Interval Computations Technology in Mathematics Research: From Help in Theoretical Breakthroughs to Practically Useful Results About Numerical Methods

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In many practical situations, we are interested in a quantity $y$ that is difficult or even impossible to measure directly. For example, it is difficult to directly measure the amount of oil in an oil field or the distance to a faraway star. To estimate the desired quantity $y$, we measure auxiliary quantities which are related to $y$ in a known way, and then use the results of measuring these quantities (and their known relation with $y$) to estimate $y$. For example, in geosciences, to estimate the density of the material at a large depth, we measure travel times of seismic waves, gravity values at different locations, etc. The corresponding estimation (data processing) is one of the main application of high performance computers.

Most data processing algorithms result in a numerical estimate for the desired quantity $y$. In practice, the relation between $y$ and the auxiliary quantities may be known only approximately, and due to inevitable measurement inaccuracies, the results of measuring the auxiliary quantities are also only approximately equal to the actual (unknown) values of these quantities. It is
therefore important to gauge how this uncertainty affects the result \( y \) of data processing, i.e., how accurate is the result of data processing.

Textbook approach to processing measurement uncertainty usually assumes that we know the probability distributions of measurement errors. In practice, however, these probabilities are often unknown; we only know the upper bounds on the measurement errors. In such situations, if we know the measurement result \( X \) and the upper bound \( D \) on the measurement error, then the only information that we can conclude about the actual (unknown) value of \( x \) is that \( x \) is in the interval \([X-D, X+D]\). It is therefore desirable to propagate this uncertainty through data processing algorithms and compute the resulting interval of possible values of \( y \). There exist many techniques for such interval computations; see, e.g., http://www.cs.utep.edu/interval-comp. These techniques use high performance computers to estimate the accuracy of \( y \).

Interval computations techniques guarantee certain bounds on the quantity \( y \) which is related to known interval-bounded quantities. Thus, they can be also used to prove (guarantee) that certain inequalities hold for all the values within given intervals. This property of interval computations has been successfully used in several recent mathematical breakthroughs.
such as J. Hass and R. Schlafly's 1995 solution to the "double
bubble" iso-perimetric problem and a more recent T. C. Hales'
solution of the Kepler conjecture about the densest arrangement
of spheres in space.

In this talk, we show that interval computations technology can
be useful not only in solving open problems of pure
mathematics, but also in proving results from applied
mathematics. As a case study, we will show how these techniques
can be used to prove results about numerical methods, e.g., to
prove a discrete non-negativity conservation principle for a
certain class of higher order finite element methods.

Keywords: interval computations; computer proofs; mathematics
research using technology; higher order finite element methods