Potential Application of Fuzzy Techniques to Math Education: Emphasizing Paradoxes as a (Seemingly Paradoxical) Way to Enhance the Learning of (Strict) Mathematics

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Abstract—**Classes are often taught in an authoritarian way, when students do not understand the need to study the material. So, a key to success is for a teacher to make sure that the students understand this need – before explaining the material itself. In mathematics, this need sometimes comes from practice; sometimes, from the fact that common sense treatment of the topics leads to contradictions (paradoxes) and therefore, a more formal treatment is needed. Our experience shows that explaining paradoxes like the irrationality of square root of 2 or paradoxes of set theory indeed enhances learning.**

I. MAINSTREAM APPROACH TO PEDAGOGY: NOT AS SUCCESSFUL AS EXPECTED

The traditional approach to teaching has indeed led to many successful ideas and techniques. It has been shown that, when properly used, information technology, active learning, constructivism, and many other ideas indeed drastically improve the teaching process.

However, with many new ideas, new successful tools, the resulting success is often not as spectacular as the innovators originally predicted and hoped.

For example, in the 1960s, a new set of teaching practices was introduced in the teaching of mathematics. These practices – known as New Math – emphasized mathematical structure through abstract concepts like set theory and number bases other than 10. The original idea of the promoters was that it would drastically speed up the knowledge of mathematics in schools and thus, help reduce the gap between the US and other countries. The original experiments indeed showed this to be a success; however, overall, the level of mathematical knowledge did not increase as drastically as expected, and the perception – that the US schoolchildren are lagging behind in math – is still a serious concern.

This phenomenon is not restricted to the United States. It may be somewhat of an irony that the practices of the New Math were originally introduced as a response to the challenges provided by the perceived supremacy of the Soviet mathematical education system. At the same time, a similar perception of the supremacy of the Western mathematical education system has led renowned Russian mathematicians such as A. N. Kolmogorov to introduce similar ideas of more abstract mathematics into the Soviet mathematical education – starting from the level of elementary mathematics; see, e.g., [7], [13].

Originally, when enthusiastic teachers, including Kolmogorov himself, started using this approach in their teaching, they indeed improved the resulting students' knowledge and skills. However, the overall result of this reform in teaching mathematics turned out to be not as spectacular as Kolmogorov and other predicted (based on their initial success). An (overall positive but) reasonably critical analysis of this situation is given, e.g., in [1].

II. NEED FOR UNDERSTANDABILITY

In schools, classes are often taught in an authoritarian way. Specifically, in many cases, when the new material starts, the students do not receive a convincing explanation of why this material is important; instead, the teacher relies on his or her authority as a professional.

This observation helps to explain why the existing pedagogical techniques are often not as successful as predicted: no matter how interested and active the students are, they do not reach their full potential simply because they do not fully understand the need to study this particular material.

In view of this observation, a key to success is for a teacher to make sure that the students understand this need – before explaining the material itself.

III. NEED FOR UNDERSTANDABILITY IS ESPECIALLY IMPORTANT IN TEACHING MATHEMATICS

In many school disciplines like English, the need to communicate clearly may not be always well understood by the students, but it is usually well understand by their parents and the population as a whole.

In mathematics, the situation with understandability is much more problematic. Not only the students do not understand the need for mathematics education; in general, most people do not understand the significance and importance of mathematics.

Most people see mathematics as a collection of useless rules and formulas that they need to memorize to get the correct answer to the textbook problems, rules and formulas that, in their opinion, have no use in real life.

Since students do not understand the need for mathematics education, they feel forced to study it and, as a result, many students develop strong negative attitude towards mathematics ("I hate math").

IV. NATURAL APPROACH TO ENHANCE UNDERSTANDABILITY: PRACTICAL APPLICATIONS

Of course, mathematics is not useless, mathematics is a foundation of science and engineering and, as a result, it lies in the foundation of technological progress. Thus, a natural way to enhance the understanding of why mathematics is important is to make sure that from the very beginning, students do not simply learn the abstract rules and formulas, they are also given numerous examples of *practical* applications of these rules and formulas.

This need is well understood in modern pedagogy. Most textbooks include numerous practical examples whose purpose is to explain the use of various mathematical ideas, rules, and formulas in real life. In spite of this, we still face a situation in which many students do not understand the need for mathematics education and, as a result, do not perform as well as they can.

V. NEED TO ENHANCE UNDERSTANDABILITY IN TEACHING MATHEMATICS: PRACTICAL APPLICATIONS ARE NOT SUFFICIENT

Why is it that the mathematics courses have numerous example of practically useful mathematical calculations, and students still do not understand the need for mathematics?

An important answer to this question is provided by constructivism. According to constructivism, one of the main reasons why students do not understand this need is that the teachers "feed" them the new material in a "pre-packaged" form, and the students passively learn this material without gaining an understanding of why this material is presented in this particular way. A natural solution is to let the students discover the new concepts, the new rules – under the skillful guidance of a teacher.

This need is well understood in modern pedagogy, and it is actually used in teaching mathematics. It is worth mentioning that one should avoid an over-simplistic implementation of this idea when the teacher does not provide enough guidance and expects the inexperienced students to come up with difficult abstract concepts without his or her help. However, under proper guidance, this is indeed a very successful approach.

VI. TWO ASPECTS OF MATHEMATICS

In spite of the successes of the constructivism approach, students often still lack understanding of why studying mathematics is important. In our opinion, one of the reasons is that when we talk about mathematics, we often somewhat confuse two different aspects of mathematics.

From the viewpoint of scientists and engineers who apply mathematics, mathematics, crudely speaking, is a science that helps them perform practically useful computations. From this viewpoint, a practicing engineer and a practicing scientists do not distinguish between well-justified proven results and practically helpful heuristic methods that work in many cases – both are part of what they perceive as mathematics.

For example, from the engineering viewpoint, methods of solving non-linear algebraic and differential equations that are based on their linearization are a legitimate and useful mathematical tool – even though a knowledgeable engineer knows that the linearity assumptions have limitations and that there are practical situations when these assumptions do not work well in practice. Linearized methods work well in the description of normal functioning of mechanical and civil engineering objects, but they are often inadequate in describing different failure or fracture modes.

From the viewpoint of a professional mathematician, however, the main difference between mathematics and all other disciplines is not in practical computations. According to the mathematician's viewpoint, the main idea behind mathematics is the idea that all its statements must be proven and, once proven, they are absolutely correct.

VII. MATHEMATICS AS VIEWED BY MATHEMATICIANS

A physical formula may be well justified by experimental data, but this justification is not a proof. In physics, it often happens that further experiments reveal that the original formula is only approximately true, is only true under certain conditions, and therefore, needs to be modified. This happened, for example, with Newton's mechanics. For several centuries, formulas of Newton's mechanics adequately described physical processes ranging from the motion of planets in a Solar system to the working of engines. However, later, it turned out that, strictly speaking, formulas of Newton's mechanics are only approximately true:

- on the other side of the range, to accurately describe the motion of celestial bodies, we need to take into account effects of Special and General Relativity;
- on the other side of the range, to accurately describe the motion of molecules and atoms (and of elementary particles that form atoms), we need to take into account effects of quantum physics.

From this viewpoint, a physical theory developed in the 19th century usually turned out to be only approximately correct, it will be adjusted (or even completely replaced) by a more accurate more adequate theory.

In contrast, in mathematics, once a theorem is proven, it remains correct.

For example, Pythagoras theorem started as an empirical fact, i.e., in effect, as a physical law. Later on, Pythagoras proved that this formula can be derived from the basic axioms (what we now call Euclidean geometry).

- From the physical viewpoint, the Pythagoras formula is still a useful physical law, actively used in computing the distances between points. However, it is now well understood that it is only approximately true. Indeed, according to General Relativity, the actual space-time is curved; hence, in general, the spatial distance between the points with coordinates (x, y, z) and (x', y', z') is, in general, different from the Pythagoras formula $d = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ – it is described by more general formulas of Riemannian geometry.
- From the mathematical viewpoint, however, Pythagoras theorem remains a correct theorem in Euclidean geometry.

The difference between useful empirical facts and mathematical theorems was well understood already in the ancient times. For example, to compute the area A of a circular-shaped region, mathematicians from the ancient Egypt, Babylon, and Greece used approximate expression for π, such as $\pi \approx 3\frac{1}{7}$. They understood, however, that these expressions were only approximately true, and that the actual value of π is different. On the other hand, the formula $A = \pi \cdot r^2$ that related the desired area to the radius r of the circular region is a proven theorem.

In other words, they used an approximate formula $A \approx 3\frac{1}{7} \cdot r^2$, but they understood that the coefficient $3\frac{1}{7}$ is only approximately true, while the coefficient 2 in r^2 cannot be changed. In contrast, in physics, when it turned out that the Newton's formula $F = G \cdot \frac{m}{r^2}$, for a gravitational force caused by a mass m at a distance r turned out to be only approximately true, physicists proposed changing both the gravitational constant G *and* the parameter 2: e.g., before General Relativity, theories with $F = G \cdot \frac{m}{r^{2+\varepsilon}}$ were the mainstream explanation of the difference between Mercury's predicted and observed motions.

VIII. BACK TO TEACHING MATHEMATICS: WHAT IS OFTEN LACKING IS UNDERSTANDING OF THE NEED FOR MATHEMATICAL RIGOR

The above analysis explains what is missing in the current students' understanding of why mathematics is useful. In short, student may (somewhat grudgingly) accept that formulas are useful, but not proofs.

From this viewpoint, what we need to enhance the students' understanding of mathematics is help them understand that not only engineering formulas are useful, rigorous mathematical proofs have their value. How can we do that?

Good news is that we do not have to invent new reasons: after all, there are reasons why rigorous mathematics was designed in the first place, and why it remains an important (and well-supported) part of science and, more generally, of our quest for knowledge. What we need to do is convincingly convey these reasons to students of mathematics.

In some sense, in mathematics education – like in biology – ontogenesis repeats philogenesis. In biology, an embryo starts as a single cell, then becomes a multi-cell organism, then gets more and more complex – in effect, copying the major steps of biological evolution that started with single-cell organisms, moved to simple multi-cell ones, etc.

Similarly, a young student of mathematics starts with commonsense notions – the same notions that formed the basic mathematics knowledge before mathematics became a science. As the students learn new material, they often repeat the same mistakes and misunderstandings that the students before them made – and, tracing back, the same mistakes that led to the corresponding development of mathematics in the first place.

So, our question is: in the history of mathematics, why was there a need for rigor?

IX. PARADOXES: THE MAIN MOTIVATION BEHIND RIGOR IN MATHEMATICS

History of mathematics (see, e.g., [5]) shows that the main reason why mathematicians started making their methods more rigorous is that heuristic methods sometimes lead to seeming contradictions (*paradoxes*).

Let us remind of some well-known paradoxes (see, e.g., [5]) that led to the development of mathematical rigor.

Heap paradox. Formal approach to mathematics was first developed by the ancient Greeks, and their paradoxes explain why they felt this need for rigor. Probably the first known paradox is the *heap* paradox:

- one little rock does not form a heap;
- if we have a pile of rocks which is not a heap, and add one more rock, then we still do not get a heap;
- by induction, we can thus conclude that no matter how many rocks we add, we will never get a heap;
- on the other hand, everyone knows that heaps are possible.

This paradox can be reformulated in terms of a crowd (one person does not form a crowd, etc.). This paradox, well known to the Greeks, explained the need to restrict ourselves to well-defined notions – in contrast to vaguely defined ("fuzzy") notion such as *heap* or *crowd.*

Heap paradox and the 20th century successes of "fuzzy logic". It is interesting to mention that this same paradox was revived in the 20th century by computer scientists who were interested in making computers understand our commonsense words and terms. To describe such "fuzzy" words in precise terms – understandable to a computer – L. A. Zadeh invented a special technique called *fuzzy logic*; see, e.g., [6]. In fuzzy logic, in contrast to the traditional two-valued logic, notions like "is a heap" or "is a crowd" are not necessarily either true or false. In many practical situations, expert may conclude that to some degree, we have a crowd. When one more person enters, this degree of "crowdedness" increases – until this degree attains its largest possible value 1, meaning that all reasonable persons will conclude that this particular collection of people is a crowd.

This solution of a paradox enabled researchers to successfully use informal (seemingly inconsistent) expert knowledge in the design of efficient automated systems for control, data processing, decision making, etc.; see, e.g. [6].

Fuzzy logic was the first example of *soft computing*, attempts to formalize parts of commonsense reasoning that were "left behind" when mathematics became more rigorous. Fuzzy logic and other soft computing techniques are used in many areas of science and engineering; see, e.g., [6]. In particular, some fuzzy logic techniques have been proposed for use in processing pedagogical data; see, e.g., [9], [4]. The relation between fuzzy logic and paradoxes is analyzed, e.g., in [6], [11].

Diagonal of a unit square: a forgotten paradox. Another known paradox is the fact that the diagonal of a square is not commeasurable with its side, i.e., for example, that the diagonal $\sqrt{2}$ of the unit square is not a rational number. Right now, university students view it as a (somewhat boring) theorem, but when this fact was discovered by the Pythagoreans, it was perceived as a paradox – to such an extent that this result was suppressed for quite some time.

This paradox comes from the fact that in practice, all the numbers come from measurements, be it the length of a line or the mass of a body. Measurements are never 100% accurate; as a result, we can only get an approximate value of the measured quantity. Since measurements are not absolutely accurate, it makes sense to present the result of the measurement only with the accuracy corresponding to the accuracy of the measurement. For example, if we measure the distance with an accuracy of ± 1 km, then we can say that the distance from the University of Texas at El Paso to the El Paso International Airport is approximately 16 km – but, based on this measurement, it does not make too much sense to claim that this distance is approximately 16.75 km. As a result of this procedure, all the numbers that we see in practical applications are rational, i.e., numbers of the type $\frac{p}{q}$, where p and q are real numbers. For us, who are accustomed to decimal numbers, these values are usually decimal fractions, i.e., values of the type $\frac{p}{10^q}$; however, in the ancient world, people used other bases as well, so we can safely consider general rational numbers.

Since all numbers coming from practice are rational, it makes sense, in practice-oriented mathematics, to only consider rational numbers. For several centuries, this idea worked well: if we start with a rational length, we can divide it into 2, 3, and more equal pieces, and still get a rational number. Many other reasonable operations also led to rational numbers. Sometimes, mathematicians did not know how exactly a given number can be represented as a fraction $\frac{p}{q}$, but they assumed that this representation is always possible – and made arguments based on this seemingly natural assumption.

Imagine their surprise when it turned out that the quantity as simple-to-define as a diagonal of a unit square is not a rational number! This discovery has lead to the need for a revolutionary reworking of many previous mathematical results and descriptions.

The amazing thing about this paradox – just like about many other paradoxes – is that it is simple to describe (and thus, not that difficult to teach to the students). Indeed, let us assume that $\sqrt{2} = \frac{p}{q}$ for some integers p and q. We can always divide both p and q by their greatest common divisor, so we can safely assume that p and q have no common factors.

By squaring both sides and multiplying both sides by q^2, we conclude that $p^2 = 2q^2$, hence, we conclude that p^2 is even. Thus, p cannot be odd – because then p^2 would be odd as well; so, p is even; in other words, $p = 2k$ for some k. Therefore, $p^2 = 4k^2$ and hence, $q^2 = 2k^2$ is also even – from which we can similarly deduce that q is even as well. Thus, p and q are both even, i.e., they have a common factor 2: a contradiction to our assumption that p and q have no factor factors.

Achilles and the Tortoise. A paradox that probably everyone knows is the Achilles and the Tortoise paradox. According to this paradox, fast Achilles cannot catch up with a slow Tortoise no matter how fast he runs: before Achilles reaches the distance separating him from the Tortoise, he needs to cover the first half of this distance – and during that time the Tortoise will move even further away.

Similarly to the $\sqrt{2}$ "paradox", to many people, this argument does not sound like a paradox at all. We know how to solve practical problems of who catched up with whom, when, and where, so this "paradox" may sound artificial. However, in the ancient world, when solutions to all these motion-related problems were not known, this paradox was a serious problem. In effect, it was not resolved until the appearance of calculus in the 17th century enabled us to solve motion-related problems.

Paradoxes related to calculus. Calculus resolved some paradoxes, but – as often happens in the history of science, it led to the discovery of several new paradoxes. An example of such a paradox was related to summation of infinite series. Before calculus, there were very few attempts to build a converging sequence or to add up an infinite series. Calculus provided a general framework within which such a summation became routine. Most functions like $\exp(x)$, $\sin(x)$, $\cos(x)$, where shown to be representable as Taylor series – and this provided a very efficient way of computing these functions, so efficient, that even nowadays, Taylor series are the main techniques used by computers to compute the values of these functions.

For many series, calculus-based ideas led to successful formulas for their sum. However, for some series, the same ideas led to paradoxes. Probably the most well known of these paradoxes was discovered by Euler who tried to compute the sum s of the infinite series

$$1 + (-1) + 1 + (-1) + 1 + \ldots$$

On the one hand, we can combine elements into pairs, and end up with

$$s = (1 + (-1)) + (1 + (-1)) + \ldots = 0 + 0 + \ldots = 0.$$

On the other hand, we can keep the first element in this infinite sum intact and combine elements starting with the second one. Then, we get

$$s = 1+((-1)+1)+((-1)+1)+\ldots = 1+0+0+\ldots = 1 \neq 0.$$

This (and similar) paradoxes has led to the need to revisit the foundations of calculus, to introduce rigor. This "revolution of rigor", largely associated with the names of Cauchy and Weierstrass, introduced the modern "epsilon–delta" definitions into calculus. For example, according to Cauchy, an infinite series $a_1 + a_2 + \ldots$ has a sum s if and only if for every $\varepsilon > 0$, there exists an N such that for all $n \geq N$, we have $|(a_1 + \ldots + a_n) - s| \leq \varepsilon$. Euler's paradox is then resolved because according to this definition, the series $1+(-1)+1+(-1)+\ldots$ does not have a sum at all.

The "epsilon–delta" definitions are not easy – but they are needed because without rigorous definitions, we can get paradoxes. Students taking calculus, however, often learn these definitions without understanding why this complexity is needed – and thus, sometimes have trouble understanding the related notions.

Paradoxes of infinity. Infinity has caused many other paradoxes – e.g., the known paradox of Galileo. According to Galileo, there are exactly as many natural numbers as there are squares of natural numbers – because natural numbers can be placed in 1-1 correspondence with their squares: $0 \leftrightarrow 0$, $1 \leftrightarrow 1$, $2 \leftrightarrow 4$, $3 \leftrightarrow 9$, $\ldots$

On the other hand, since most natural numbers are not squares, the number of squares is much smaller than the number of natural numbers.

This paradox helped mathematicians understand that they have to be careful when dealing with infinite sets – and it was not resolved until Georg Cantor invented set theory in 1879. In Cantor's set theory, Galileo's paradox becomes a theorem – an infinite set can have exactly as many elements as its own proper subset.

Russell's paradox and the 20th century revolution of rigor. Set theory sounded like a convenient and consistent new foundations for mathematics – until the famous philosopher Bertran Russell showed that, in its original form, set theory also has paradoxes. Namely, in set theory, some sets are elements of themselves: e.g., a set U of all possible sets is itself a set and thus, its own element: $U \in U$. Other sets are not elements of themselves: e.g., the set N of all natural numbers is not itself a natural number and thus, it is not its own element: $N \notin N$. It is natural to consider the set $V \stackrel{\text{def}}{=} \{x : x \notin x\}$ of all the sets which are not their own elements. Russell asked a natural question: is this set V its own element or not?

There are only two options: $V \in V$ and $V \notin V$ and in both cases, as Russell has shown, we get a contradiction.

Indeed, if $V \in V$, this, by definition of V as a set of all sets that are not their own elements, means that $V \notin V$. This contradicts to the assumption that $V \in V$.

If $V \notin V$, this means that V is an example of a set that we combine into V, so $V \in V$ – also a contradiction.

This paradox was made even clearer by its "barber" reformulation: a barber attached to a military regiment is ordered to regularly shave those (and only those) who do not shave themselves. The paradox appears when we try to find out whether he is required to shave himself or not. If he does not shave himself, then, according to the order, he has to shave himself. If he shaves himself, then the same order prohibits him from shaving himself – again a contradiction.

The main answer to Russell's paradox was similar to the answer to Galileo's paradox: when we deal with infinite sets, we should not rely on common sense, we should only use formal methods. Several axiomatic approaches to set theory have been developed, and in all of them, Russell's paradox is resolved – usually, by showing that the existence of Russell's set V cannot be deduced from the axioms – moreover, this paradox can be viewed as a theorem proving that such a set cannot exist.

In terms of sets, this non-existence may sound not very intuitive, but in terms of the barber reformulation, this solution makes perfect sense: the above instructions to the barber are clearly inconsistent, so there cannot exist a barber who always follows these instructions.

Paradoxes beyond pure mathematics. Similar paradoxes occur not only in pure mathematics, they also occur in applications of mathematics to physics and other disciplines. The most well known example is quantum mechanics – and its mathematical counterpart, Hilbert spaces and operators in Hilbert spaces. Many counter-intuitive properties of quantum mechanics come, crudely speaking, from the fact that the same elementary particle (e.g., photons, electrons) can exhibit seemingly inconsistent properties. For example, the same particle can sometimes behave as a point particle, with no spatial distribution; on the other hand, sometimes, it can behave as a wave, a spatially distributed process.

It is interesting that in the last two decades, it turned out that these seemingly paradoxical properties of quantum objects can be used to drastically speed up computations and provide additional communication security; see, e.g., [12].

X. OUR RECOMMENDATION: TEACHING PARADOXES

Our proposal. In short, what we propose is to teach the students paradoxes. The same paradoxes that motivated mathematicians to introduce rigor in the first place can help in teaching students why rigor is necessary.

This proposal is realistic and helpful. I have explained paradoxes to students in Russia and in the US – as part of the after-school program. My anecdotal evidence – and the experience of several of my Russian colleagues who taught similar paradoxes to students – shows that the paradoxes, with their unpredicted result, do raise the students' interest in the related mathematical concepts.

Teaching paradoxes in mathematics may seem counter-intuitive. For many people who like science and mathematics, one of the main attractions is that science and mathematics are logical, precise, consistent, and clear – in contrast to, e.g., humanities and art, where seemingly inconsistent ideas can (and do) co-exist. From this viewpoint, teaching paradoxes in mathematics may seem a counter-intuitive way of enhancing students interest. Indeed, from this viewpoint, the revelation that mathematics has paradoxes can destroy exactly the motivations that we try to cultivate.

Paradoxes as one of the main sources of evolution in mathematical thought. The viewpoint of science and mathematics as (mostly) paradox-free is usually combined with an understanding of the progress in science and mathematics as a reasonably smooth gradual process. This understanding was shattered by the work of Kuhn [10]. Kuhn's examples were mainly from physics, but history of mathematics confirms that similar revolutionary changes occurred in mathematics as well; see, e.g., [5], [3].

XI. FUZZY TECHNIQUES AS A WAY OF FORMALIZING PARADOXES

Need for paradoxes: reminder. In some cases, students have no prior idea about some objects or techniques. In this case, all we need to do is to explain these ideas to a student. In such simple cases in which students can be viewed as "tabula rasa", there is probably no need for paradoxes – because paradoxes may only confuse the students.

The need for paradoxes comes from the fact that students are rarely "tabula rasa". Whether it comes to statements about the physical world or mathematical objects, students usually have their own preconceptions. For example, before studying physics, students erroneously believe that a moving body will by itself come to a halt, and we need to constantly apply a force to keep it moving. One of the objectives of teaching is to help students learn the correct statement – law of inertia. In such cases, a paradox approach may be useful.

Why fuzzy techniques: general idea. A paradox means that in some sense, both a statement S and its negation $\neg S$ are true. In other words, a paradox is when a composite statement $S \,\&\, \neg S$ holds. The possibility for such statements to be to some degree true is one of the well-known features of fuzzy logic. It is therefore reasonable to use fuzzy logic to describe such paradoxical statements.

Why fuzzy techniques: details. Our objective is to move a student from a state in which he or she (erroneously) believes in a statement S into a state in which he or she believes in the correct statement $\neg S$. Both original state S and the new state $\neg S$ are often very clear and precise, there is nothing fuzzy about them. The need to consider fuzziness comes from the fact that the students' opinions often evolve continually. So, when a student changes his or her opinion from S to $\neg S$, the student's degree of belief in S changes continuously from 1 (absolute belief in S) to 0 (absolute belief in $\neg S$).

Let us describe this transition in more detail. We start with a student absolutely believing in S; his or her degree of belief in S is exactly 1. Then, we instill some elements of doubt in S in the student's mind. At this stage, the student's degree of belief in S decreases – although at the beginning, it is still close to the "absolute" value 1.

If we stop at this point, the student will go back to believing in S: indeed, if a student needs to choose between S and $\neg S$, then, since the student's degree of belief in S in larger than his or her degree of belief in $\neg S$, the student will choose S. The critical transition happens when the student's degrees of belief in S and $\neg S$ are exactly equal – i.e., when the degree of belief in S is exactly 0.5. After this stage, the degree of belief in S becomes smaller than the degree of belief in $\neg S$. Thus, the student may not have yet fully mastered this subject – i.e., his degree of belief in $\neg S$ in still smaller than 1 – but if given a choice between S and $\neg S$, the student will choose the alternative with the larger degree of belief, i.e., the correct statement $\neg S$.

So, to transit from the original incorrect belief S into the desired correct belief $\neg S$, it is crucial to lead the student to the point when his or her degrees of belief in S and in $\neg S$ are exactly equal – i.e., when his original clear idea will be replaced by a paradox.

How fuzzy techniques can be useful. Fuzzy techniques enable us to gauge the person's degree of belief in different statements and thus, to better understand how close we are to the desired paradoxical situation – situation which will bring us on the path to correct understanding.

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