Everything Is a Matter of Degree: A New Theoretical Justification of Zadeh's Principle

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***Abstract*—One of the main ideas behind fuzzy logic and its applications is that everything is a matter of degree. We are often accustomed to think that every statement about a physical world is true or false – that an object is either a particle or a wave, that a person is either young or not, either well or ill – but in reality, we sometimes encounter intermediate situations. In this paper, we show that the existence of such intermediate situations can be theoretically explained – by a natural assumption that the real world is cognizable.**

I. Everything Is a Matter of Degree: One of the Main Ideas Behind Fuzzy Logic

One of the main ideas behind Zadeh's fuzzy logic and its applications is that everything is a matter of degree.

We are often accustomed to think that every statement about a physical world is true or false:

- that an object is either a particle or a wave,
- that a person is either young or not,
- that a person is either well or ill,

but in reality, we sometimes encounter intermediate situations.

II. Formulation of the Problem

That everything is a matter of degree is a convincing empirical fact, but a natural question is: why? How can we explain this fact?

This is what we will try to do in this paper: come up with a theoretical explanation of this empirical fact.

III. There Should be an Objective Theoretical Explanation

Most traditional examples of fuzziness come from the analysis of commonsense reasoning. When we reason, we use words from natural language like "young", "well". In many practical situations, these words do not have a precise true-or-false meaning, they are fuzzy. One may therefore be left with an impression that fuzziness is a subjective characteristic, it is caused by the specific way our brains work.

However, the fact that that we are the result of billions of years of successful adjusting-to-the-environment evolution makes us conclude that everything about us humans is not accidental. In particular, the way we reason is not accidental, this way must reflect some real-life phenomena – otherwise, this feature of our reasoning would have been useless and would not have been abandoned long ago.

In other words, the fuzziness in our reasoning must have an objective explanation – in fuzziness of the real world.

IV. What We Plan to Do

In this paper, we first give examples of objective real-world fuzziness. After these example, we provide an explanation of this fuzziness – in terms of cognizability of the world.

V. First Example of Objective "Fuzziness" – Fractals

The notion of dimension has existed for centuries. Already the ancient researchers made a clear distinction between 0-dimensional objects (points), 1-dimensional objects (lines), 2-dimensional objects (surfaces), 3-dimensional objects (bodies), etc.

In all these examples, dimension is a natural number: 0, 1, 2, 3, ...

Since the 19th century, mathematicians have provided a mathematical extension of the notion of dimension that allowed them to classify some weird mathematical sets as being of fractional (non-integer) dimension, but for a long time, these weird sets remained anomalies.

In the 1970s, B. Mandlebrot noticed that actually, many real-life objects have fractional dimension, ranging from the shoreline of England to the shape of the clouds and mountains to noises in electric circuits (to social phenomena such as stock prices). He called such sets of fractional (non-integer) dimension *fractals*; see, e.g., [7], [8], [9].

It is now clear that fractals play an important role in nature. So, what we originally thought of as an integer-valued variable turned out to be real-valued.

VI. Second Example of Objective "Fuzziness" – Quantum Physics

Until the 19th century, physical phenomena were described by classical physics. In classical physics, some variables are continuous, some are discrete.

For example, the coordinates and velocities of particles usually take continuous values. However, if we are interested in stable states or periodic trajectories, we often end up with a discrete set of stable states.

This discreteness underlies most engineering implementations of computers: to represent 0 or 1, we select an object with 2 possible states, and use one of these states to represent 0 and another to represent 1.

In the 20th century, however, it turned out that a more adequate description of the physical world comes from *quantum physics*. One of the peculiar features of quantum physics is the so-called *superposition principle* (see, e.g., [2]) according to which with every two states $\langle 0|$ and $\langle 1|$, it is also possible to have "intermediate" states (superpositions) $c_0 \cdot \langle 0| + c_1 \cdot \langle 1|$ for all complex values c_0 and c_1 for which $|c_0|^2 + |c_1|^2 = 1$.

So, what we originally thought of as an integer-valued variable turned out to be real-valued.

Comment. It is worth mentioning that these quantum combinations of 0 and 1 states are not only happening in real life, but, as it was discovered in the 1990s, their use can drastically speed up computations. For example:

- we can search in an unsorted list of n elements in time $\sqrt{n}$ – which is much faster than the time n which is needed on non-quantum computers [5], [6], [12];
- we can factor a large integer in time which does not exceed a polynomial of the length of this integer – and thus, we can break most existing cryptographic codes like widely used RSA codes which are based on the difficulty of such a factorization on non-quantum computers [12], [15], [16].

These techniques form the basis of *quantum computing*; see, e.g., [12].

VII. Third Example of Objective "Fuzziness" – Fractional Charges of Quarks

In the late 19th century and early 20th century, it was experimentally confirmed that seemingly continuous matter is actually discrete: it consists of molecules, molecules consist of atoms, and atoms consist of elementary particles.

A part of this confirmation came from an experimental discovery that all electric charges are proportional to a single charge – which was later revealed to be equal to the charge of an electron.

Based on this proportionality, physicists concluded that many observed elementary particles ranging from (relatively) stables particles such as protons and neutrons to numerous unstable ones – like many mesons and baryons discovered in super-collides and in cosmic rays – cannot be further decomposed into "more elementary" objects.

In the 1960s, M. Gell-Mann [2], [4], [14] discovered that if we allow particles with fractional electronic charge, then we can describe protons, neutrons, mesons, and baryons as composed of 3 (now more) even more elementary particles called *quarks*. At first, quarks were often viewed as purely mathematical constructions, but experiments with particle-particle collisions revealed that, within a proton, there are three areas (*partons*) where the reflection seems to be the largest – in perfect accordance with the fact that in the quark model, a proton consists of exactly three quarks.

So, what we originally thought of as an integer-valued variable turned out to be real-valued.

VIII. There Exist Other Examples of Objective "Fuzziness"

In physics, there are many other examples when what we originally thought of as an integer-valued variable turned out to be real-valued. In this paper, we just described the most well known ones.

IX. Our Explanation of Why Physical Quantities Originally Thought to Be Integer-Valued Turned out to Be Real-Valued: Main Idea

In philosophical terms, what we are doing is "cognizing" the world, i.e., understanding how it works and trying to predict consequences of different actions – so that we will be able to select an action which is the most beneficial for us.

Of course, our knowledge is far from complete, there are many real-world phenomena which we have not cognized yet – and many philosophers believe that some of these phenomena are not cognizable at all.

If a phenomenon is not cognizable, there is nothing we can do about it. What we are interested in is phenomena which are cognizable. This is what we will base our explanation on – that in such phenomena, it is reasonable to expect continuous-valued variables, i.e., to expect that properties originally thought to be discrete are actually matters of degree.

X. First Explanation: Goedel's Theorem vs. Tarski's Algorithm

A. Goedel's theorem: a brief reminder

Our first explanation of "objective fuzziness" is based on the historically first result in which something was actually proven to be not cognizable – the well-known 1931 Goedel's theorem; see, e.g., [3].

This theorem can be formulated in terms of arithmetic. Specifically, we have variables which run over natural numbers 0, 1, 2, ... A term is anything that can be obtained from these variables and natural-valued constants by using addition and multiplication, e.g., $2 \cdot x \cdot y + 3 \cdot z$ (subtraction is also OK).

Elementary formulas are defined as expressions of the type $t = t'$, $t < t'$, $t > t'$, $t \le t'$, $t \ge t'$, and $t \ne t'$ for some terms t and t'. Examples are $2 \cdot x \cdot y + 3 \cdot z = 0$ or $x < y + z$.

Finally, a formula is anything which is obtained from elementary formulas by using logical connectives "and" (&), "or" ($\vee$), "implies" ($\rightarrow$), "not" ($\neg$), and quantifiers "for all x" ($\forall x$) and "there exists x" ($\exists x$). Example:

$$\forall x\, \forall y (x < y \rightarrow \exists z (y = x + y)).$$

Many statements about the physical world can be formulated in terms of such formulas. Our objective is therefore to find out whether a given formula is true or false.

Goedel's theorem states that no algorithm is possible that would, given a formula, check whether this formula is true or false. In other words, if we allow variables with discrete values, then it is not possible to have an algorithm which would solve all the problems.

B. Tarksi's result

In the 1940s, another well-known logician, Alfred Tarski, raised an interesting question: what if we only allow continuous variables? In other words, what if we consider the same formulas as Goedel considered, but we change their interpretation: now every variable can take arbitrary real values. It turns out that in this case, it is possible to have an algorithm that, given a formula, checks whether this formula is true or false. [17].

C. Conclusion

Thus, in a cognizable situations, we cannot have variables which only take discrete values – these variables must be able to take arbitrary real values.

Comment. It is worth mentioning that the original Tarski's algorithm required an unrealistically large amount of computation time; however, later, faster, practically useful algorithms have been invented; see, e.g., [1], [10].

XI. Second Explanation: Efficient Algorithms for Linear Algebra vs. NP-Hardness of Integer Programming

A. Not all algorithms are practical

Our first explanation of continuity (and "fuzziness") was that with the discrete variables, we cannot have a deciding algorithm, but with continuous variables, we can.

The existence of an algorithm is necessary for cognition, but not sufficient. It is well known that some theoretical algorithms are not practical at all. For example, if an algorithm requires an exponential number of computational steps 2^n on an input of size n, this means that for inputs of a reasonable size $n \approx 300-400$, the required computation time exceeds the lifetime of the Universe.

B. Feasible vs. non-feasible algorithms

There is still no perfect formalization of this difference between "practical" (*feasible*) and impractical (*non-feasible*) algorithms. Usually:

- algorithms for which the computation time $t_A(x)$ is bounded by some polynomial $P(n)$ of the length $n = \text{len}(x)$ of the input (e.g., linear-time, quadratic-time, etc.) are practically useful, while
- for practically useless algorithms, the computation time grows with the size of the input much faster than a polynomial.

In view of this empirical fact, in theoretical computer science, algorithms are usually considered *feasible* if their running time is bounded by a polynomial of n. The class of problems which can be solved in polynomial time is usually denoted by P; see, e.g., [13].

C. Notion of NP-hardness

Not all practically useful problems can be solved in polynomial time. To describe such problems, researchers have defined several more general classes of problems. One of the most well known classes is the class NP. By definition, this class consists of all the problems which can be solved in *non-deterministic* polynomial time – meaning that if we have a guess, we can check, in polynomial time, whether this guess is a solution to our problem.

Most computer scientists believe that NP≠P, i.e., that some problems from the class NP cannot be solved in polynomial time. However, this inequality has not been proven, it is still an open problem. What *is* known is that some problems are *NP-hard*, i.e., any problem from the class NP can be reduced to each of these problems in polynomial time. One of such NP-hard problems is the problem SAT of propositional satisfiability: given a propositional formula F, i.e., a formula obtained from Boolean (yes-no) variables $x_1, \ldots, x_n$ by using &, $\vee$, and $\neg$, check whether there exist values $x_1, \ldots, x_n$ which make this formula true.

NP-hardness of SAT means that if NP≠P (i.e., if at least one problem from the class NP cannot be solved in polynomial time), then SAT also cannot be solved in polynomial time. In other words, SAT is the hardest of the problems from this class.

It is known that all the problems from the class NP can be solved in exponential time. Indeed, for a problem of size n, there are $\leq a^n$ possible guesses, where a is the size of the corresponding alphabet, so we can simply try all these guesses one by one.

D. Systems of linear equations

One of the simplest-to-solve numerical problems is the solution to a system of linear equations

$$a_{11} \cdot x_1 + \ldots + a_{1n} \cdot x_n = b_1;$$

$$a_{21} \cdot x_1 + \ldots + a_{2n} \cdot x_n = b_2;$$

$$\ldots$$

$$a_{m1} \cdot x_1 + \ldots + a_{mn} \cdot x_n = b_m.$$

In the situation when all the unknowns x_i can take arbitrary real values, there exist efficient algorithms for solving such systems of equations – even the well-known Gauss elimination method, while not the fastest, it still feasible.

However, as soon as we restrict ourselves to discrete (e.g., integer-valued) variables x_i, the solution of such a system becomes an NP-hard problem [13].

E. Conclusion

So, we end up with the same conclusion: that in a cognizable situations, we cannot have variables which only take discrete values – these variables must be able to take arbitrary real values.

XII. Symmetry: Another Fundamental Reason for Continuity ("Fuzziness")

A. Case study: benzene

To explain why symmetry leads to continuity, let us start with a chemical example. In the traditional chemistry, a molecule is composed from atoms that exchange electrons with each other. If an atom borrows one electron from another atom, we say that they have a connection of valence 1, if two electrons, there is a connection of valence 2, etc.

From the analysis of benzene, it has been clear that it consists of 6 carbon and six hydrogen atoms, i.e., that its chemical formula is C_6H_6. However, for a long time, it was not clear how exactly they are connected to each other. The solution came in the 19th century to a chemist August Kekule in a dream. He dreamed of six monkeys that form a circle in which each monkey holds to the previous monkey's tail. According to this solution, the six C atoms form a circle. To each of these atoms, a H atom is attached. Each C atom has a 1 valence connection to H, 1 valence connection to one of its neighbors, and 2 to another neighbor.

The resulting chemical structure is still routinely described in chemical textbooks – because a benzene loop is a basis of organic chemistry and life. However, now we understand that this formula is not fully adequate. Indeed, according to this formula, the connections between C atoms are of two different types: of valence 1 and of valence 2. In reality, the benzene molecule is completely symmetric, there is no difference between the strengths of different connections.

It is not possible to have a symmetric configuration is we require that valencies are integers. To equally split the remaining valence of 3 (1 is taken for H) between the two neighbors, we need a valence of 3/2. This is not possible in classical chemistry – but this is possible, in some sense, in quantum chemistry where, as we have mentioned, we have a continuum of intermediate states; see, e.g., [2].

B. Fuzzy logic itself is such an example

Fuzzy logic itself can be viewed as an example where symmetries leads to values intermediate between the original discrete values.

Indeed, in traditional logic, we have two possible truth values: 1 ("true") and 0 ("false"). How can we use this logic to describe the absence of knowledge? If we do not know whether a given statement A is true or not, this means that we have the exact same degree of belief in the statement A as we have in its negation $\neg A$. In the traditional logic, none of the two truth values are symmetric (invariant) under such transformation $A \to \neg A$. Thus, to adequately describe this situation, we need to also consider additional (intermediate) truth values.

And indeed, in fuzzy logic with the set of truth values $[0,1]$ and the negation operation $f_\neg(x) = 1 - x$, there s a value which is invariant under the operation $A \to \neg A$: the value 0.5.

XIII. Case Study: Territory Division

A. Formulation of the problem

In many conflict situations, several participants want to divide a territory between themselves. It may be farmer's children dividing his farm, it may be countries dividing a disputed territory.

B. Traditional (non-fuzzy) formalization of the problem

Let us follow [11] and describe a traditional (non-fuzzy) formalization of this problem. Let us denote the disputed territory (i.e., to be more precise, the set of all the points in this territory) by T. Our objective is to divide this territory between n participants, i.e., to select a division of the set T into the sets $T_1, T_2, \ldots, T_n$ for which $T_i \cap T_j = \emptyset$ for $i \neq j$ and

$$T_1 \cup T_2 \cup \ldots \cup T_n = T.$$

It is reasonable to assume that the utility u_i of the i-th participant in acquiring the territory T_i is linear in T_i, i.e., has the form

$$u_i(T_i) = \int_{T_i} U_i(x)\,dx$$

for some appropriate function $U_i(x)$. As we mentioned in [11], it is reasonable to use Nash's criterion to select the optimal division, i.e., to select the division for which the product

$$u \stackrel{\text{def}}{=} u_1(T_1) \cdot u_2(T_2) \cdot \ldots \cdot u_n(T_n)$$

attains the largest possible value. According to [11], in the optimal solution, for every participants i, there is a weight c_i such that each point x is assigned to the participant with the largest weighted utility $c_i \cdot U_i(x)$.

In particular, for two participants, there is a threshold c such that all the points x for which $U_1(x)/U_2(x) > c$ go to the first participant, and all the points x for which $U_1(x)/U_2(x) < c$ go to the second participant.

C. Possibility of a "fuzzy" solution

From the commonsense viewpoint, why do we have to necessarily divide all the disputed territory? Why cannot we control some parts of it together? In other words, instead of dividing the set T into subsets T_i, why cannot we assign, to every point $x \in T$ and to every i, the degree $d_i(x)$ to which the i-th participant will control the neighborhood of this point – in such a way that for every point x,

$$d_1(x) + \ldots + d_n(x) = 1.$$

In other words, instead of a crisp partition we have a fuzzy partition.

In this setting, the utility u_i of the i-th participant has the form

$$u_i(d_i) = \int U_i(x) \cdot d_i(x)\,dx,$$

and our objective is to find a fuzzy partition for which the product

$$u \stackrel{\text{def}}{=} u_1(d_1) \cdot u_2(d_2) \cdot \ldots \cdot u_n(d_n)$$

attains the largest possible value.

D. Observation: the above "fuzzy" problem always has a crisp optimal solution

The derivation from [11] was based on the idea that if we attain a maximum, then a small change of assignment in the vicinity of each point will only decrease (or not change) the desired product. For the fuzzy problem, a similar argument shows that there are weights c_i such that in the optimal solution, every point x for which the weighted utility each point x is assigned to the participant with the largest weighted utility $c_i \cdot U_i(x)$ of the i-th participant is larger than the weighted utility of all other participants is assigned to this i-th participant.

The only points about which we cannot make a definite assignment are the ones in which two or more participants have exactly the same weighted utility. How we divide these points between these participants does not matter – as long as the overall degree of all the points assigned to each of these participants remains the same. In particular, this means that it is always possible to have a crisp division with the optimal value of the desired product.

So, we arrive at a somewhat paradoxical situation: even when we allow "fuzzy" divisions, the corresponding optimization problem always have a crisp solution. So, at first glance, it may seem that fuzzy solutions are not needed at all.

As we will see, the situation changes if we consider symmetry.

E. Symmetry leads to fuzziness

For the territory division problem, a symmetry means a transformation $f : T \to T$ that preserves the area of each (crisp) subset and that preserves the utility of each subarea to each participant. Preserving area means that f has to be a measure-preserving transformation. Preserving utility means that we must have $U_i(x) = U_i(f(x))$ for all x.

It is reasonable to require that if the original situation allows a symmetry, then the desired division should be invariant with respect to this symmetry. Let us show that this requirement leads to a fuzzy solution.

Indeed, let us consider the simplest situation in which we have only two participants, and both assign equal value to all the points $U_1(x) = U_2(x) = 1$. In this case, the utility of each set T_i is simply equal to its area A_i, so the optimization problem takes the form

$$A_1 \cdot A_2 \to \max .$$

Since the sum $A_1 + A_2$ is equal to the area A of the original territory T, this problem takes the form

$$A_1 \cdot (A - A_1) \to \max .$$

One can easily check that the optimal crisp solution means that $A_1 = A/2$, i.e., that we divide the area T into two equal halves.

This solution is optimal but it is not symmetric. Indeed, in this case, symmetries are simply area-preserving transformations. Symmetry of the division means that $f(T_1) = T_1$ for all such transformations f. However, for every two points $x, y \in T$, we can have an area-preserving transformation f that maps x into y: $f(x) = y$. In particular, we can have sauch a transformation for $x \in T_1$ and $y \in T_2$, in which case $f(T_1) \neq T_1$. Thus, a crisp symmetric solution is impossible.

In contrast, a fuzzy symmetric solution is quite possible – and uniquely determined: we simply assign to each point x equal degrees $d_1(x) = d_2(x) = 1/2$. Then, $f(d_1) = d_1$ and $f(d_2) = d_2$ for all area-preserving transformations f.

In general, we always have an optimal symmetric solution: in this solution, equally desired points – for which $c_i \cdot U_i(x) = c_j \cdot U_j(x)$ – are all assigned a joint control with the same degree of ownership depending only on ia and j.

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REFERENCES

[1] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, Springer-Verlag, Berlin, 2006.

[2] R. P. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, 2005.

[3] T. Franzen, *Gödel's Theorem: An Incomplete Guide to its Use and Abuse*, A.K. Peters, 2005.

[4] M. Gell-Mann, *The Quark and the Jaguar*, Owl Books, 1995.

[5] L. K. Grover, "A fast quantum mechanical algorithm for database search", *Proceedings of the 28th Annual ACM Symposium on the Theory of Computing*, May 1996, pp. 212-ff.

[6] L. K. Grover, "From Schrödinger's equation to quantum search algorithm", *American Journal of Physics*, 2001, Vol. 69, No. 7, pp. 769–777, 2001.

[7] B. Mandelbrot, *Fractals: Form, Chance and Dimension*, W. H. Freeman and Co, 1977.

[8] B. Mandelbrot, *The Fractal Geometry of Nature*. W. H. Freeman & Co, 1982.

[9] B. Mandelbrot and and R. L. Hudson, *The (Mis)Behavior of Markets: A Fractal View of Risk, Ruin, and Reward*, Basic Books, 2004.

[10] B. Mishra, "Computational real algebraic geometry", in: *Handbook on Discreet and Computational Geometry*, CRC Press, Boca Raton, Florida, 1997.

[11] H. T. Nguyen and V. Kreinovich, "How to Divide a Territory? A New Simple Differential Formalism for Optimization of Set Functions", *International Journal of Intelligent Systems*, 1999, Vol. 14, No. 3, pp. 223–251.

[12] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.

[13] C. H. Papadimitriou, *Computational Complexity*, Addison Wesley, San Diego, 1994.

[14] B. Povh, *Particles and Nuclei: An Introduction to the Physical Concepts*, Springer-Verlag, 1995.

[15] P. Shor, "Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer", *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, Santa Fe, NM, Nov. 20–22, 1994.

[16] P. Shor, "Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer", *SIAM J. Sci. Statist. Comput.*, 1997, Vol. 26, pp. 1484-ff.

[17] A. Tarski, *A Decision Method for Elementary Algebra and Geometry*, 2nd ed., Berkeley and Los Angeles, 1951, 63 pp.