Decision Making Beyond Arrow's "Impossibility Theorem", With the Analysis of Effects of Collusion and Mutual Attraction

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***Abstract*— In 1951, K. J. Arrow proved that, under certain assumptions, it is impossible to have group decision making rules which satisfy reasonable conditions like symmetry. This Impossibility Theorem is often cited as a proof that reasonable group decision making is impossible.**

We start our paper by remarking that Arrow's result only covers the situations when the only information we have about individual preferences is their binary preferences between the alternatives. If we follow the main ideas of modern decision making and game theory and also collect information about the preferences between lotteries (i.e., collect the utility values of different alternatives), then reasonable decision making rules are possible: e.g., Nash's rule in which we select an alternative for which the product of utilities is the largest possible.

We also deal with two related issues: how we can detect individual preferences if all we have is preferences of a subgroup, and how we take into account mutual attraction between participants.

I. Group Decision Making and Arrow's Impossibility Theorem

In 1951, Kenneth J. Arrow published his famous result about group decision making [1], a result that became one of the main reasons for his 1972 Nobel Prize; see also [16], [22], [23], [34].

The problem. Arrow's result deals with the following setting. A group of n participants $P_1, \ldots, P_n$ needs to select between one of m alternatives $A_1, \ldots, A_m$. To find individual preferences, we ask each participant P_i to rank the alternatives A_j from the most desirable to the least desirable:

$$A_{j_1} \succ_i A_{j_2} \succ_i \ldots \succ_i A_{j_n}.$$

Based on these n rankings, we must form a single group ranking (in the group ranking, equivalence $\sim$ is allowed).

Case of two alternatives is easy. In the simplest case when we have only two alternatives A_1 and A_2, each participant either prefers A_1 or prefers A_2. In this case, it is reasonable, for a group:

- to prefer A_1 if the majority prefers A_1,
- to prefer A_2 if the majority prefers A_2, and
- to claim A_1 and A_2 to be of equal quality for the group (denoted $A_1 \sim A_2$) if there is a tie.

Case of three or more alternatives is not easy. When we have three or more alternatives, there is no such simple rule; to be more precise, we can still come up with many possible group decision rules, but all these rules will be, in some sense, counter-intuitive.

Arrow's result. Arrow has explicitly formulated several reasonable conditions and showed that no group decision rule can satisfy all these conditions. Arrow's conditions are very straightforward and very natural.

The first is the Pareto condition: that if all participants prefer A_j to A_k, then the group should also prefer A_j to A_k.

The second condition is Independence from Irrelevant Alternatives: the group ranking between A_j and A_k should depend only on how participants rank A_j and A_k – and should not depend on how they rank other alternatives.

Arrow has shown that every group decision rule which satisfies these two condition is a *dictatorship* rule – the rule according to which the group accepts the preferences of one of the participants as the group decision and ignores the preferences of all other participants. This clearly violates another reasonable condition of symmetry: that the group decision rules should not depend on the order in which we list the participants.

II. Beyond Arrow's Impossibility Theorem: Nash's Bargaining Solution

It is sometimes claimed that reasonable group decision making is impossible. Arrow's Impossibility Theorem is often cited as a proof that reasonable group decision making is impossible – e.g., that a perfect voting procedure is impossible; see, e.g., [34].

Arrow's result is only valid if we have binary (partial) information about individual preferences. We will see that the pessimistic interpretation of Arrow's result is, well, too pessimistic.

Indeed, Arrow's result assumes that the only information we have about individual preferences is their binary ("yes"-"no") preferences between the alternatives. This information does not fully describe a persons' preferences: e.g., the same preference $A_1 \succ A_2 \succ A_3$ may indicate that a person strongly prefers A_1 to A_2, and A_2 to A_3, and it may also indicate that this person strongly prefers A_1 to A_2, and at the same time, A_2 is almost of the same quality as A_3.

To describe this degree of preference, researchers in decision making use the notion of *utility*; see, e.g., [22], [23].

What is utility: a reminder. A person's rational decisions are based on the relative values to the person of different outcomes. In financial applications, the value is usually measured in monetary units such as dollars. However, the same monetary amount may have different values for different people: e.g., a single dollar is likely to have more value to a poor person than to a rich one. In view of this difference, in decision theory, to describe the relative values of different outcomes, researchers use a special *utility* scale instead of the more traditional monetary scales.

There are many different ways to elicit utility from decision makers. A common approach is based on preferences of a decision maker among *lotteries*. A simple way to define a lottery is as follows. Take a very undesirable outcome A^- and a very desirable outcome A^+, and then consider the lottery $A(p)$ in which we get A^+ with probability p and A^- with probability $1-p$ (p is given and is usually understood as an "objective" probability). Clearly, the larger p, the more preferable $A(p)$: $p < p'$ implies $A(p) < A(p')$. Traditional decision theory is based on assumptions concerning preferences over lotteries. For example, the following two assumptions are usually adopted as axioms:

- the comparison amongst lotteries is a linear order – i.e., a person can always select one of the two alternatives, and
- the comparison is Archimedean – i.e. if for all $\varepsilon > 0$, an outcome B is better than $A(p-\varepsilon)$ and worse than $A(p+\varepsilon)$, then it is of the same quality as $A(p)$: $B \sim A(p)$ (where $A \sim B$ means that A and B are of the same quality).

Because of our selection of A^- and A^+, most reasonable outcomes are better than $A^- = A(0)$ and worse than $A^+ = A(1)$. Due to linearity, for every p, either $A(p) < B$, or $B \sim A(p)$, or $B < A(p)$. If we define the *utility* of outcome B as $u(B) \stackrel{\text{def}}{=} \sup\{p \mid A(p) < B\}$, we have $A(u(B)-\varepsilon) < B$ and $A(u(B)+\varepsilon) > B$; thus, due to the Archimedean property, we have $A(u(B)) \sim B$. This value $u(B)$ is called the *utility* of the outcome B.

As defined above utility always takes values within the interval $[0, 1]$. It is also possible to define utility to take values within other intervals. Indeed, note that the numerical value $u(B)$ of the utility depends on the choice of reference outcomes A^- and A^+. If we select a different pair of reference outcomes, then the resulting numerical utility value $u'(B)$ is different. The usual axioms of utility theory guarantee that two utility functions $u(B)$ and $u'(B)$ corresponding to different choices of the reference pair are related by a linear transformation: $u'(B) = a \cdot u(B) + b$ for some real numbers $a > 0$ and b. By using appropriate values a and b, we can then re-scale utilities to make the scale more convenient (e.g. in financial applications, closer to the monetary scale).

Expected utility. Often, we have a "branching" situation involving n incompatible events $E_1, \ldots, E_n$ with probabilities $p_1, \ldots, p_n$ such that exactly one of them will occur. E.g. coins can fall heads or tails, dice can show 1 to 6, etc. In such situations, for every n outcomes $B_1, \ldots, B_n$, we can form a lottery by assigning outcome B_i if event E_i occurs. If we know the utility $u_i = u(B_i)$ of each outcome B_i, and we know the probability $p_i = P(E_i)$ of each event E_i, then the utility of the corresponding lottery may be determined as follows.

We know the probability p_i of each event E_i. Thus, the lottery "B_i if E_i" is equivalent to the lottery in which we get B_i with probability p_i. The fact that $u(B_i) = u_i$ means that each B_i is equivalent to getting A^+ with probability u_i and A^- with probability $1 - u_i$. By replacing each B_i with this new "lottery", we conclude that the lottery "if E_i then B_i" is equivalent to a two-step lottery in which we:

- first select E_i with probability p_i, and
- then, for each i, select A^+ with probability u_i and A^- with the probability $1 - u_i$.

In this two-step lottery, the probability of getting A^+ is equal to $p_1 \cdot u_1 + \ldots + p_n \cdot u_n$ (often this is obtained by adding suitable axioms on combination of lotteries, but the meaning should be intuitive here). Thus, by our definition of utility, the utility of the lottery "if E_i then B_i" is equal to $u = \sum\limits_{i=1}^{n} p_i \cdot u_i = \sum\limits_{i=1}^{n} p(E_i) \cdot u(B_i)$. In mathematical terms, u is the expected value of the utility, so this approach is often called the *expected utility approach*.

In the traditional approach, between several alternatives we select the one with the largest utility u, hence the one with the largest value of the expected utility.

Nash's bargaining solution. So, for each participant P_i, instead of knowing this participants' preferences, we can determine the utility $u_{ij} \stackrel{\text{def}}{=} u_i(A_j)$ of all the alternatives $A_1, \ldots, A_m$. Once we know such utilities, we can ask the same question: how to transform these known utilities into a reasonable group decision rule?

The answer to this question was, in effect, provided by another future Nobelist John Nash who, in his 1950 paper [24], has shown that under reasonable assumptions like symmetry, independence from irrelevant alternatives, and *scale invariance* (i.e., invariance under replacing the original utility function $u_i(A)$ with an equivalent function $a \cdot u_i(A)$), the only group decision rule is selecting an alternative A for which the product $\prod\limits_{i=1}^{n} u_i(A)$ is the largest possible.

Here, the utility functions must be scaled in such a way that the "status quo" situation $A^{(0)}$ is assigned the utility 0. This re-scaling can be achieved, e.g., by replacing the original utility values $u_i(A)$ with re-scaled values $u_i'(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A^{(0)})$.

For a more detailed discussion on Nash's bargaining solution and its application to group decision making, see, e.g., [18], [22], [23], [26].

It is easy to see that the Pareto condition and Independence condition are both satisfied for Nash's solution. Let us start with the Pareto condition. If all participants prefer A_j to A_k, this means that $u_i(A_j) > u_j(A_k)$ for every i, hence $\prod\limits_{i=1}^{n} u_i(A_j) > \prod\limits_{i=1}^{n} u_i(A_k)$ – which means that the group would prefer A_j to A_k.

The Independence condition is even easier to check: according to Nash's solution, we prefer A_j to A_k if $\prod\limits_{i=1}^{n} u_i(A_j) > \prod\limits_{i=1}^{n} u_i(A_k)$. From this formula, once can easily see that the group ranking between A_j and A_k depends only on how participants rank A_j and A_k – and does not depend on how they rank other alternatives.

Comment. Nash's solution can be easily explained in terms of *fuzzy logic* (see, e.g., [17], [29]: We want all participants to be happy, so we want the first participant to be happy *and* the second participant to be happy, etc. We can take $u_1(A)$ as the "degree of happiness" of the first participant, $u_2(A)$ as the "degree of happiness" of the second participant, etc. If, in order to formalize "and", we use the operation $d \cdot d'$ (one of the two operations originally proposed by L. Zadeh to describe "and"), then the degree to which all n participants are satisfied is equal to the product $u_1(A) \cdot u_2(A) \cdot \ldots \cdot u_n(A)$. So, if we look for the alternative which leads to the largest possible degree of mutual satisfaction, then we must look for the alternative A for which the product $u_1(A) \cdot u_2(A) \cdot \ldots \cdot u_n(A)$ attains the largest possible value.

Potential applications. This idea can be applied to various problems ranging from global problems such as the division of a disputed territory [19], [20], [26] to more down-to-Earth problems such as dividing a cake (or, in general, an inheritance). Many ingenious decisions of this problems are known; see, e.g., [7], [8], [12], [13], [31]; our point is that Nash's solution can work as well.

III. How we Can Determine Utilities

It is easy to determine, for each participants P_i, his or her utility u_{ij} for a given alternative A_j (with a given accuracy 2^{-k}). For example, we can use the iterative *bisection* method in which, at every step, we have an interval $[\underline{u}, \overline{u}]$ that is guaranteed to contain the actual (unknown) value of the utility u.

As we have mentioned, in the standard scale, $u \in [0,1]$, so we can start with the interval $[\underline{u}, \overline{u}] = [0,1]$.

At each iteration, once we have an interval $[\underline{u}, \overline{u}]$ that contains u, we compute its midpoint $u_{\text{mid}} \stackrel{\text{def}}{=} (\underline{u} + \overline{u})/2$ and compare the alternative A_j with the lottery "A^+ with probability u_{mid}, otherwise A^-". Depending on the result of this comparison, we can now halve the interval $[\underline{u}, \overline{u}]$:

- If, for the participant, the alternative A_j is better than this lottery, then we know that $u \in [u_{\text{mid}}, \overline{u}]$, so we have a new interval $[u_{\text{mid}}, \overline{u}]$ of half-width which is guaranteed to contain u.
- If, for the participant, the alternative A_j is worse than this lottery, then we know that $u \in [\underline{u}, u_{\text{mid}}]$, so we have a new interval $[\underline{u}, u_{\text{mid}}]$ of half-width which is guaranteed to contain u.

After each iteration, we decrease the width of the interval $[\underline{u}, \overline{u}]$ by half. Thus, after k iterations, we get an interval of width 2^{-k} which contains the actual value u – i.e., we have determined u with the accuracy 2^{-k}.

IV. What is the Guarantee That Participants Will Provide Correct Utility Values?

Problem: sometimes it is beneficial to cheat. The above description relies on the fact that we can elicit true preferences (and hence, true utility functions) from the participants. However, sometimes, it is beneficial for a participant to cheat and provide false utility values.

For example, if a participant P_1 know the utilities of all the other participants, then it is sometimes advantageous to supply false utility values. Indeed, an ideal situation for P_1 is when, out of m alternatives $A_1, \ldots, A_m$, the group as a whole selects an alternative A_{m_1} which is the best for P_1, i.e, for which $u_1(A_{m_1}) \geq u_1(A_j)$ for all $j \neq m_1$.

It is not necessarily true that the product $\prod\limits_{i=1}^{n} u_i(A_j)$ computed based on P_1's true utility is the largest for the alternative A_{m_1}. However, we can force this product to attain the maximum for A_{m_1} if we report, e.g., a "fake" utility function $u_1'(A)$ for which $u_1'(A_{m_1}) = 1$ and $u_1'(A_j) = 0$ for all $j \neq m_1$.

In case of uncertainty, cheating may hurt the cheater: an observation. In practice, we rarely encounter a situation in which one person is familiar with the preferences of all the others while others have no information about this person's preferences. Usually, if other participants have no information about this person's preferences, then this person has no information about the preferences of the others as well.

In this case, cheating may be disadvantageous. For example, if we report the above false utility function, then if others have similar utility functions with $u_i(A_{m_i}) > 0$ for some $m_i \neq m_1$ and $u_i(A_j) = 0$ for all other j, then for every alternative A_j, Nash's product is equal to 0. From this viewpoint, all alternatives are equally good, so each of them can be chosen. In particular, it may be possible that the group selects an alternative A_q which is the worst for P_1 – i.e., for which $u_1(A_q) < u_1(A_j)$ for all $j \neq p$.

On the other hand, by reporting the actual utility function, P_1 may lead to the selection of an alternative A_k which is better than A_q.

So, in this example, by reporting a false utility function $u_1'(A_j)$ instead of the correct one $u_1(A_j)$, the participant P_1

may hurt himself by reducing his payoff from $u_1(A_k)$ to $u_1(A_q) < u_1(A_k)$.

Territorial division problem: a reminder. Let us show that in the reasonable case of dividing the territory, it is beneficial to report the correct utility values.

Comment. This result was partly announced in [19], [20].

In this case [26], we have a set A to divide. Here, each alternative corresponds to a partition of the set A into n subsets $X_1, \ldots, X_n$ such that $\bigcup\limits_{i=1}^{n} X_i = A$ and $X_i \cap X_j = \emptyset$ when $i \neq j$. The utility functions have the form $u_i(X) = \int_X v_i(t)\,dt$ for given functions $v_i(t)$ from the set A to the set of non-negative real numbers. Based on the utility functions $v_i(t)$, we find a partition $X_1, \ldots, X_n$ for which Nash's product $u_1(X) \cdot \ldots \cdot u_n(X_n)$ attains the largest possible value.

Without losing generality, let us concentrate on the actions of the first participant P_1. Let us assume that $v_1(t)$ is the actual utility function of this participant. The participant P_1 can either report his/her actual function $v_1(t)$, or he/she can report a different utility function $v_1'(t) \neq v_1(t)$. For each reported function $v_1'(t)$, we can find the partition $X_1, \ldots, X_n$ that maximizes the corresponding product

$$\left(\int_{X_1} v_1'(t)\,dt\right) \cdot \left(\int_{X_2} v_2(t)\,dt\right) \cdot \ldots \cdot \left(\int_{X_n} v_n(t)\,dt\right).$$

As a result, the participant P_1 gets the set X_1, so its actual utility is equal to $\int_{X_1} v_1(t)\,dt$. Let us denote this actual utility by $u(v_1', v_1, v_2, \ldots, v_n)$.

The question is: which utility function $v_1'(t)$ should the participant P_1 report in order to maximize his gain $u(v_1', v_1, v_2, \ldots, v_n)$? We assume that we do not know the utility functions $v_2(t), \ldots, v_n(t)$ of other participants. For different $v_i(t)$, different selections $v_1'(t)$ may lead to better gain for P_1.

Decision making under uncertainty: a reminder. The situation of decision making under uncertainty is typical in decision making; see, e.g., [22]. There are several known approaches to solving a general problem of decision making under uncertainty.

We can choose an *optimistic* approach in which, for each action A, we only consider its most optimistic outcome, with the largest possible gain $u^+(A)$ – and choose an action for which this luckiest outcome is the largest.

Alternatively, we can choose a *pessimistic* approach in which, for each action A, we only consider its most pessimistic outcome, with the smallest possible gain $u^-(A)$ – and choose an action for which this worst-case outcome is the largest.

Realistically, both approaches appear to be too extreme. In real life, it is more reasonable to use, as an objective function, a combination of pessimistic and optimistic cases. Such a combined pessimism-optimism criterion was originally proposed in [15]: namely, we choose a real number $\alpha \in [0, 1]$, and choose an alternative A for which the combination

$$u(A) = \alpha \cdot u^-(A) + (1 - \alpha) \cdot u^+(A)$$

takes the largest possible value.

Pessimism corresponds to $\alpha = 1$, optimistic corresponds to $\alpha = 0$, realistic approaches correspond to $\alpha \in (0, 1)$.

Comment. While this combination may sound arbitrary, it is actually the only rule which satisfied reasonable scale-invariance conditions; see, e.g., [25], [28].

For our problem, Hurwicz's criterion means that we select a utility function $v_1'(t)$ for which the combination

$$J(v_1') \stackrel{\text{def}}{=} \alpha \cdot \min_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n) + (1 - \alpha) \cdot \max_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n) \qquad (2)$$

attains the largest possible value.

For territorial division, it is beneficial to report the correct utilities: result. It turns out that unless we select the optimistic criterion, it is always best to select $v_1'(t) = v_1(t)$, i.e., to tell the truth.

Theorem 1. *When $\alpha > 0$, the objective function $J(v_1')$ attains its largest possible value for $v_1'(t) = v_1(t)$.*

Comment. In the optimistic case, all choices are equivalent.

In such situations, when we have several different alternatives that lead to the same value of the objective function, a natural idea is to use some other criterion to select between these optimal alternatives; see, e.g., [27]. In our case, a natural other criterion is to consider pessimism or Hurwicz's pessimism-optimism criterion. In both cases, we come to a conclusion that telling the truth is the best strategy.

Comment. For reader's convenience, the proof of this result is presented in the appendix.

V. How to Find Individual Preferences from Collective Decision Making: Inverse Problem of Game Theory

Problem. We have mentioned that usually, it is relatively easy to elicit preferences from the participants, and to determine utility values based on these preferences.

In some cases, however, we have a subgroup ("clique") of participants who do their best to make joint decisions and who do not want to disclose their differences. This is a frequent situation, e.g., within political groups – who are afraid that any internal differences can be exploited by the competing groups. In such situations, it is extremely difficult to determine individual preferences based on the group decisions.

For example, during the Cold War, this is what kremlinologists tried to do – with different degrees of success.

In this section, we will show how this determination can be done.

Comment. Decision making and game theory are usually trying, given individual preferences, to find the appropriate group decision. Here, we encounter an *inverse problem:* given the decisions, we want to reconstruct individual preferences.

Towards an algorithm for solving the inverse problem. Let us assume that we have a group of n participants $P_1, \ldots, P_n$ that does not want to reveal its individual preferences. We can, however, ask the group as a whole to compare different preferences; we must use the result of this comparison to determine individual utility functions.

We assume that when making group decisions, the group uses Nash's solution. Of course, since Nash's solution depends only on the product of the utility functions, in the best case, we will be able to determine n individual utility functions without knowing which of these functions corresponds to which individual.

Comment. This is OK, because the main objective of our determining these utility functions is to be able to make decision of a larger group based on Nash's solution – and in making this decision, it is irrelevant who has what utility function.

In this sense, our problem is easier than the problem solved by political analysts: from our viewpoint, it is sufficient to know that one member of the ruling clique is more conservative and another is more liberal, but a political analyst would also be interesting in knowing who exactly is conservative and who is more liberal.

We have mentioned that the utility function is determined modulo an arbitrary linear transformation $u(A) \to a \cdot u(A) + b$. Thus, without losing generality, we can assume that the individual utility functions $u_i(A)$ are re-scaled in such a way that for the status quo $A^{(0)}$, we have $u_i(A^{(0)}) = 0$, and for a pre-selected very favorable outcome A^+, we have $u_i(A^+) = 1$.

Let us now select an alternative A and let us show how we can determine the values $u_1(A), \ldots, u_n(A)$. For each real number $q \in [0, 1]$, we can form a lottery $L(q)$ in which we have A^+ with probability q and A with probability $1 - q$. For this lottery $L(q)$, the individual utility is equal to

$$u(L(q)) = q \cdot u(A^+) + (1 - q) \cdot u(A) = q + (1 - q) \cdot u_i(A);$$

therefore, Nash's product is equal to $\prod_{i=1}^{n} (q + (1 - q) \cdot u_i(A))$.

For the group, the quality of a lottery "A^+ with probability p, otherwise $A^{(0)}$" increases with p. When $p = 0$, this lottery is simply a status quo $A^{(0)}$, so it is clearly worse than the lottery $L(q)$. For $p = 1$, this lottery is simply A^+, so it is clearly preferable to $L(q)$. Thus, by using the above-described bisection method, for each q, we can find (with arbitrary accuracy) the value $p(q)$ for which $L(q)$ is equivalent to this lottery.

For this new lottery, the individual utilities are equal to

$$u_i(A) = p(q) \cdot u(A^+) + (1 - p(q)) \cdot u(A^{(0)}) =$$

$$p(q) \cdot 1 + (1 - p(q)) \cdot 0 = p(q),$$

so Nash's product is equal to $p(q)^n$. Thus, from the fact that the new lottery is equivalent to the $L(q)$, we conclude that the corresponding Nash's products are equal, i.e., that

$$\prod_{i=1}^{n} (q + (1 - q) \cdot u_i(A)) = p(q)^n$$

for the known values $p(q)^n$.

Dividing both sides of this equality by $(1-q)^n$, we conclude that $\prod_{i=1}^{n} (z + u_i(A)) = p(q)^n/(1 - q)^n$. We can repeat this procedure for n different values $q = 0, 1/n, 2/n, \ldots, (n - 1)/n$, and get n different values of the function

$$F(z) \stackrel{\text{def}}{=} \prod_{i=1}^{n} (z + u_i(A)).$$

This function $F(z)$ is a product of n linear functions with coefficient 1 at z; it is, therefore, a polynomial of n-th order in terms of the unknown z:

$$F(z) = a_0 + a_1 \cdot z + \ldots + a_{n-1} \cdot z^{n-1} + z^n.$$

Since we know n values $p(l/n)^n/(1-l/n)^n$ $(0 \le i \le n-1)$ of this function for the values $z_l = q_l/(1 - q_l)$ (where $q_l = l/n$), we can thus determine the coefficients a_i of this polynomial by solving the corresponding system of n linear equations with n unknowns a_i:

$$a_0 + a_1 \cdot z_l + \ldots + a_{n-1} \cdot z_l^{n-1} + z_l^n = \frac{p(q_l)^n}{(1 - q_l)^n} \qquad (1)$$

$$0 \le l \le n - 1$$

Once we have found these coefficients and thus, the polynomial $F(z) = \prod_{i=1}^{n} (z + u_i(A))$, we can then determine the values $-u_i(A)$ as the roots of this polynomial – i.e., the values for which $F(-u_i(A)) = 0$.

We can find one of the roots; there exist efficient algorithms for that; see, e.g., [2]. Once we find a root $-u_1(A)$, we can divide the polynomial by $z + u_1(A)$, and get a new polynomial of order $n - 1$. We can then use the same algorithm to find the root of the new polynomial, etc., until we find all n roots of the original polynomial $F(z)$.

Thus, we arrive at the following algorithm.

Algorithm for determining individual utility values. Let us assume that we have a group of n participants. We can ask this group to make joint decisions. Based on these decisions, we want to find the individual utility values $u_1(A), \ldots, u_n(A)$ of a given alternative A.

For that, we do the following. For each l from 0 to $n - 1$, we form the value $q_l = l/n$, and we ask the group to compare the lottery "A^+ with probability q_l, otherwise $A^{(0)}$" with the lotteries "A^+ with probability p, otherwise $A^{(0)}$" for different p. By using bisection, we can find the value $p(q_l)$ for which the lottery "A^+ with probability q_l, otherwise A" is, for this group, equivalent to the lottery "A^+ with probability $p(q_l)$, otherwise $A^{(0)}$".

After we find n values $p(q_l)$ $(0 \le l \le n - 1)$, we solve the system (1) of n linear equations with n unknowns, and get the coefficients $a_0, a_1, \ldots, a_{n-1}$. Based on these coefficients, we form a polynomial $F(z) = a_0 + a_1 \cdot z + \ldots + a_{n-1} \cdot z^{n-1} + z^n$.

Then, we apply one of the known factorization algorithms to factorize the resulting polynomial $F(z)$. It factors are $z + u_i(A)$, where $u_i(A)$ are the desired values.

From individual utility values to individual utility profiles. From the viewpoint of group decision making, it is sufficient to find out individual utility values $u_i(A_j)$ for all alternatives A_j. However, from the more general viewpoint of solving the inverse problem, it is desirable to find out the individual utility profiles. For example, if we have two alternatives A_j and A_k, we want not only to know n values $u_i(A_j)$ and n values $u_i(A_k)$, we also want to know which value $u_i(A_k)$ goes with which value $u_i(A_j)$.

For that, we pick a real number $\alpha \in [0, 1]$ and repeat the same procedure for the lottery $A \stackrel{\text{def}}{=}$"$A_j$ with probability α, otherwise A_k". Thus, we determine n individual utilities $u_i(A)$ of this lottery.

For the individual utilities,

$$u_i(A) = \alpha \cdot u_i(A_j) + (1 - \alpha) \cdot u_i(A_k).$$

Thus, if we only know n values $u_i(A_j)$, n values $u_i(A_k)$, and n values $u_i(A)$ – without knowing how these values match – we can then, for each of n values $u_i(A_j)$, determine the corresponding utility value $u_i(A_k)$ as the only one of n values $u_p(A_k)$ for which the value

$$\alpha \cdot u_i(A_j) + (1 - \alpha) \cdot A_p(A_k)$$

is equal to one of the n values $u_i(A)$.

If we select α to be a random value uniformly distributed on the interval $[0, 1]$, then the probability that

$$\alpha \cdot u_i(A_j) + (1 - \alpha) \cdot A_p(A_k)$$

for some wrong $p' \neq p$ is also accidentally equal to one of the n values $u_i(A)$ is 0, so this methods leads us to a guaranteed profile.

What if we do not know how many people are in a group? In some cases, not only we do not know individual preferences, but we also do not know how many people are in a group.

In this case, we can repeat the above procedure for $n = 1, 2, \ldots$ until we stop getting a meaningful solution for the corresponding system of linear equations (1); the largest such n is the number of participants.

Uniqueness in precise mathematical terms. Let us describe the uniqueness result in precise mathematical terms.

Definition. *Let integers n and m be fixed. The value n will be called* number of participants *and m will be called* number of alternatives.

- *By a* lottery, *we mean a vector $p = (p^{(0)}, p^+, p_1, \ldots, p_m)$ for which $p_j \geq 0$ and $p^{(0)} + p^+ + p_1 + \ldots + p_m = 1$.*
- *By an* individual utility function, *we mean a vector $u_1, \ldots, u_m$ of positive numbers.*
- *By a group utility function, we mean a collection of n utility functions $(u_{i1}, u_{i2}, \ldots, u_{im})$.*
- *We say that a group utility function u* leads *to the following preference relation $<$ between the lotteries: $p < q$ if and only if*

$$\prod_{i=1}^{n} \left(p^+ + \sum_{j=1}^{m} p_j \cdot u_{ij} \right) < \prod_{i=1}^{n} \left(p^+ + \sum_{j=1}^{m} q_j \cdot u_{ij} \right).$$

Comment. Here, the probability $p^{(0)}$ means the probability of the status quo state $A^{(0)}$, p^+ means the probability of the outcome A^+, and the utilities are scaled in such a way that for each participant, $u_i(A^{(0)}) = 0$ and $u_i(A^+) = 1$.

Our main result is that after this re-scaling, the utility values are uniquely determined by the observed group preferences – of course, modulo possible renaming (permutation) of the participants, because the Nash group decision model does not change if two participants simply swap their utility functions (and their preferences).

Theorem 2. *If two group utility functions u_{ij} and u'_{ij} lead to the same preference, then they differ only by permutation, i.e., $u'_{ij} = u_{\pi(i),j}$ for some permutation π of the set $\{1, \ldots, n\}$ of participants.*

In other words, modulo permutation of participants, we can uniquely determine the utility values from the group preferences.

The proof of this result is also given in the Appendix.

VI. Description of Altruism and Paradoxes of Love

Interdependence of utilities: idea. In the previous text, we implicitly assumed that the utility $u_i(A_j)$ of a participant P_i depends only on the objective situation, i.e., on the alternative A_j. In real-life situations, however, the degree of a person's happiness is determined not only by the objective factors – like what this person gets and what others get – but also by the degree of happiness of other people.

Normally, this dependence is positive, i.e., we feel happier if other people are happy. However, negative emotions such as jealousy are also common, when someone else's happiness makes a person not happy.

The idea that a utility of a person depends on utilities of others was first described in [32], [33]. It was further developed by another future Nobelist Gary Becker; see, e.g., [3]; see also [5], [9], [14], [38].

Interdependence of utilities: general description. In general, the utility u_i of i-th person under interdependence can be described as $u_i = f_i(u_i^{(0)}, u_j)$, where $u_i^{(0)}$ is the utility that does not take interdependence into account, and u_j are utilities of other people.

Interdependence of utilities: linear approximation. The effects of interdependence can be illustrated on the example of linear approximation, when we approximate the dependence by the first (linear) terms in its expansion into Taylor series, i.e., when the utility u_i of i-th person is equal to

$$u_i = u_i^{(0)} + \sum_{j \neq i} \alpha_{ij} \cdot u_j,$$

where the interdependence is described by the coefficients a_{ij}.

Paradoxes of love. This simple and seemingly natural model leads to interesting and somewhat paradoxical conclusions; see, e.g., [4], [6], [21].

For example, mutual affection between persons P_1 and P_2 means that $\alpha_{12} > 0$ and $\alpha_{21} > 0$. In particular, selfless love, when someone else's happiness means more than one's own, corresponds to $\alpha_{12} > 1$.

In general, for two persons, we thus have

$$u_1 = u_1^{(0)} + \alpha_{12} \cdot u_2;$$
$$u_2 = u_2^{(0)} + \alpha_{21} \cdot u_1.$$

Once we know the original utility values $u_1^{(0)}$ and $u_2^{(0)}$, we can solve this system of linear equations and find the resulting values of utility:

$$u_1 = \frac{u_1^{(0)} + \alpha_{12} \cdot u_2^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}};$$
$$u_2 = \frac{u_2^{(0)} + \alpha_{21} \cdot u_1^{(0)}}{1 - \alpha_{12} \cdot \alpha_{21}}.$$

As a result, when two people are deeply in love with each other ($\alpha_{12} > 1$ and $\alpha_{21} > 1$), then positive original pleasures $u_i^{(0)} > 0$ lead to $u_i < 0$ – i.e., to unhappiness. This phenomenon may be one of the reasons why people in love often experience deep negative emotions.

From this viewpoint, a situation when one person loves deeply and another rather allows him- or herself to be loved may lead to more happiness than mutual passionate love.

A similar negative consequence of love can also happen in situations like selfless Mother's love when $\alpha_{12} > 0$ may be not so large but α_{21} is so large that $\alpha_{12} \cdot \alpha_{21} > 1$.

There are also interesting consequences when we try to generalize these results to more than 2 persons. For example, we can define an ideal love, when each person treats other's emotions almost the same way as one's own, i.e., $\alpha_{12} = \alpha = 1 - \varepsilon$ for a small $\varepsilon > 0$. For two people, from $u_i^{(0)} > 0$, we get $u_i > 0$ – i.e., we can still have happiness. However, if we have three or more people in the state of mutual affection, i.e., if

$$u_i = u_i^{(0)} + \alpha \cdot \sum_{j \neq i} u_j,$$

then in case when everything is fine – e.g., $u_i^{(0)} = u^{(0)} > 0$ – we have

$$u_i \cdot (1 - \alpha \cdot (n-1)) = u_i \cdot (2 - \varepsilon - (1 - \varepsilon) \cdot n) = u^{(0)},$$

hence

$$u_i = \frac{u^{(0)}}{2 - \varepsilon - (1 - \varepsilon) \cdot n} < 0,$$

i.e., we have unhappiness. This may be the reason why 2-person families are the main form – or, in other words, if two people care about the same person (e.g., his mother and his wife), all there of them are happier if there is some negative feeling (e.g., jealousy) between them.

Comment. It is important to distinguish between emotional interdependence in which one's utility is determined by the utility of other people, and "objective" altruism, in which one's utility depends on the material gain of other people – but not on their subjective utility values, i.e., in which (in the linearized case)

$$u_i = u_i^{(0)} + \sum_j \alpha_j \cdot u_j^{(0)}.$$

In this approach, when we care about others' well-being and not their emotions, no paradoxes arise, and any degree of altruism only improves the situation; see, e.g, [10], [11], [30].

This objective approach to interdependence was proposed and actively used by yet another Nobel Prize winner: Amartya K. Sen; see, e.g., [35], [36], [37].

An alternative explanation of the paradoxes of love is that there is a time delay Δt between the emotions of a person and the reaction of the other persons to these emotions. This time delay can be very small, in factions of a second need to process the information, but still, the utility of a person at a moment time t depends not on the utility of others at the very same moment of time, but rather on the utility at some previous moment of time $t - \Delta t$. In this case, even if we have mutual affection, we avoid negative values of utility

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Appendix 1: Proof that For Territorial Division, It Is Beneficial to Report the Correct Utilities

Structure of the proof. We start by re-scaling the values of the utility functions so that all the gains become between 0 and 1. Then, we compute the optimistic gain u^+, the pessimistic gain u^-, and come up with the conclusion.

Re-scaling. Since the utility function is defined modulo an arbitrary multiplicative constant, we can always re-scale the utility functions in such a way that $u_i(A) = 1$ for the entire space A, i.e., that $\int_A v_i(t)\,dt = 1$ for all i.

Proof that $u^+ \le 1$. Let us show that for every function $v_1'(t)$, the optimistic estimate

$$u^+(v_1') \stackrel{\text{def}}{=} \max_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n)$$

cannot exceed 1.

Indeed, no matter what partition $X_1, \ldots, X_n$ we have, the resulting utility of the first participant $u_1(X_1) = \int_{X_1} v_1(t)\,dt$ cannot exceed the whole interval $\int_A v_1(t)\,dt$. We have re-scaled all the utility functions in such a way that this whole integral is 1, so the gain of P_1 cannot exceed 1.

Proof that $u^+ = 1$. Let us show that for every function $v_1'(t)$, the optimistic estimate $u^+(v_1')$ is equal to 1.

For that, let us show that we can get the gain which is as close to 1 as possible.

Indeed, suppose that the participant P_2 is only interested in a small neighborhood of a point t_2, so that, for some small number $\varepsilon > 0$, we have $v_2(t) = 0$ for all the points at distance $> \varepsilon$ from this point t_2. Suppose also that the participant P_3 is only interested in a small neighborhood of a point $t_3 \ne t_2$, …, and the participant P_n is only interested in a small neighborhood of a point t_n – and all these $n-1$ neighborhoods $N_2, \ldots, N_n$ are disjoint.

Then, in an optimal partition, it does not make sense to assign to P_2 any points outside N_2 – because adding these points to X_2 would not change the utility $u_2 = \int_{X_2} v_2(t)\,dt$, but assigning them to X_1 would increase u_1 – if $v_1(t_2) > 0$. Similarly, in the Nash optimal partition, participant P_3 is only assigned the points in N_3, etc. So, in the Nash optimal partition, all the points outside $n-1$ small neighborhoods $N_2, \ldots, N_n$ are assigned to P_1. So, the resulting utility of P_1 is equal to the integral of $v_1(t)$ over the complement to these neighborhoods. When $\varepsilon \to 0$, this integral tends to $\int_A v_1(t)\,dt$ – i.e., to 1.

Comment. We have shown that the optimistic gain u^+ is always equal to 1 – irrespective of what utility function we report. Thus, in the purely optimistic case, we can report any utility function and get the same result. So, we have proved the comment after Proposition 1.

Proof that when we report the correct utility function $v_1'(t) = v_1(t)$, then $u^- \ge 1/n$. Let us now prove that when $v_1'(t) = v_1(t)$, then the pessimistic estimate

$$u^-(v_1') \stackrel{\text{def}}{=} \min_{v_2,\ldots,v_n} u(v_1', v_1, v_2, \ldots, v_n)$$

is larger than or equal to $1/n$.

Let us show that for the optimal partition, $u_1(X_1) \ge u_1(X_2)$. Indeed, according to [26], in the optimal partition there exists a threshold value λ such that of all the points x from the union $X_1 \cup X_2$, points with $v_2(x)/v_1(x) < \lambda$ are assigned to X_1 and points with $v_2(x)/v_1(x) > \lambda$ are assigned to X_2.

Similar to the proofs from [26], let us add a small neighborhood of a point $x_0 \in X_2$ where $v_2(x_0)/v_1(x_0) \approx \lambda$ to the set X_1. This adds $\varepsilon \cdot v_1(x_0)$, where ε is the volume of this neighborhood, to the utility $u_1(X_1)$ of the first participant, and subtracts $\varepsilon \cdot v_2(x_0) \approx \varepsilon \cdot v_1(x_0) \cdot \lambda$ from $u_2(X_2)$. Thus, in Nash's product, the subproduct $u_1(X_1) \cdot u_2(X_2)$ is replaced by

$$(u_1(X_1) + \varepsilon \cdot v_1(x_0)) \cdot (u_2(X_2) - \varepsilon \cdot v_1(x_0) \cdot \lambda) =$$

$$u_1(X_1) \cdot u_2(X_2) \cdot v_1(x_0) \cdot \varepsilon \cdot (u_2(X_2) - \lambda \cdot u_1(X_1)) + o(\varepsilon).$$

Since the partition $X_1, X_2, \ldots$ was maximizing the Nash product, this change can only decrease the value of the product; so, we conclude that

$$\lambda \cdot u_1(X_1) \geq u_2(X_2).$$

For values $x \in X_2$, we have $v_2(x)/v_1(x) \geq \lambda$, so $v_2(x) \geq \lambda \cdot v_1(x)$. Integrating this inequality over X_2, we conclude that

$$u_2(X_2) = \int_{X_2} v_2(t)\,dt \geq \lambda \cdot \int_{X_2} v_1(t)\,dt = \lambda \cdot u_1(X_2).$$

So, from $\lambda \cdot u_1(X_1) \geq u_2(X_2)$, we conclude that

$$\lambda \cdot u_1(X_1) \geq u_2(X_2) \geq \lambda \cdot u_1(X_2),$$

hence $u_1(X_1) \geq u_1(X_2)$.

Similarly, $u_1(X_1) \geq u_1(X_i)$ for all i. By adding the inequalities corresponding to $i = 1, 2, \ldots, n$, we conclude that

$$n \cdot u_1(X_1) \geq u_1(X_1) + u_1(X_2) + \ldots + u_1(X_n) =$$

$$\int_{X_1} v_1(t)\,dt + \ldots + \int_{X_n} v_1(t)\,dt = \int_A v_1(t)\,dt = 1,$$

hence $u_1(X_1) \geq 1/n$.

Proof that when we report the correct utility function $v_1'(t) = v_1(t)$, then $u^- = 1/n$. We have shown that for all utility functions, $u^- \geq 1/n$. Let us prove that there exist utility functions for which $u^- = 1/n$.

Indeed, if we take $v_2(t) = \ldots = v_n(t) = v_1(t)$, then each point has the same value for all n participants; so, we simply divide the overall utility of 1 into n parts $u_1 + \ldots + u_n = 1$ for which the product $u_1 \cdot \ldots \cdot u_n$ is the largest possible. It is well known (and easy to prove) that the largest value of this product is attained when all u_i are equal: $u_1 = \ldots = u_n = 1/n$. In this case, each participant gets exactly the utility $1/n$.

Thus, indeed $u^- = 1/n$.

Proof that when we report a false utility function $v_1'(t) \neq v_1(t)$, then $u^- < 1/n$. If we report a utility function $v_1'(t)$, then, as we have just mentioned, when $v_2(t) = \ldots = v_n(t) = v_1'(t)$, the participant P_1 can get, as X_1, any set for which $\int_{X_1} v_1'(t)\,dt = 1/n$.

Since $v_1(t) \neq v_1'(t)$, let us take, as X_1, the set of all the values for which $v_1(t)/v_1'(t) \leq \lambda$, where the threshold λ is determined by the condition that $\int_{X_1} v_1'(t)\,dt = 1/n$. For $t \in X_1$, we have $v_1(t) \leq \lambda \cdot v_1'(t)$, hence

$$u_1(X_1) = \int_{X_1} v_1(t)\,dt \leq \lambda \cdot \int_{X_1} v_1'(t)\,dt = \lambda \cdot (1/n),$$

and the equality is only possible if $v_1'(t) = \lambda \cdot v_1(t)$ for all $t \in X_1$.

Similarly, for $t \notin X_1$, we have $v_1(t) \geq \lambda \cdot v_1'(t)$, hence

$$1 - u_1(X) = \int_{-X_1} v_1(t)\,dt \geq \lambda \cdot \int_{-X_1} v_1'(t)\,dt = \lambda \cdot (1 - 1/n),$$

and the equality is only possible if $v_1'(t) = \lambda \cdot v_1(t)$ for all $t \notin X_1$.

By dividing the inequalities $u_1(X_1) \leq \lambda \cdot (1/n)$ and $1 - u_1(X_1) \geq \lambda \cdot (1 - 1/n)$, we conclude that

$$\frac{u_1(X_1)}{1 - u_1(X_1)} \leq \frac{1/n}{1 - 1/n},$$

where the inequality is only possible when $v_1(t) = \lambda \cdot v_1'(t)$ for all t.

Since both $v_1(t)$ and $v_1'(t)$ are normalized to 1 in the sense that $\int_A v_1(t)\,dt = \int_A v_1'(t)\,dt = 1$, the only way to have $v_1'(t) = \lambda \cdot v_1(t)$ for all $t \in A$ is to have $\lambda = 1$ and thus, $v_1'(t) = v_1(t)$. We know that $v_1'(t) \neq v_1(t)$, hence equality is impossible, and

$$\frac{u_1(X_1)}{1 - u_1(X_1)} < \frac{1/n}{1 - 1/n}.$$

Reversing both sides in this inequality, we get

$$\frac{1 - u_1(X_1)}{u_1(X_1)} > \frac{1 - 1/n}{1/n}.$$

Adding 1 to both sides, we get $1/u_1(X_1) > 1/(1/n) = n$, hence $u_1(X_1) < 1/n$. The statement is proven.

Conclusion. When we report a correct utility function, we get $u^+ = 1$ and $u^- = 1/n$. When we report a false utility function, then we get $u^+ = 1$v and $u^- < 1/n$. Thus, for every $\alpha > 0$, the value of $\alpha \cdot u^- + (1 - \alpha) \cdot u^+$ is the largest when we report the correct utility function. The theorem is proven.

Appendix 2: Proof that, Modulo Permutation, We Can Uniquely Reconstruct Individual Preferences from Group Decisions

For every lottery, we can compare this lottery with lotteries in which $q_1 = \ldots = q_m = 0$, and thus, get a value q^+ for which

$$\left(q^+\right)^n = \prod_{i=1}^n (p^+ + p_1 \cdot u_{i1} + \ldots + p_m \cdot u_{im}).$$

In other words, based on the group preferences, for every $m+1$ non-negative values $p^+, p_1, \ldots, p_m$ for which $p^+ + p_1 + \ldots + p_m \leq 1$, we can determine the value of the function

$$F(p^+, p_1, \ldots, p_m) \stackrel{\text{def}}{=} \prod_{i=1}^n (p^+ + p_1 \cdot u_{i1} + \ldots + p_m \cdot u_{im}). \quad (2)$$

This function is a polynomial of n-th order in terms of $m+1$ variables. Each such polynomial has the form

$$\sum_{i^+=1}^{n}\sum_{i_1=1}^{n}\ldots\sum_{i_m=1}^{n} a_{i^+,i_1,\ldots,i_n} a_{i^+,i_1,\ldots,i_m} \cdot (p^+)^{i^+} \cdot p_1^{i_1} \cdot \ldots \cdot p_n^{i_n}.$$

Thus, each such polynomial is uniquely determined by finitely many coefficients $a_{i^+,i_1,\ldots,i_m}$.

Based on the group preferences, we know the values of this polynomial at infinitely many points; based on these points, we can uniquely reconstruct the coefficients – by solving the corresponding system of linear equations in terms of the unknown $a_{i^+,i_1,\ldots,i_n}$.

So, based on the group preferences, we can uniquely reconstruct the polynomial $F(p^+,p_1,\ldots,p_m)$. The above representation (2) means that we factorize the polynomial into n linear factors. Factorization of a polynomial into irreducible factors is known to be unique modulo scalar factors: i.e., if $F=\prod F_i$ and $F=\prod F'_j$, then each factor F_i is equal to $c\cdot F'_j$ for some constant c and some factor F'_j. In our case, all factors have a coefficient 1 at p^+, so $c=1$. Thus, modulo permutation, the factors $p^+ + p_1\cdot u_{i1}+\ldots+p_m\cdot u_{im}$ – hence the values u_{ij} – are uniquely determined by the group preferences.

The theorem is proven.