The mathematical analysis of t-norms is logically non-trivial

Sándor Jenei[1], Vladik Kreinovich[2],
David Ponevac[2], and Yaffa Al Bayyari[2]
[1]Institute of Mathematics and Informatics
University of Pécs H-7624
Pécs, Ifjúság útja 6., Hungary
jenei@ttk.pte.hu
[2]Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
vladik@utep.edu

Abstract

It is well-known that t-norms are widely applicable in certain models, which describe human reasoning about uncertainty, and that for different applications, different t-norms fit better. Thus, given a practical problem, it is important to be able to find a t-norm which is the most suitable for that particular problem. To solve such optimization problems, it would be desirable to know the structure of the class of all possible t-norms. Toward this – probably unreachable – goal there are many interesting open problems. If the corresponding mathematical problems are expressed in terms of quantifiers and logical connectives, then we get formulas which are very similar to formulas about real numbers. A. Tarski has proved that there is a *deciding* algorithm – i.e., an algorithm that, given a formula for real numbers, decides whether it is true or not – for real numbers. So, the natural question is whether we can extend Tarski's algorithm to a class of mathematical statements about t-norms? The answer is "no": once we allow quantifiers over t-norms, no deciding algorithm exists. In this sense, in general, the analysis of the mathematical properties of t-norms is logically non-trivial.

1 Formulation of the Problem

t-norms are practically important. The traditional two-valued logic is well suited for describing statements about which we are 100% sure. Very often, however, we have to supplement this absolute knowledge with expert knowledge.

Experts are rarely 100% confident in their rules and recommendations; at best, for each statement, they can provide their degree of confidence in this statement. Usually, this degree is described either by a linguistic term or by a number from the interval $[0, 1]$: 1 means full confidence, 0 means no confidence, and values in between describe partial confidence.

One of the main reasons for gathering expert knowledge is that we want to make *logical deductions* based on this knowledge. The simplest case of logical deduction is when we deduce a new statement S based on two expert statements S_1 and S_2. The natural question is: we know the expert's degrees of confidence s_1 and s_2 in the statements S_1 and S_2; what is the resulting degree of confidence of the statement S?

For the new statement S to be true, both original statements S_1 and S_2 must be true. The degree of confidence in S is determined as the degree of confidence that the conjunction $S_1 \,\&\, S_2$ is true.

This degree of confidence depends not only on the expert's degrees s_1 and s_2[1] but also on the extent to which the statements S_1 and S_2 are related.

So, in the ideal world, we should ask the same expert not only about the individual degrees s_i of different statements S_i, but also about the degree of confidence in all possible combinations $S_i \,\&\, S_j$, $S_i \,\&\, S_j \,\&\, S_k$, etc. However, there are exponentially many possible combination, so it is practically impossible to elicit the degrees in all of them.

As a result, we face the following problem: given degrees of confidence s_1 and s_2 in two statements S_1 and S_2 (and no other information about S_1 and S_2), we must use these two numbers to provide an estimate $s = T(s_1, s_2)$ of the expert's degree of confidence in $S_1 \,\&\, S_2$. Which function T should we use to compute this degree? The following conditions are natural to be posed.

- If S_1 is absolutely false, i.e., if $s_1 = 0$, then $S_1 \,\&\, S_2$ should also be absolutely false, i.e., we must have $T(a, 0) = 0$ for all a.

- If S_1 is absolutely true, i.e., if $s_1 = 1$, then $S_1 \,\&\, S_2$ should be equivalent to S_2, i.e., we must have $T(a, 1) = a$ for all a.

- Statements $S_1 \,\&\, S_2$ and $S_2 \,\&\, S_1$ have the same meaning, so we must have $T(a, b) = T(b, a)$ for all a and b.

- Similarly, statements $S_1 \,\&\, (S_2 \,\&\, S_3)$ and $(S_1 \,\&\, S_2) \,\&\, S_3$ also have the same meaning, so we must have $T(a, T(b, c)) = T(T(a, b), c)$ for all a, b, and c.

- Finally, if we increase the degree of confidence of one or both of the statements S_i, then our degree of confidence in $S_1 \,\&\, S_2$ must also increase (or at least remain the same, but not decrease). In other words, if $s_1 \le s_1'$ and $s_2 \le s_2'$, then we should have $T(s_1, s_2) \le T(s_1', s_2')$.

Definition 1. *A function $T : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies these conditions, i.e., for which $T(a, 0) = 0$, $T(a, 1) = a$, $T(a, b) = T(b, a)$, $T(a, T(b, c)) =$*

[1] In mathematical fuzzy logic this phenomena is usually referred to as "truth-functionality".

$T(T(a,b),c)$*, and which is monotonic in each of the two variables, is called a* t-norm.

T-norms are used in many practical applications; see, e.g., [6, 15, 23, 26, 27, 32]

In different situations, different t-norms are most adequate. It is well known that there are infinitely many (in fact, continuum many) different t-norms. A natural question is: Which t-norm fits better to a certain problem?

It is known that in different areas of human expertise (and even in different problems within the same area of expertise), different t-norms work better: e.g., in medical diagnostic, where caution is important, the t-norm which most adequately describes human reasoning is different from, e.g., geophysics where we need to make bold conclusions fast; see, e.g., surveys [18, 24] and references therein.

It is important to analyze the class of t-norms. Since in different practical applications, different t-norms are better, it is important to be able to find, for each practical situation, the most appropriate t-norm. To solve the corresponding optimization problems, it would be desirable to know the structure of the class of t-norms.

For example, it is known that optimization problems are much easier to solve if we are optimizing a convex function over a convex domain D, i.e., a domain for which for every x and y and for every $\alpha \in [0,1]$, the convex combination $\alpha \cdot x + (1-\alpha)\cdot y$ also belongs to D. As a result, even if the domain is not convex but has some convex subset, we may still be able to enhance the solution of our optimization problem.

Thus, it is reasonable to ask: is the set of all t-norms convex? That is, if we have two t-norms $T(a,b)$ and $T'(a,b)$, is their convex combination $\alpha \cdot T(a,b) + (1-\alpha)\cdot T'(a,b)$ also a t-norm?

It is conjectured in [3] that the answer to this question is "never": the resulting convex combination is never a t-norm (except for trivial cases, e.g., when $\alpha = 0$, or $T = T'$) because the convex combination is not associative.[2] Contrary to the conjecture, there are many examples where the convex combination *is* a t-norm if the underlying t-norms are *not* left-continuous. Unfortunately, such t-norms are absolutely uninteresting from the viewpoint of logical applications. A partial solution for the class of left-continuous t-norms is in [13]. The general question is open even for the class of continuous Archimedean t-norms [3]. Moreover, it is not even known whether the product t-norm $T(a,b) = a \cdot b$ – one of the simplest and most widely used t-norms – can be represented as the convex combination of two different t-norms.

Is there a general algorithm for solving such problems? The above open problem about the product t-norm, when described formally, has the following form:

[2] It is worth mentioning that in some practical situations, such non-associative combination functions may also adequately describe expert reasoning; see, e.g., [31] and references therein.

$$\exists T \exists T' \exists \alpha\, ((\exists a \exists b\, (T(a,b) \neq T'(a,b)))\, \&\, 0 \leq \alpha \leq 1\, \&$$
$$\forall c \forall d\, (\alpha \cdot T(c,d) + (1-\alpha) \cdot T'(c,d) = c \cdot d)).$$

Here, T and T' run over arbitrary t-norms, while the variables a, b, c, d, and α are real-valued.

This statement is similar to the statements from a class for which there is a known algorithm (first designed by Tarski) for deciding whether a given formula is true or not. Tarksi's algorithm [29, 30] (see also [4, 17]) deals with the *first order theory of real numbers*, i.e., with formulas of the following type:

- We start with *variables* x, y, z, ..., that run over real numbers, and with two constants: 0 and 1.
- From these variables and constants, we can form *expressions* (also called *terms*) by applying addition and multiplication. For example, $x \cdot x + y \cdot y$ is an expression in this sense.
- From expressions e, e', ..., we can form *elementary formulas* of the type $e = e'$, $e \neq e'$, $e > e'$, $e < e'$, $e \leq e'$, and $e \geq e'$. For example, $x \cdot x + y \cdot y = z \cdot z$ is an elementary formula in our language.
- From elementary formulas, we can form *formulas* by applying logical connectives & ("and"), $\vee$ ("or"), $\Rightarrow$ ("implies"), $\neg$ ("not"), and quantifiers $\forall x$ and $\exists x$.

(Readers who are interested in technical details related to logic in general, and first order logic in particular, can also consult [5, 11, 28].)

The original Tarski's algorithm is not always practically useful: it sometimes takes time $\approx 2^{2^n}$ for an input of size n [9] (see also [7, 8]). However, for this class of formulas, there exist more efficient algorithms that enable us to solve many practical problems by reducing them to the first order theory of real numbers. In particular, there are applications to transportation problems [20], to control system design [1], etc. (see also [2]).

Tarksi's algorithm has also been applied to t-norms: namely, for a fixed algebraic t-norm such as $t_1(a,b) = a \cdot b$ or $t_2(a,b) = \min(a,b)$, we can add the expression $t_i(a,b)$ to the formulas and still get a *decidable* theory, i.e., a theory in which we have an algorithm that decides whether a given formula holds or not [25].

Can we extend Tarski's algorithm to the above formula? The only difference between the above formula (describing an open problem) and Tarski's formulas is that we also allow quantifiers over t-norms. It is known that if we allow quantifiers over *arbitrary* functions, then the problem stops being decidable; see, e.g., [16]. What if we allow quantifiers over t-norms instead?

What we do in this paper. We prove that the theory remains undecidable if we allow quantifiers over *classes* of t-norms. Moreover, our results reveal that even the theory of certain *single* t-norms is undecidable.

Preliminaries. A t-norm is called *continuous* if it is continuous as a two-place function. A usual condition on continuous t-norms is that if we have two different statements S_1 and S_2 with the same degree of confidence s $(0 < s < 1)$, then our degree of confidence that both of these statements are true should be smaller than our degree of confidence in each of these statements. More formally, this condition means that $T(s, s) < s$ for all such $s \in]0, 1[< 1$. Such continuous t-norms are called *Archimedean*; see, e.g., [6, 15, 23, 26, 27]. A continuous Archimedean t-norm is called *nilpotent* if it has zero divisors (that is, if there exists $x \in]0, 1]$ such that there exists $y \in]0, 1]$ with $T(x, y) = 0$). A prototype of nilpotent t-norms is the so-called Lukasiewicz t-norm, given by

$$T_{\mathbf{L}}(x, y) = \max(0, x + y - 1).$$

A continuous Archimedean t-norm is called *strict* if it has no zero divisors. An example is the product t-norm, given by

$$T_{\mathbf{P}}(x, y) = x \cdot y.$$

In fact, these are the unique examples for nilpotent and for strict t-norms up to φ-transformation, as shown by the following theorem.

Theorem 1. [19] *Any nilpotent t-norm T is isomorphic to $T_{\mathbf{L}}$, that is, there exists φ, which is an increasing bijection of* $[0, 1]$*, such that T_φ, the φ-transform of T, is the Lukasiewicz t-norm. That is,*

$$T_\varphi(x, y) := \varphi^{-1}\left(T(\varphi(x), \varphi(y))\right) = T_{\mathbf{L}}(x, y).$$

Any strict t-norm T is isomorphic to $T_{\mathbf{P}}$, that is, there exists φ, which is an increasing bijection of $[0, 1]$*, such that T_φ, the φ-transform of T, is the product t-norm. That is,*

$$T_\varphi(x, y) := \varphi^{-1}\left(T(\varphi(x), \varphi(y))\right) = T_{\mathbf{P}}(x, y).$$

In some situations, one may want to consider discontinuous t-norms. In this case, at each discontinuity point, since the function T is monotonic, we can select either the left limit or the right limit as the value for this point. It is customary to consider left-continuous t-norms; see, e.g., [3, 13]. Left-continuous t-norms are widely used in non-classical logics. The important role of left-continuity is that this condition is equivalent to that $([0, 1], T)$ can be equipped with the structure of a *residuated lattice.*

2 Main Result

Definition 2. *Let T be a class of t-norms. We define a* first order theory of t-norms from the class T *as follows:*

- *Let* V_r *and* V_t *be two sets. Elements of* V_r *will be called* variables which run over real numbers *and elements of* V_t *will be called* variables that run over t-norms.
- *A* term *is defined inductively:*
  - *constants 0, 1, and variables* $v \in V_r$ *are terms;*
  - *if* e_1 *and* e_2 *are terms and* $t \in V_t$*, then* (e_1)*,* $e_1 + e_2$*,* $e_1 \cdot e_2$*, and* $T(e_1, e_2)$ *are also terms.*
- *An* elementary formula *is an expression of the type* $e_1 = e_2$*,* $e_1 < e_2$*,* $e_1 > e_2$*,* $e_1 \le e_2$*,* $e_1 \ge e_2$*, and* $e_1 \ne e_2$*.*
- *A* formula *is defined inductively:*
  - *every elementary formula is a formula;*
  - *if* F *and* F' *are formulas, then* $\neg F$*,* $F \vee F'$*, and* $F \,\&\, F'$ *are also formulas;*
  - *if* F *is a formula with a variable* $x \in V_r \cup V_t$*, then* $\exists x\, F$ *and* $\forall x\, F$ *are also formulas.*

Definition 3. *Let* T *be a class of t-norms.*

- *We say that a theory of t-norms from this class is* decidable *if there exists an algorithm that, given a formula (without free variables), tells whether this formula is true (when all the quantifiers* $\forall t$ *and* $\exists t$ *are interpreted as going over t-norms from the class* T*).*
- *We say that a theory of t-norms from this class is* undecidable *if no such algorithm exists.*

Theorem 2. *We have the following undecidability results.*

1. *The theory of strict t-norms is undecidable.*
2. *The theory of nilpotent t-norms is undecidable.*
3. *The theory of continuous Archimedean t-norms is undecidable.*
4. *The theory of continuous t-norms is undecidable.*
5. *The theory of left-continuous t-norms is undecidable.*
6. *The theory of t-norms is undecidable.*

Proof.

1°. It is known that while the first order theory of *real* numbers is decidable, the first order theory of *natural numbers* is not decidable. In our proof, we refer to the result of Matiyasevich et al. [10, 21, 22] saying that no algorithm exists which solves Diophantine equations with 13 variables, i.e., no algorithm can decide whether a formula

$$\exists x_1 \ldots \exists x_{13}[(x_1 \in \mathbf{N}) \,\&\, \ldots \,\&\, (x_{13} \in \mathbf{N}) \,\&\, Q(x_1, \ldots, x_{13}) = 0]$$

is true, where $\mathbf{N}$ denotes the set of all natural numbers and Q is a polynomial with integer coefficients. (This result solved Hilbert's tenth problem [12].)

So, if we can express the property $x \in \mathbf{N}$ in terms of t-norms, then we will be able to describe the formulas from the above undecidable class in terms of t-norms – and thus, prove that the corresponding theory is undecidable.

2°. We propose the following formula to describe $x \in \mathbf{N}$:

$$x = 0 \vee \forall T \left(P(T) \Rightarrow T\left(1 - \frac{1}{x+3}, 1 - \frac{1}{x+3}\right) = 1 - \frac{1}{x+2}\right), \qquad (1)$$

where $P(T)$ is the following formula:

$$T\left(1 - \frac{1}{0+3}, 1 - \frac{1}{0+3}\right) = 1 - \frac{1}{0+2} \,\&$$

$$\forall a \left(T\left(1 - \frac{1}{a+3}, 1 - \frac{1}{a+3}\right) = 1 - \frac{1}{a+2} \Rightarrow \right.$$

$$T\left(1 - \frac{1}{a+3}, 1 - \frac{1}{a+3}\right) = 1 - \frac{1}{a+2} \,\&$$

$$\left.\forall b \left(a < b < a+1 \Rightarrow T\left(1 - \frac{1}{b+3}, 1 - \frac{1}{b+3}\right) \neq 1 - \frac{1}{b+2}\right)\right).$$

3°. If there exists a t-norm T which satisfies property $P(T)$, then for this t-norm, by the definition of this property, we have:

- $T\left(1 - \frac{1}{x+3}, 1 - \frac{1}{x+3}\right) = 1 - \frac{1}{x+2}$ for $x = 0$;
- hence, the property $T\left(1 - \frac{1}{x+3}, 1 - \frac{1}{x+3}\right) = 1 - \frac{1}{x+2}$ does not hold for any x between 0 and $0 + 1 = 1$, but it does hold for $x = 1$;
- since the property $T\left(1 - \frac{1}{x+3}, 1 - \frac{1}{x+3}\right) = 1 - \frac{1}{x+2}$ holds for $x = 1$, it does not hold for any x between 1 and $1 + 1 = 2$, but it does hold for $x = 2$;
- ...

By induction, one can easily prove that

$$T\left(1-\frac{1}{x+3},1-\frac{1}{x+3}\right)=1-\frac{1}{x+2} \tag{2}$$

holds if and only if $x \in \mathbf{N}$. Thus, the formula in (1) is true if and only if x is a natural number.

4°. That is, if there exists a t-norm T which satisfies property $P(T)$, then for this t-norm, by the definition of this property, we have (2) if and only if $x \in \mathbf{N}$. Thus, by defining $f(y) = T(y, y)$ to be the diagonal of T ($y \in [0, 1]$) first we need to construct a function $f : [0, 1] \to [0, 1]$ such that

$$f\left(1-\frac{1}{x+3}\right)=1-\frac{1}{x+2} \tag{3}$$

holds if and only if $x \in \mathbf{N}$. If x runs in $[0, \infty[$ then $y = 1 - \dfrac{1}{x+2}$ runs in $\left[\dfrac{2}{3}, 1\right[$, and (3) has the following equivalent form, as easily verified:

$$f\,(y)=2-\frac{1}{y} \tag{4}$$

holds if and only if $y \in H := \left\{\dfrac{2}{3}, \dfrac{3}{4}, \ldots, 1-\dfrac{1}{x+3}, \ldots\right\}$.

5°. First we shall prove Statement 1 of the Theorem. We shall use that any strictly increasing continuous $[0, 1] \to [0, 1]$ function with $f(0) = 0$, $f(1) = 1$, can be a diagonal of a strict t-norm (see, e.g., [14] and references therein).

Consider any strictly increasing continuous function $f : [0, 1] \to [0, 1]$ with $f(0) = 0$, $f(1) = 1$, which, for $x \in [0, 1[$, satisfies (4) if and only if $y \in H$. That is, which satisfies $f\left(\dfrac{2}{3}\right) = \dfrac{1}{2}$, $f\left(\dfrac{3}{4}\right) = \dfrac{2}{3}$, $f\left(\dfrac{4}{5}\right) = \dfrac{3}{4}$, …, but which is at $y \in [0, 1[\setminus H$ different from $2 - \dfrac{1}{y}$. Such a function clearly exists. Therefore, for the f, which is constructed above there exists a strict t-norm T, such that f is its diagonal. This proves Statement 1.

Next we prove Statement 2 of the Theorem. We shall use that any increasing continuous $[0, 1] \to [0, 1]$ function with $f(0) = 0$, $f(1) = 1$, which is not strictly increasing, but which is strictly increasing on the preimage of $]0, 1]$, can be a diagonal of a nilpotent t-norm (see, e.g., [14] and references therein).

Consider any such function $f : [0, 1] \to [0, 1]$ which, for $y \in [0, 1[$, satisfies (4) if and only if $y \in H$, and which, in addition, satisfies $f\left(\dfrac{1}{3}\right) = 0$

(observe that $2 - \dfrac{1}{1/3} \neq 0$). That is, f satisfies $f\left(\dfrac{2}{3}\right) = \dfrac{1}{2}$, $f\left(\dfrac{3}{4}\right) = \dfrac{2}{3}$, $f\left(\dfrac{4}{5}\right) = \dfrac{3}{4}$, ..., but at $y \in [0,1[\backslash H$ it is different from $2 - \dfrac{1}{y}$. Such a function clearly exists. Therefore, for the f, which is constructed above there exists a nilpotent t-norm T, such that f is its diagonal. This proves Statement 2.

To conclude the proof of Statements 3–6, observe that, e.g., the nilpotent t-norm, which is constructed above is continuous Archimedean, continuous, left-continuous, respectively.

The theorem is proven.

Remark. Observe that in our proof, we have proven even more. Namely, consider a *single* t-norm, which has property (2) if and only if $x \in \mathbf{N}$. There are many such t-norms, it follows from 5° (indeed, there exists at least one for each suitable diagonal). Then it follows from the proof of Theorem 2 that even the theory of this single t-norm is undecidable. Moreover, one may choose another logical formula instead of (1), which captures $x \in \mathbf{N}$. Then it may yield another property instead of (2), and the theory of any single t-norm, which satisfies that property is undecidable.

3 Auxiliary Results

As we have mentioned, t-norms describe the degree of confidence in statements of the type $S_1 \,\&\, S_2$. To describe degree of confidence in a statement of the type $S_1 \vee S_2$, *t-conorms* are used, which are functions $S : [0,1] \times [0,1] \to [0,1]$ for which $S(a,0) = a$, $S(a,1) = 1$, $S(a,b) = S(b,a)$, $S(a,S(b,c)) = S(S(a,b),c)$, and which is monotonic in each of the two variables. T-conorms are *dual* to t-norms, since for any t-norm T, the function $S(x,y) = 1 - T(1-x, 1-y)$ is a t-conorm, and vice versa. Of course, the properties that are defined for t-norms can be carried over to t-conorms as well. Note that for continuous t-conorms, the Archimedean property takes the form $T(s,s) > s$ for all $s \in]0,1[$.

Similar to the case of t-norms, we can define theories for different classes of t-conorms by allowing quantifiers $\forall S$ and $\exists S$ that run over all t-conorms from the corresponding class. These theories are also undecidable:

Theorem 3. *We have the following undecidability results.*

1. *The theory of strict t-conorms is undecidable.*
2. *The theory of nilpotent t-conorms is undecidable.*
3. *The theory of continuous Archimedean t-conorms is undecidable.*
4. *The theory of continuous t-conorms is undecidable.*
5. *The theory of right-continuous t-norms is undecidable.*

6. The theory of t-conorms is undecidable.

Proof. It follows from the duality between t-norms and t-conorms. From the logical viewpoint, every formula related to t-norms can be described in terms of t-conorms, and vice versa. Hence, Theorem 2 implies Theorem 3. The theorem is thus proven.

A similar result holds for *strong negations*, which are strictly increasing continuous functions $N : [0,1] \to [0,1]$ for which $N(0) = 1$, $N(1) = 0$, and $N(N(x)) = x$ ($x \in [0,1]$) holds.

Theorem 4. *The theory of strong negations is undecidable.*

Proof. We propose the following formula to describe $x \in \mathbf{N}$:

$$x = 0 \vee \left(\forall n \left(P(N) \Rightarrow N\left(\frac{1}{x+2}\right) = 1 - \frac{1}{x+2}\right)\right), \tag{5}$$

where $P(N)$ is the following formula:

$$N\left(\frac{1}{0+2}\right) = 1 - \frac{1}{0+2} \,\&$$

$$\forall a \left(\left(a \geq 0 \,\&\, N\left(\frac{1}{a+2}\right) = 1 - \frac{1}{a+2}\right) \Rightarrow\right.$$

$$N\left(\frac{1}{(a+1)+2}\right) = 1 - \frac{1}{(a+1)+2} \,\&$$

$$\left.\forall b \left(a < b < a+1 \Rightarrow N\left(\frac{1}{b+2}\right) \neq 1 - \frac{1}{b+2}\right)\right).$$

If there exists a strong negation N which satisfies property $P(N)$, then for this strong negation, just like in the proof of Theorem 2, it holds true that the property

$$x \geq 0 \,\&\, N\left(\frac{1}{x+2}\right) = 1 - \frac{1}{x+2} \tag{6}$$

holds if and only if $x \in \mathbf{N}$. Thus, the formula in (5) is true if and only if x is a natural number.

To complete the proof, we need to show that there exists a strong negation which satisfies $P(N)$, i.e., a strong negation for which (6) holds if and only if $x \in \mathbf{N}$. If x runs in $[0, \infty[$ then $y = \dfrac{1}{x+2}$ runs in $\left]0, \dfrac{1}{2}\right]$, and (6) has the following equivalent form, as easily verified: for $y \leq \dfrac{1}{2}$,

$$f(y) = 1 - y \tag{7}$$

if and only if $y \in H := \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$.

Let $f : \left[0, \frac{1}{2}\right] \to \left[\frac{1}{2}, 1\right]$ be any such function, that is, which satisfies $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f\left(\frac{1}{3}\right) = \frac{2}{3}$, $f\left(\frac{1}{4}\right) = \frac{3}{4}$, $\ldots$, but which is at $y \in \left]0, \frac{1}{2}\right] \setminus H$ different from $1 - y$. In addition, we may safely assume that f is strictly decreasing and continuous. Such a function clearly exists. Define $\bar{f} : [0,1] \to [0,1]$ by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \left[0, \frac{1}{2}\right] \\ f^{-1}(x) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

It is a straightforward exercise to check that $\bar{f}$ is well-defined, and is a strong negation, such that (7) holds if and only if $y \in H := \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$. The theorem is proven.

Acknowledgments

This work was supported in part by NSF grants EAR-0225670, DMS-0532645, and HRD-1242122 (Cyber-ShARE Center of Excellence), and by the Texas Department of Transportation grant No. 0-5453. S. Jenei was supported by a Bolyai János Research Grant.

References

[1] C. Abdallah, P. Dorato, R. Liska, S. Steinberg, and W. Yang, "Applications of quantifier elimination theory to control system design", *Proceedings of the 4th IEEE Mediterranean Symposium on Control and Automation*, 1996.

[2] G. Alefeld, G. Mayer, and V. Kreinovich, "The shape of the symmetric solution set", In: R. B. Kearfott and V. Kreinovich (eds.), *Applications of Interval Computations*, Kluwer, Dordrecht, 1996, pp. 61–79.

[3] C. Alsina, M. J. Frank, and B. Schweizer, "Problems on associative functions", *Aequationes Mathematicae*, 2003, Vol. 66, pp. 128–140.

[4] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1983.

[5] J. Barwise (ed.). *Handbook of Mathematical Logic*. North-Holland, Amsterdam, 1977.

[6] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[7] N. Ben-Or, D. Kozen, and J. Reif, "The complexity of elementary algebra and geometry", *Journal of Computer and System Sciences*, 1986, Vol. 32, pp. 251–264.

[8] J. Canny, "Improved algorithms for sign determination and existential quantifier elimination", *The Computer Journal*, 1993, Vol. 36, No. 5, pp. 409–418.

[9] J. H. Davenport and J. Heintz, "Real quantifier elimination is doubly exponential", *Journal of Symbolic Computations*, 1988, Vol. 5, No. 1/2, pp. 29–35.

[10] M. Davis, Yu. V. Matiyasevich, and J. Robinson, "Hilbert's tenth problem. Diophantine equations: positive aspects of a negative solution", In: *Mathematical developments arising from Hilbert's problems*, Proceedings of Symposia in Pure Mathematics, Vol. 28, American Math. Society, Providence, Rhode Island, 1976, Part 2, pp. 323–378.

[11] H. B. Enderton. *A Mathematical Introduction to Logic*, Academic Press, New York, 1972.

[12] D. Hilbert, "Mathematical problems" (lecture delivered before the International Congress of Mathematics in Paris in 1900), translated in *Bull. Amer. Math, Soc.*, 1902, Vol. 8, pp. 437–479; reprinted in *Mathematical Developments Arising from Hilbert's Problems*, Proceedings of Symposia in Pure Mathematics, Vol. 28, American Math. Society, Providence, Rhode Island, 1976, Part 1, pp. 1–34.

[13] S. Jenei, "On the convex combination of left-continuous t-norms", *Aequationes Mathematicae*, 2006, Vol. 72, pp. 47-59.

[14] E. P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[15] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[16] O. M. Kosheleva and V. Kreinovich, "Unit-distance preserving theorem is locally non-trivial", *Geombinatorics*, 1995, Vol. 4, No. 4, pp. 119–128.

[17] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational Complexity and Feasibility of Data Processing and Interval Computations*, Kluwer, Dordrecht, 1997.

[18] V. Kreinovich, G. C. Mouzouris, and H. T. Nguyen, "Fuzzy rule based modeling as a universal approximation tool", In: H. T. Nguyen and M. Sugeno (eds.), *Fuzzy Systems: Modeling and Control*, Kluwer, Boston, Massachusetts, 1998, pp. 135–195.

[19] C-H. Ling, "Representation of associative functions", *Publ. Math. Debrecen*, 1965, Vol. 12, pp. 189–212.

[20] R. Loos and V. Weispfenning, "Applying linear quantifier elimination", *Computer Journal*, 1993, Vol. 36, No. 5, pp. 450–462.

[21] Yu. V. Matiyasevich, "Enumerable sets are diophantine", *Soviet Math. Doklady*, 1970, Vol. 11, pp. 354–357.

[22] Yu. V. Matiyasevich and J. Robinson, "Reduction of an arbitrary Diophantine equation to one in 13 unknowns", *Acta Arithmetica*, 1974, Vol. 27, pp. 521–553.

[23] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

[24] H. T. Nguyen and V. Kreinovich, "Methodology of fuzzy control: an introduction", In: H. T. Nguyen and M. Sugeno (eds.), *Fuzzy Systems: Modeling and Control*, Kluwer, Boston, Massachusetts, 1998, pp. 19–62.

[25] H. T. Nguyen, V. Kreinovich, and A. Di Nola, "Which truth values in fuzzy logics are definable?", *International Journal of Intelligent Systems*, 2003, Vol. 18, No. 10, pp. 1057–1064.

[26] H. T. Nguyen, C. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, CRC Press, Boca Raton, Florida, 2009.

[27] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.

[28] J. R. Schoenfield. *Mathematical Logic*, Addison-Wesley, 1967.

[29] A. Seidenberg, "A new decision method for elementary algebra", *Annals of Mathematics*, 1954, Vol. 60, pp. 365–374.

[30] A. Tarski, *A Decision Method for Elementary Algebra and Geometry*, 2nd ed., University of California Press, Berkeley and Los Angeles, 1951, 63 pp.

[31] R. Trejo, V. Kreinovich, I. R. Goodman, J. Martinez, and R. Gonzalez, "A realistic (non-associative) logic and a possible explanations of 7 ± 2 law", *International Journal of Approximate Reasoning*, 2002, Vol. 29, pp. 235–266.

[32] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.