes performed by combined measuring instruments (like, for example, in sensor data fusion), based on some special constructors of domains of coherent simplexes that were defined on the basis of the standard coherence spaces constructors [12].

6.1. Competitive Measurements

Competitive *measurement processes* perform the fusion of partially redundant information about the same aspect of the subject, obtaining the information using different instruments, in order to simulate a more accurate measuring instrument (by reducing the uncertainty, mainly systematic errors, limited resolution etc. present in the individual measuring instrument).

$\lhd\rhd_1, \lhd\rhd_2 \subseteq \mathbb{S} \times \mathbb{V}$ are said to be compatible standard representation relations with respect to a standard relation $\lhd\rhd \subseteq \mathbb{S} \times \mathbb{V}$ if and only if $\lhd\rhd_1, \lhd\rhd_2 \subseteq \lhd\rhd$ and $\forall v_1, v_2 \in \mathbb{V}\ \forall s_1, s_2 \in \mathbb{S}$

$$\left(s_1 \lhd\rhd_1 v_1 \wedge s_2 \lhd\rhd_2 v_2 \wedge \exists\left(v_1 \sqcup v_1\right) \in \mathbb{V} \Rightarrow s_1 \sqcup s_2 \lhd\rhd v_1 \sqcup v_2\right). \tag{16}$$

Two calibrated measuring instruments $\mathbf{M}_1$ and $\mathbf{M}_2$ are said to be compatible if and only if their standard representation relations $\lhd\rhd_1, \lhd\rhd_2$ are compatible with respect to some standard $\lhd\rhd$. Consider then the *disjoint union* of the token scales M_1 and M_2, given by $M_1 \dot\cup M_2 = \left(\{1\} \times M_1\right) \cup \left(\{2\} \times M_2\right)$ and an associated interpretation function of indexed tokens $\left(i_1, i_2\right) : M_1 \dot\cup M_2 \to \mathbb{S}$, defined by

$$\left(i_1, i_2\right)\left(n, \beta\right) = \begin{cases} i_1\left(\beta\right) & \text{if } n = 1; \\ i_2\left(\beta\right) & \text{if } n = 2. \end{cases} \tag{17}$$

A competitive simplex $\Phi \subseteq \left(M_1 \dot\cup M_2\right)$ is coherent if $\sqcup\left\{ \left(i_1, i_2\right)\left(n, \beta\right) \middle| \left(n, \beta\right) \in \Phi \right\} \neq \top_{\mathbb{S}}$ and its interpretation is $\mathbf{i}_{\dot\cup}\left(\Phi\right) = \sqcup\left\{ \left(i_1, i_2\right)\left(n, \beta\right) \middle| \left(n, \beta\right) \in \Phi \right\}$.

Theorem 4. The competitive combination of calibrated measuring instruments $\mathbf{M}_1$ and $\mathbf{M}_2$ is given by a domain $\mathbf{M}_1 \,\&\, \mathbf{M}_2 = \left(CohSimp\left(M_1 \dot\cup M_2\right), \left(i_1, i_2\right) \right)$ which is also calibrated for the standard $\lhd\rhd$. □

6.2. Complementary Measurements

Complementary measuring processes perform the fusion of independent information about separate aspects of a subject, obtaining the information using different measuring instruments, in order to simulate a larger, multi-faceted measuring instrument.

Consider two calibrated measuring instruments $\mathbf{M}_1$ and $\mathbf{M}_2$ and the cartesian product of their token scales $M_1 \times M_2$, with and an associated interpretation function $i_1 \times i_2 : M_1 \times M_2 \to \mathbb{S}_1 \times \mathbb{S}_2$, defined by $(i_1 \times i_2)(\beta_1, \beta_2) = (i_1(\beta_1), i_2(\beta_2))$, where $\mathbb{S}_1 \times \mathbb{S}_2$ is the semantic domain. A subset $\Psi \subseteq (M_1 \times M_2)$ is a coherent complementary simplex if $\sqcup \{(i_1 \times i_2)(\beta_1, \beta_2) | (\beta_1, \beta_2) \in \Psi\} \neq (\top_{\mathbb{S}_1}, \top_{\mathbb{S}_2})$ and its interpretation is given by

$$\mathbf{i}_\times(\Psi) = \sqcup \{(i_1 \times i_2)(\beta_1, \beta_2) | (\beta_1, \beta_2) \in \Psi\} . \quad (19)$$

Theorem5. The complementary fusion of the measuring instruments $\mathbf{M}_1$ and $\mathbf{M}_2$ is the domain $\mathbf{M}_1 \otimes \mathbf{M}_2 = (CohSimp(M_1 \times M_2), (i_1 \times i_2))$, in each component calibrated for the standards $\lhd\rhd_1$, $\lhd\rhd_2$. □

6.3. Cooperative Measurements

Cooperative measuring processes perform the fusion of independent information about separate aspects of a subject, obtaining the information using different measuring instruments, in order to estimate, at the interpretation level, a single measurement that derivates from the original ones. Because of lack of space, we omit here further details of its formulation. See Section 7 for a contextualized example.

7. A SAMPLE MEASURING PROCESS

Consider the case of a box of which we want to measure the height *H*, the width *W* and the area *A* of its frontal face[6]. To obtain a standard for calibrating measuring instruments for *H* and *W*, consider a semantic domain of real intervals given by $\mathbb{S} = \{[x_1; x_2] | 0 \le x_1 \le x_2 \le 50\} \cup \{\varnothing\}$, ordered under the reverse inclusion relation (meaning that the least diameter, the best information), with $\bot = [0, 50]$ (meaning no information) and $\top = \varnothing$ (meaning contradictory information). The standard representation relations between $\mathbb{S}$ and the height *H* and the width *W* of the box are given by $s \lhd\rhd H(W) \Leftrightarrow H(W) \in s$.

Assume that we have three calibrated instruments $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ that we want to combine competitively ($\mathbf{M}_1 \& \mathbf{M}_2$) to measure the height *H*, complementarily $(\mathbf{M}_1 \& \mathbf{M}_2) \otimes \mathbf{M}_3$ to measure also the width *W*. We also shall combine them cooperatively to obtain the area *A*, at the interpretation level. Each instrument $\mathbf{M}_j$ can produce readings (tokens) $\beta_1^j, \beta_2^j, \ldots$ that are interpreted in the semantic domain $\mathbb{S}$. The interpretation functions of the instruments $\mathbf{M}_j$ are shown in Table I.

Table II shows one possible reading process for each instrument $\mathbf{M}_j$, operating during the time instants $T = \{t_0, t_1, t_2, t_3\}$, the maximal simplexes of coherent readings and the respective best semantic values that such processes produce at time t_3.

TABLE I. Interpretation functions i_j for tokens β_k^j of measuring instruments $\mathbf{M}_j$

	β_1^j	β_2^j	β_3^j	β_4^j	⋯
i_1	$[2;5]$	$[3;6]$	$[4;7]$	$[5;8]$	⋯
i_2	$[1,5;5,5]$	$[2,5;4,2]$	$[3,8;4,3]$	$[1,5;6,5]$	⋯
i_3	$[0,5;1,5]$	$[0,8;1,8]$	$[1,1;2,1]$	$[1,4;2,4]$	⋯

TABLE II. Reading processes for instruments $\mathbf{M}_j$ and best results at time t_3

$\mathbf{M}_j$	t_0	t_1	t_2	t_3	Maximal Readings	Best Semantic Value
$\mathbf{M}_1$	-	β_1^1	β_2^1	β_3^1	$\{\beta_{1t_1}^1, \beta_{2t_2}^1, \beta_{3t_3}^1\}$	$[4,0;5,0]$
$\mathbf{M}_2$	-	β_2^2	β_3^2	β_3^2	$\{\beta_{2t_1}^2, \beta_{3t_2}^2, \beta_{3t_3}^2\}$	$[3,8;4,2]$
$\mathbf{M}_3$	-	β_3^3	β_3^3	β_4^3	$\{\beta_{3t_1}^3, \beta_{3t_2}^3, \beta_{4t_3}^3\}$	$[1,4;2,1]$

The maximal coherent simplexes of readings obtained at time t_3 are registered by each exponential domain $!\mathbf{M}_j$ in the maximal simplicial complexes that represent the complete histories of the reading processes ending at time t_3 in each instrument $\mathbf{M}_j$. For example, for the instrument $\mathbf{M}_1$, in the exponential domain $!\mathbf{M}_1$, we find that the initial fragments of the history of measuring *H* are given by the simplicial complexes shown in Table III. The maximal coherent complex among them has time duration $t_3 - t_0$. It represents the complete history of the measuring process, and the interpretation of its maximal simplex $\{\beta_{1t_1}^1, \beta_{2t_2}^1, \beta_{3t_3}^1\}$ (also shown in Table II) represents the best information gathered about the height *H*, when using only the measuring instrument $\mathbf{M}_1$.

The maximal coherent competitive simplexes produced by the fusion process performed by the combination $\mathbf{M}_1 \& \mathbf{M}_2$, at each time instant, and the respective interpretations in the semantic domain $\mathbb{S}$, are shown in Table IV. In Table V, we present the maximal coherent tensor simplexes produced by the fusion process performed by the complementary combination $(\mathbf{M}_1 \& \mathbf{M}_2) \otimes \mathbf{M}_3$, at each time instant, and the respective interpretations in the semantic domain $\mathbb{S} \times \mathbb{S}$.

---

[6] *H* and *W* lay in a flat cpo of real numbers $r \le 50$.

TABLE III. Initial fragments of the measuring process of the instrument $\mathtt{M}_1$

Time	Simplicial Complex
t_0	$\{\sigma^{\varepsilon}\}$
t_1	$\{\sigma^{\varepsilon},\{\beta^1_{1t_1}\}\}$
t_2	$\{\sigma^{\varepsilon},\{\beta^1_{1t_1}\},\{\beta^1_{2t_2}\},\{\beta^1_{1t_1},\beta^1_{2t_2}\}\}$
t_3	$\{\sigma^{\varepsilon},\{\beta^1_{1t_1}\},\{\beta^1_{2t_2}\},\{\beta^1_{1t_1},\beta^1_{2t_2}\},\{\beta^1_{3t_3}\},$ $\{\beta^1_{1t_1},\beta^1_{3t_3}\},\{\beta^1_{2t_2},\beta^1_{3t_3}\},\{\beta^1_{1t_1},\beta^1_{2t_2},\beta^1_{3t_3}\}\}$

TABLE IV. Competitive fusion process in the domain $\mathtt{M}_1 \,\&\, \mathtt{M}_2$, for measuring H

Time	Maximal Competitive Fusion Simplex	Interpretation $(\mathbf{i}_{\cup} \lhd\rhd H)$
t_0	σ^{ε}	$\bot=[0;50]$
t_1	$\{\beta^1_{1t_1},\beta^2_{2t_1}\}$	$[2,5;4,2]$
t_2	$\{\beta^1_{1t_1},\beta^2_{2t_1},\beta^1_{2t_2},\beta^2_{3t_2}\}$	$[3,8;4,2]$
t_3	$\{\beta^1_{1t_1},\beta^2_{2t_1},\beta^1_{2t_2},\beta^2_{3t_2},\beta^1_{3t_3},\beta^2_{3t_3}\}$	$[4,0;4,2]$

TABLE V. Complemetary fusion frocess in the domain $(\mathtt{M}_1 \,\&\, \mathtt{M}_2)\otimes\mathtt{M}_3$

T	Maximal Complementary Fusion Simplex	Interpretation $(\mathbf{i}_{\times} \lhd\rhd (H,W))$
t_0	$(\sigma^{\varepsilon},\sigma^{\varepsilon})$	$\bot=([0;50],[0;50])$
t_1	$\{(\beta^1_{1t_1},\beta^3_{3t_1}),(\beta^2_{2t_1},\beta^3_{3t_1})\}$	$([2,5;4,2],[1,1;2,1])$
t_2	$\{(\beta^1_{1t_1},\beta^3_{3t_1}),(\beta^2_{2t_1},\beta^3_{3t_1}),$ $(\beta^1_{2t_2},\beta^3_{3t2}),(\beta^2_{3t_2},\beta^3_{3t2})\}$	$([3,8;4,2],[1,1;2,1])$
t_3	$\{(\beta^1_{1t_1},\beta^3_{3t_1}),(\beta^2_{2t_1},\beta^3_{3t_1}),(\beta^1_{2t_2},\beta^3_{3t2}),$ $(\beta^2_{3t_2},\beta^3_{3t2}),(\beta^1_{3t_3},\beta^3_{4t_3}),(\beta^2_{3t_3},\beta^3_{4t_3})\}$	$([4,0;4,2],[1,4;2,1])$

Table VI shows the cooperative measurement process performed in the domain $(\mathtt{M}_1 \,\&\, \mathtt{M}_2)\otimes\mathtt{M}_3$, obtaining the frontal area A using, at the interpretation level, the interval operation [13] $f:\mathbb{S}\times\mathbb{S}\rightarrow\mathbb{S}$, defined by $f(H,W)=H\cdot W$, that estimates the area A.

TABLE VI. Cooperative fusion process in the domain $(\mathtt{M}_1 \,\&\, \mathtt{M}_2)\otimes\mathtt{M}_3$, where $\mathbf{i}_f \lhd\rhd A$

T	t_0	t_1	t_2	t_3
$\mathbf{i}_f$	$[0;2500]$	$[2,75;8,82]$	$[4,18;8,82]$	$[5,6;8,82]$

8. CONCLUSION

This paper introduced a domain-theoretic framework to model measuring processes. We define the notion of calibration of measuring instruments and developed the necessary formal machinery to allow the modelling of measuring processes performed by combined instruments. Such instruments can operate in parallel, and synchronized in time, to attend different kinds of requirements: to obtain better results by reducing the uncertainty; to reduce systematic measurement errors; to simulate a multi-faceted instrument etc. A sample (competitive, complementary, and cooperative) fusion process was modelled, to help figuring out in a concrete way how the elements of the various domains look like in the constructions of the formal model.

REFERENCES

[1] D.S. Scott, C. Strachey, *Towards a mathematical semantics of computer languages*, Oxford University Computing Lab., 1971.

[2] M.H. Escardó, "PCF extended with real numbers", *Theoretical Computer Science*, **162**(1), 79-115, 1996.

[3] G.P. Dimuro, A.C.R. Costa, D.M. Claudio, "A coherence space of rational intervals for a construction of IR", *Reliable Computing*, **6**(2), 139-178, 2000.

[4] A. Edalat, "Domains for computation in mathematics, physics and exact real arithmetic", Bulletin of Symbolic Logic, 3(4), 401-452, 1997.

[5] G.P. Dimuro, A.C.R. Costa, "Toward a domain-theoretic modelling of measuring processes" In: J.M. Abe and J.I.S. Filho (eds.) *Logic, Artificial Intelligence and Robotics, Frontiers in Artificial Intelligence and Applications*, **71**, IOS Press, Amsterdam, 2001, 68-75.

[6] G.P. Dimuro, A.C.R. Costa, "Toward a domain-theoretic modelling of sensor data fusion" In: G.L. Torres et al. (eds.) *Advances in Intelligent Systems and Robotics, Frontiers in Artificial Intelligence and Applications*, **101**, IOS Press, Amsterdam, 2003, 103-110.

[7] H.F Durrant-Whyte, *Integration, coordination, control of multisensor robot systems*, Kluwer, Boston, 1988.

[8] J.V. Dam, *Environment modelling for mobile robots: neural learning for sensor fusion*, Amsterdam, 1998.

[9] V. Stoltenberg-Hansen, L. Linström, E. Griffor, *Mathematical theory of domains*, Cambridge University Press, Cambridge, 1994.

[10] J-Y. Girard, "The system *F* of variable types - 15 years later", *Theoretical Comput. Science*, **45**, 159-192, 1986.

[11] S. Lefschetz, *Introduction to Topology*, Princeton University Press, 1949.

[12] A.S Troelstra, "Lectures on Linear Logic", *CLSI Lectures Notes*, **29**, 1992.

[13] R.E. Moore, *Methods and Applications of Interval Analysis*, SIAM Publ., Philadelphia, 1979.

Authors: G.P. Dimuro and A.C.R. Costa are partially supported by CTINFO/CNPq and FAPERGS. This work was done during their stay at UTEP, in July 2003. V. Kreinovich is supported in part by NASA under cooperative agreement NCC5-209, the Air Force Office of Scientific Research grant F49620-00-1-0365, NSF grants EAR-0112968, EAR-0225670, and by the ARL grant DATM-05-02-C-0046.

e-mails: {liz,rocha}@atlas.ucpel.tche.br, vladik@cs.utep.br.